Coalition games on interaction graphs

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joint work with Zhentao Li and Adrian Vetta

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- Unfortunately, people are selfish : if it is is more interesting for them, they will create a project of their own.
- Solution : distribute payoff in such a way people do not want to leave the grand coalition.

Coalition games

Coalition game

- A set I of n agents.
- A valuation function $v : 2^n \rightarrow \mathbb{N}$. (the value generated by any group deciding to work on another project)

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A subset S of agents is a coalition if $v(S)$ is positive.

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Definition (coalition)

A subset S of agents is a coalition if $v(S)$ is positive.

Goal :
Distribute money to the agents in such a way, every coalition S receiced at least $v(S)$ euros.
$\Rightarrow$ No one wants to leave the *grand coalition* and work on its own project.

Definition (core)

The core of the coalition game is the set of vectors of payoff satisfying the following constraints :

$$\begin{aligned} \textstyle\sum_{i \in I} x_i &= v(I) & & \\ \text{and } \textstyle\sum_{i \in S} x_i &\geq v(S) & & \forall S \subseteq I \\ x_i &\geq 0 & & \forall i \in I \end{aligned}$$

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Linear Programming

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Why Linear Programming ?

- Real Numbers Linear Programming can be solved in polynomial time.
- Integral Linear Programming are NP-hard to solve.

Relative cost of stability

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$$\begin{aligned} \nu(\mathcal{G}) = \max \quad & \sum_{S:S\subseteq I} v(S)\cdot y_S \\ \text{s.t.} \quad & \sum_{S\subseteq I: i\in S} y_S \ \leq\ 1 \quad \forall i\in I \\ & y_S \text{ is integral and } \ \geq\ 0 \quad \forall S\subseteq I \end{aligned}$$

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Remark : It is an (integral) LP.

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Simplification :

$\Rightarrow$ Since $\tau^* - \nu$ is not "stable" (by disjoint copy for instance), we consider $\frac{\tau^*}{\nu}$ instead.

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Game theoretical interpretation :
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By Strong Duality Theorem, we have :

$$\nu(\mathcal{G}) \leq \nu^*(\mathcal{G}) = \tau^*(\mathcal{G}) \leq \tau(\mathcal{G})$$

Thus

$$\frac{\tau^*}{\nu} = \frac{\nu^*}{\nu} \leq \frac{\tau}{\nu}$$

Hypergraph representation

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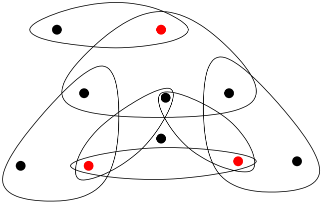

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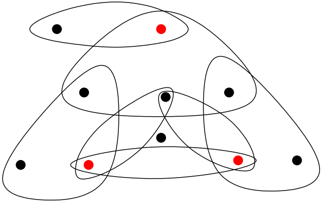
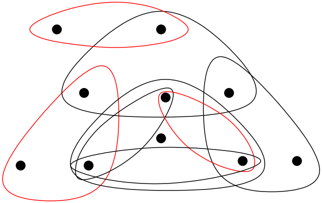


$$\nu(\mathcal{G}) = \max \sum_{S : S \subseteq I} v(S) \cdot y_S \quad \text{s.t.} \sum_{S \subseteq I : i \in S} y_S \leq 1 \quad \forall i \in I$$



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Lemma

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Example :

- A set of agents $I = \{1, \ldots, n\}$
- A subset S of I is a coalition if and only if $|S| > |I|/2$.

- Maximum Packing : 1.
- Minimum Covering : $\geq \frac{|I|}{2} - 1$.

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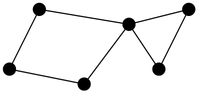
Question

When can we bound $\frac{\tau}{\nu}$?

Graphs

A graph is a pair (V, E) where :

- V is a set of “points” called vertices ;
- E is a set of links between points called edges.



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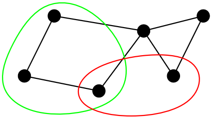

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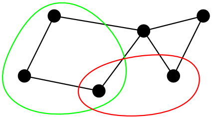
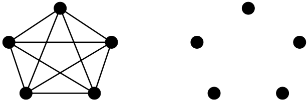


A clique is a graph where vertices are pairwise incident.

