Recoloring bounded treewidth graphs

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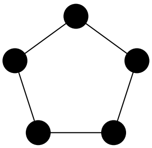
Recoloring graphs



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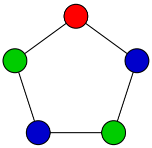

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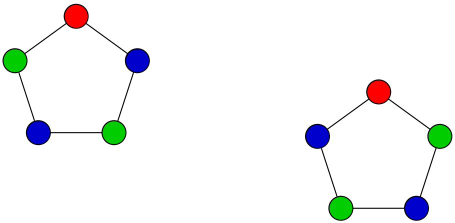

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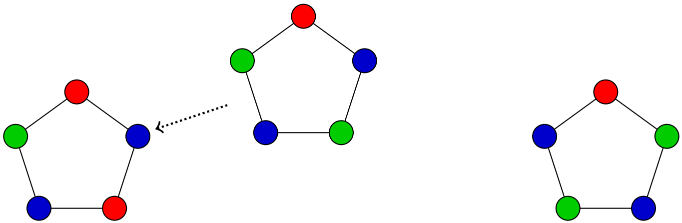

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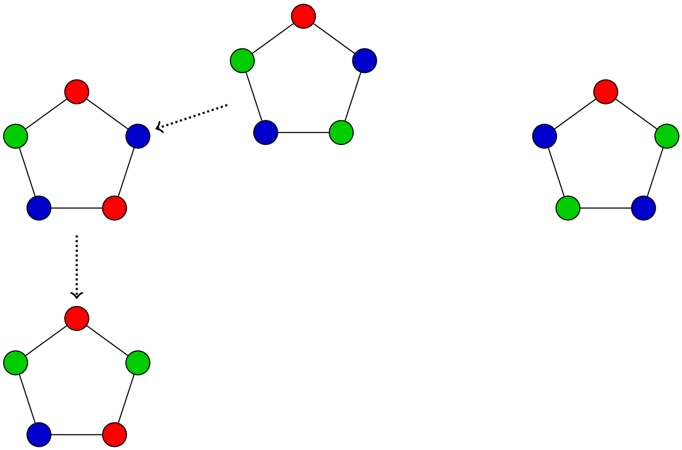

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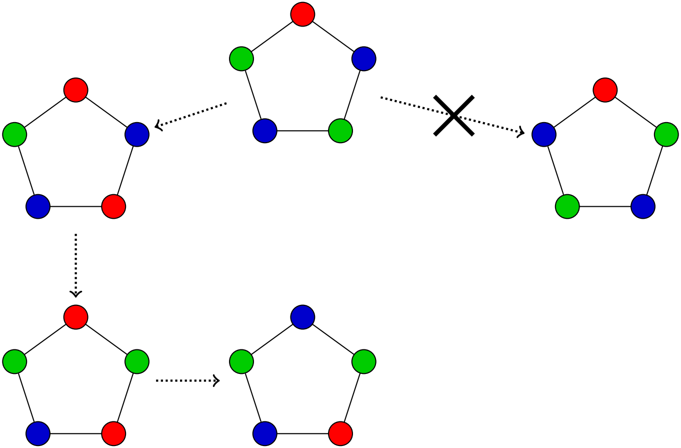

