Reconfiguration of Spanning Trees with Many or Few Leaves

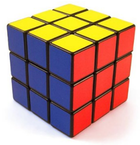
Nicolas Bousquet Takehiro Ito Yusuke Kobayashi
Haruka Mizuta Paul Ouvrard Akira Suzuki Kunihiro Wasa

ESA'20

Reconfiguration problems

Informal framework :

- Two solutions of an instance of a problem.
- An elementary transformation between two solutions.



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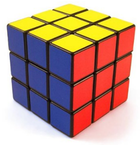

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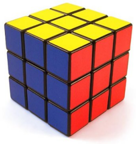

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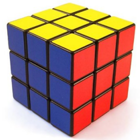

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Many motivations :

- One-player games (puzzles).
- Markov chains (ergodicity and mixing time).
- Applications to many problems : statistical physics, biology, motion of robots, discrete geometry...

Main questions

- **Reachability problem.** Given two configurations, is it possible to transform the one into the other ?
- **Connectivity problem.** Given any pair of configurations, is it possible to transform the one into the other ?
- **Minimization.** Given two configurations, what is the length of a shortest sequence ?

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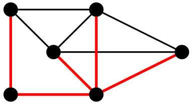
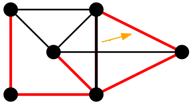
- **Minimization.** Given two configurations, what is the length of a shortest sequence ?
- **Algorithmics.** Can we efficiently solve these problems ? (In polynomial time, FPT-time...).

Spanning tree reconfiguration



Spanning Tree : Subset of edges that induces a spanning connected graph without cycle.



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T_1 can be transformed into T_2 via an edge flip if there exists $e_1 \in T_1$ and $e_2 \in T_2$ such that $T_2 = (T_1 \cup e_2) \setminus e_1$.

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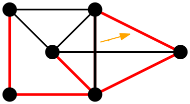

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SPANNING TREE RECONFIGURATION (STR)
Input : A graph G, two spanning trees T_1, T_2.
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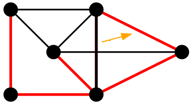

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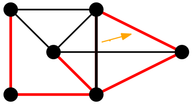

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What if we add some constraints on the the spanning trees ?
$\rightarrow$ In this paper : Number of leaves.

STR with few leaves

Spanning Tree Reconfiguration (STR) with $\leq k$ Leaves
Input : A graph G, two spanning trees T_1, T_2 with $\leq k$ leaves.
Output : YES iff $T_1 \rightsquigarrow T_2$ via spanning trees with $\leq k$ leaves.

Our Result 1 :
STR with ≤ 3 Leaves is PSPACE-complete.

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- 3 can be replaced by any integer ≥ 3 in the statement.

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Spanning Tree Reconfiguration with Many Leaves
Input : A graph G, an integer k and two spanning trees T_1, T_2 with $\geq k$ leaves.
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Our results 2 :
STR WITH MANY LEAVES :

- is PSPACE-complete even restricted to bipartite graphs or split graphs ;
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STR WITH MANY LEAVES :

- is PSPACE-complete even restricted to bipartite graphs or split graphs ;
- is PSPACE-complete restricted to planar graphs ;
- can be decided in polynomial time on cographs and interval graphs.

Reduction for STR with ≤ 3 leaves

Adaptation of a reduction from VERTEX COVER to HAMILTONIAN PATH.

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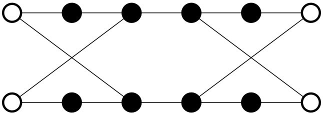
Adaptation of a reduction from Vertex Cover to Hamiltonian Path.

Edge gadget :



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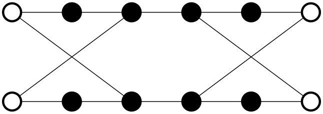
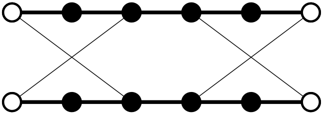
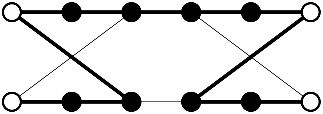


Lemma (folklore) : Any hamiltonian path must intersect the edge gadget in one of the two following ways (up to symmetry) :





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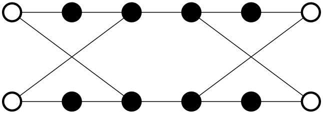