A stable set is a graph where vertices are pairwise not incident.



Interaction graph

Myerson proposed the following model :

Definition (interaction graph)

Let G be a graph where the vertices of G are the agents of the coalition game $\mathcal{G}$.
The game $\mathcal{G}$ has interaction graph G if every coalition is connected (*i.e.*, if $v(S) > 0$ then S is connected).

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Interpretation : The agents must be able to communicate if they want to create a coalition.

Examples :

- G is a clique : any coalition may exist.
- G is a stable set : coalitions have size one.

Treewidth and coalition game

Theorem (Meir et al.)

Let G be a graph. We have the following inequality :

$$\frac{\tau(\mathcal{G})}{\nu(\mathcal{G})} \leq_\forall tw(G) + 1$$

Moreover there exist graphs for which this bound is tight.

By $\leq_\forall$, we mean that every game $\mathcal{G}$ on interaction graph G satisfies this inequality.

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Our work : Improve this result, and bounds on the relative cost of stability.

Treewidth

A tree T and a (bag) function $f : T \to 2^V$ is a tree decomposition of $G = (V, E)$ if :

- For every $v \in V$, the set of nodes containing v in their bags is a subtree T_v of T.
- For every edge uv, T_u and T_v intersects.

The width of a decomposition is the maximum size of a bag of the tree-decomposition minus one.

Definition (treewidth)

The treewidth of G, $tw(G)$, is the minimum width of a tree-decomposition of G.

Examples

- K_n has a tree decomposition of width $n-1$ (all the vertices are in the same bag).

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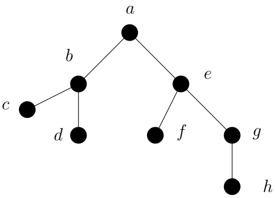
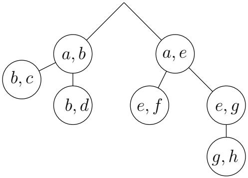
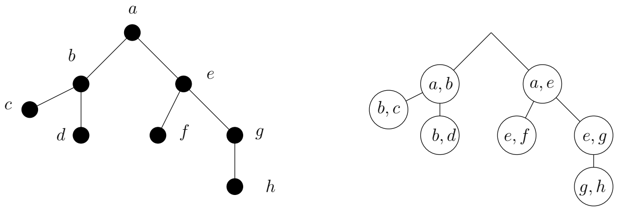
- Trees have tree decompositions of width 1.

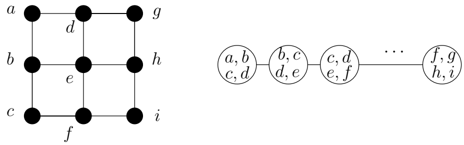




Examples

- K_n has a tree decomposition of width $n-1$ (all the vertices are in the same bag).
- Trees have tree decompositions of width 1.
- The $d \times d$ grid has a tree decomposition of width d.





A new invariant : vinewidth

A tree T and a function $f : T \to 2^V$ is a vine decomposition of $G = (V, E)$ if :

- For every $v \in V$, the set of nodes containing v in their bag is a subtree T_v of T.
- For every edge uv, T_u and T_v **intersects or share an edge**.

The width of a decomposition is the maximum size of a bag of the vine-decomposition.

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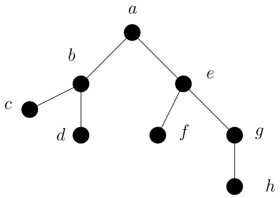

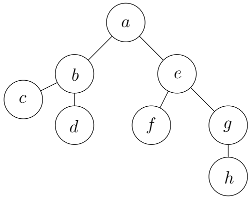

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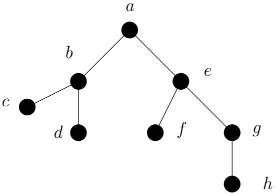

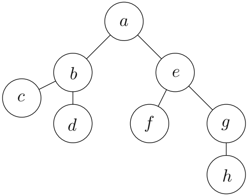

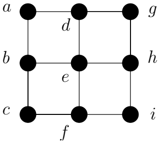

Link between treewidth and vinewidth

Lemma

Every graph G satisfies

$$\frac{tw(G)+1}{2} \leq vw(G) \leq tw(G)+1$$

Moreover both inequalities are tight.