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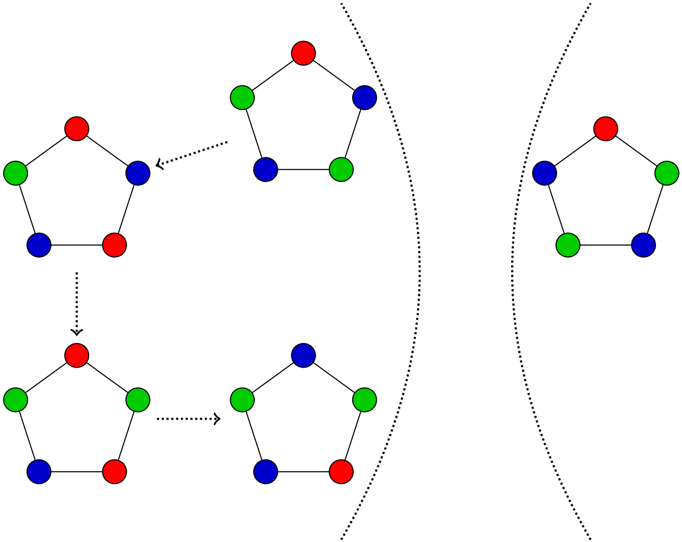
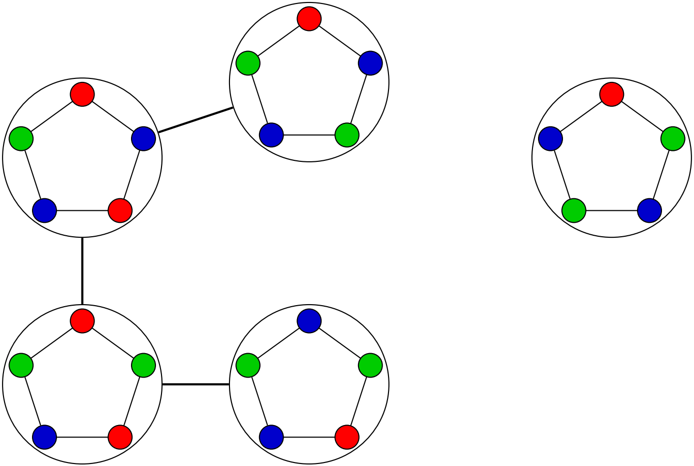


Recoloring graphs ⇒ Reconfiguration graphs

Solutions // Vertices. Adjacent solutions // Neighbors.



Reconfiguration graph

More formally

k-Reconfiguration graph of G

- Vertices: Proper k-colorings of G
- Edges between any two k-colorings which differ on exactly one vertex.

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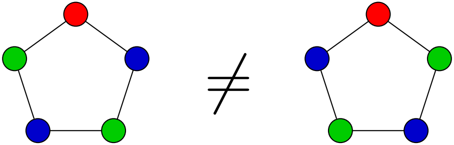
Remark

Two colorings equivalent up to color permutation are distinct.



Interesting questions

- Two solutions:
  - Are in the same connected component?
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  - Are in the same connected component?
  - What distance between them?
- Reconfiguration graphs:
  - Connex?
  - What diameter?

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A graph is k-mixing if its k-reconfiguration graph is connected.

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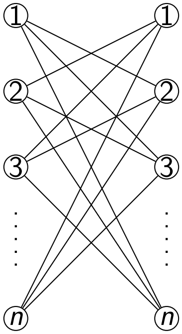

Gap

No function f on the chromatic number ensures that G is k-mixing if $k \geq f(\chi)$.

State of the art

Theorem (Cereceda, van den Heuvel, Johnson '07)

Determining if a bipartite graph is 3-mixing is co-NP hard.

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Recoloring diameter

Given a k-mixing graph, the recoloring diameter is in $\mathcal{O}(A(n))$ if the diameter of the k-reconfiguration graph is bounded by $C \times A(n)$. (n is the number of vertices)

Upper bounds on recoloring

Theorem (Cereceda)

As long as $k \geq n + 1$, the clique K_n is k-mixing in $\mathcal{O}(n)$.

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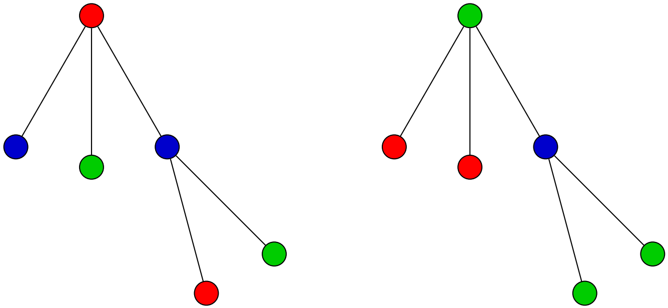

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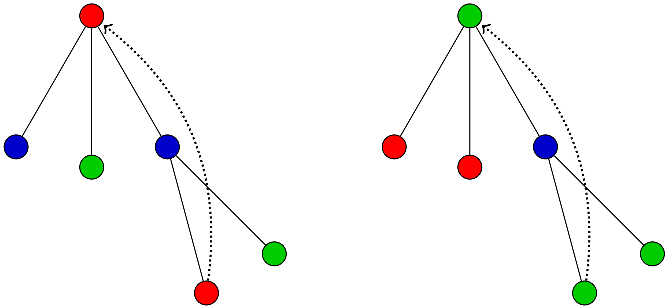

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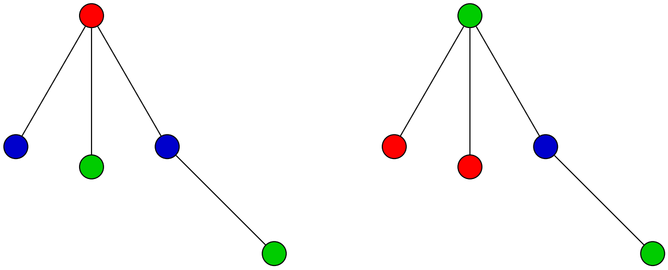

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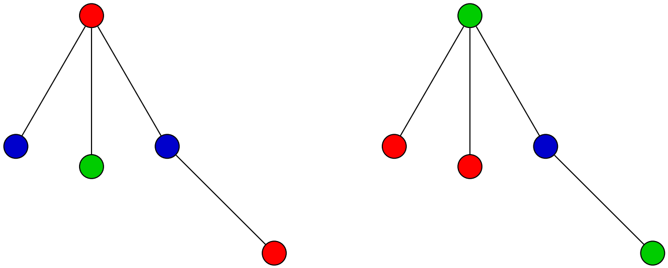

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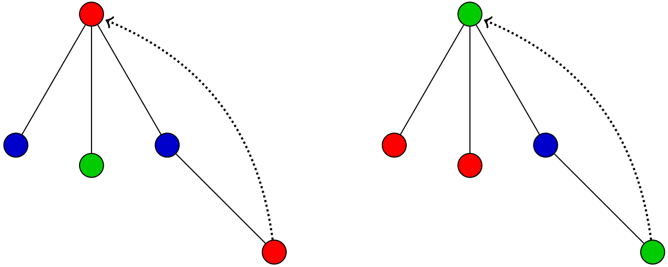

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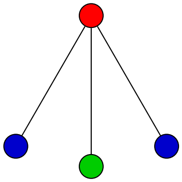

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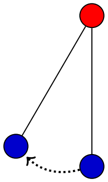

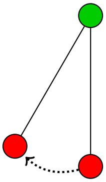

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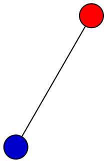

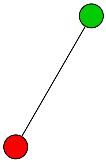

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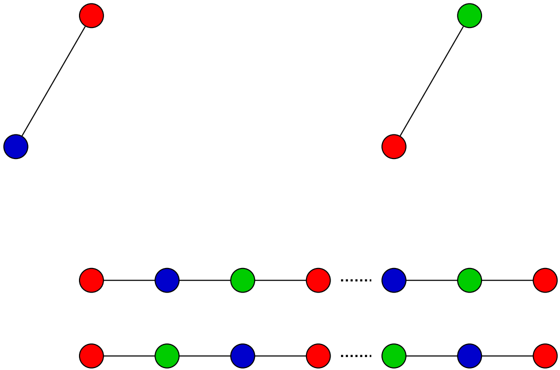

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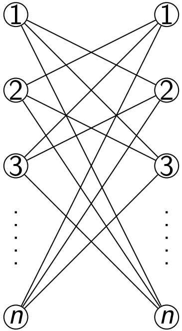
Questions

- Does the same hold for bounded treewidth graphs?
- And for perfect graphs?

Perfect graphs: a counter-example



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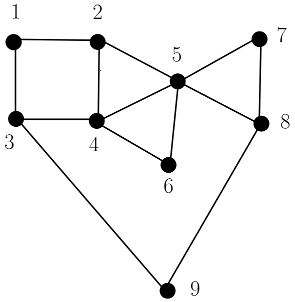

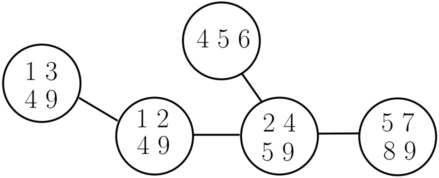

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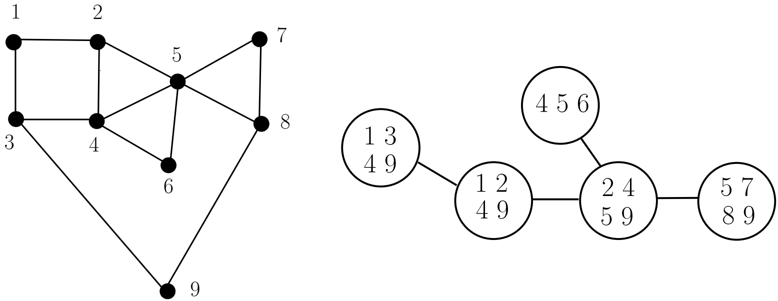

Theorem (Cereceda et al.)