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Proof sketch :

- Any tree decomposition is a vine decomposition.
  $\Rightarrow vw(G) \leq tw(G)+1$.
- We make the "union" of every pair of adjacent bags to be sure that T_u and T_v intersect.
  $\Rightarrow tw(G) \leq 2 \cdot vw(G) - 1$.

Main statement

$$\tau = \min \sum_{i \in I} x_i \quad \text{s.t.} \sum_{i : i \in S} x_i \geq v(S) \quad \forall S \subseteq I$$

$$\nu = \max \sum_{S : S \subseteq I} v(S) \cdot y_S \quad \text{s.t.} \sum_{S \subseteq I : i \in S} y_S \leq 1 \quad \forall i \in I$$

Theorem (BLV'14)

For every graph G, we have :

$$vw(G) \leq_{\exists} \frac{\tau(\mathcal{G})}{\nu(\mathcal{G})} \leq_{\forall} vw(G)$$

By $\leq_{\forall}$, we mean that every game $\mathcal{G}$ on interaction graph G satisfies this inequality.
By $\leq_{\exists}$, we mean that there exists a game $\mathcal{G}$ on interaction graph G which satisfies this inequality.

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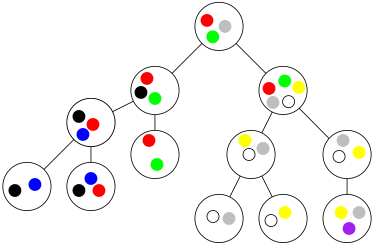
Bottom-up from the leaves of a (rooted) vine decomposition.

- If all the vertices of a coalition C are in the bag B_f of a leaf f, add vertices of B_f in the covering and C in the packing. Delete the coalitions containing one vertex of B_f.
- Otherwise, delete all the leaves of T.



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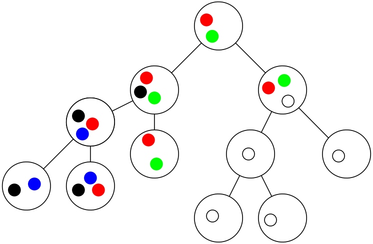

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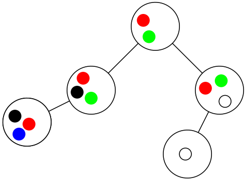

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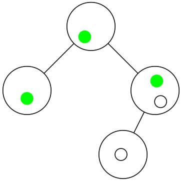

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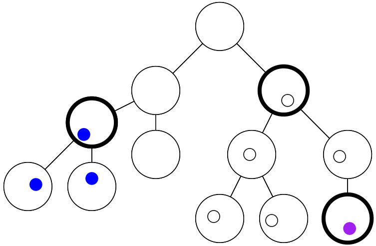

- We have selected k disjoint coalitions : $\nu(\mathcal{G}) \geq k$.
- We have deleted at most $vw(G) \cdot k$ vertices.

Relative cost of stability

Reminder of the motivation :
We want to bound the relative cost of stability $\frac{\tau^*}{\nu}$.

Theorem

Let G be a graph. There exists $\delta > 0$ such that

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Proof of $\frac{\nu^*(\mathcal{G})}{\nu(\mathcal{G})} \leq_{\forall} vw(G)$:

Follows from the previous theorem since $\frac{\nu^*(\mathcal{G})}{\nu(\mathcal{G})} \leq \frac{\tau(\mathcal{G})}{\nu(\mathcal{G})} \leq_{\forall} vw(G)$.

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Proof of $vw(G)^\delta \leq_\exists \frac{\nu^*(\mathcal{G})}{\nu(\mathcal{G})}$:

- Prove that the gap is linear for grids.
- Use grid minor theorem.

Conclusion

- What is the best constant δ (we cannot beat $\frac{1}{2}$, on cliques) ?
- Does it exist a "good" invariant which characterizes the relative cost of stability ?
- On which interaction graph can we obtain a linear bound in terms of vinewidth ?

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Thanks for your attention !