Every k-degenerate graph is $(k + 2)$-mixing in 2^n.

Our main result

Theorem (Bonamy, B.)

Every graph G is $(tw(G)+2)$-mixing in $\mathcal{O}(n^2)$.

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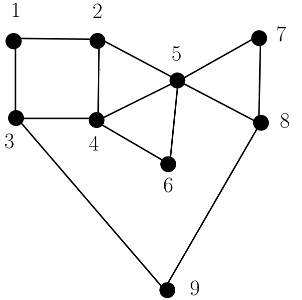
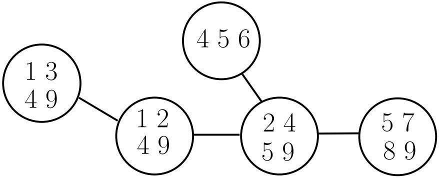
Optimal

- $tw(K_n) = n-1$, so $tw(G)+2$ colors are necessary.
- Since 3-colorings of paths have a quadratic recoloring diameter, the quadratic bound is necessary.

Sketch of the proof





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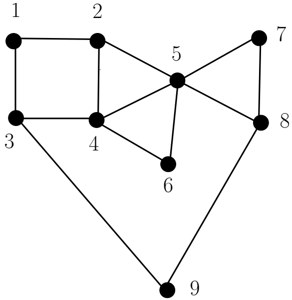

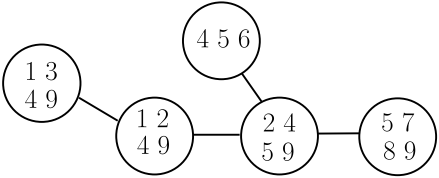

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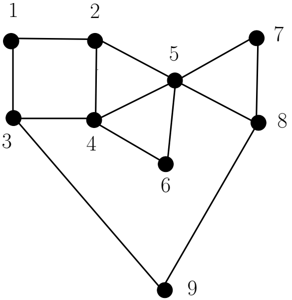

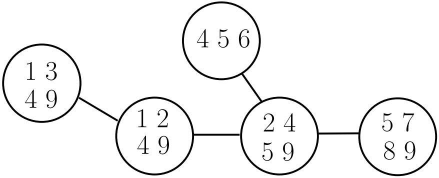

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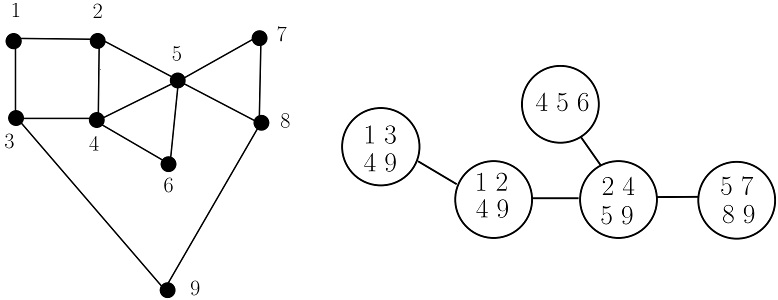

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- Waiting for the identification with a not too costful operation.

Conclusion

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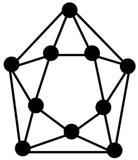

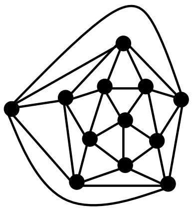

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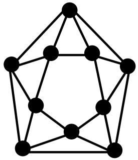

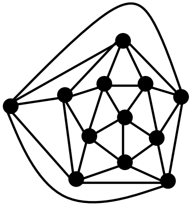

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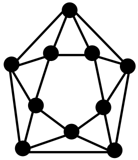

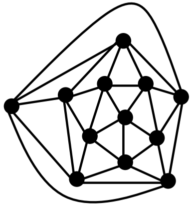

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Thanks for your attention !