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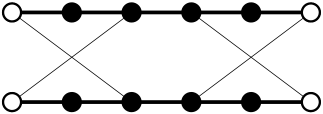
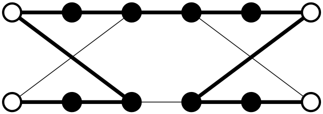
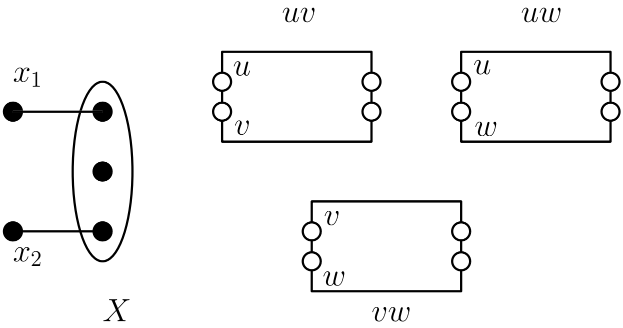




Lemma : In any spanning tree with ≤ 3 leaves, ≥ 1 white vertex has degree one in the subgraph induced by the edge-gadget.

Reduction for STR with ≤ 3 leaves (cont.)



Reduction from *k*-Vertex Cover :

- Two vertices of degree 1, an independant set X of size $k+1$, an edge gadget for every edge.

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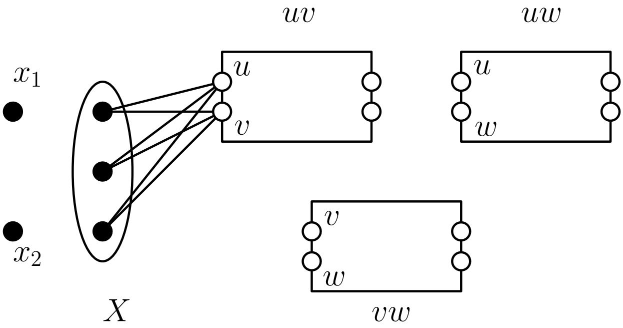

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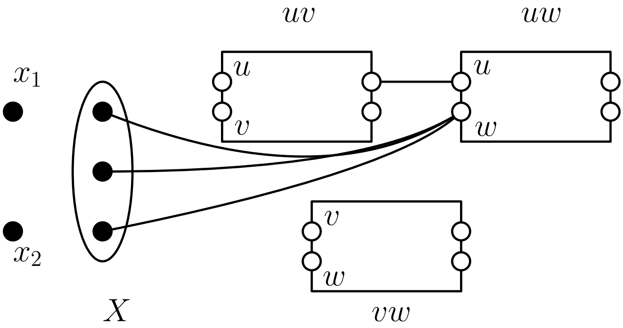

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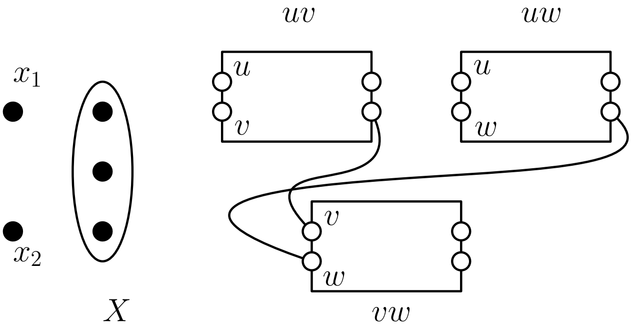

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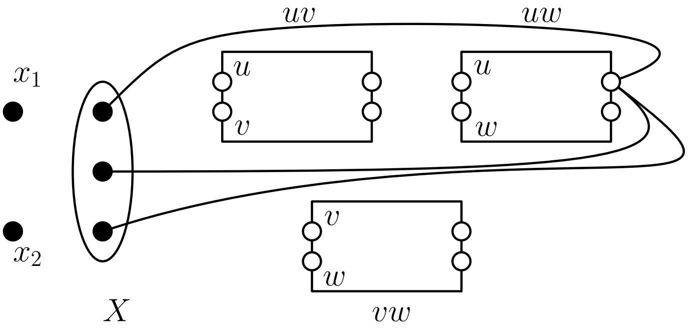

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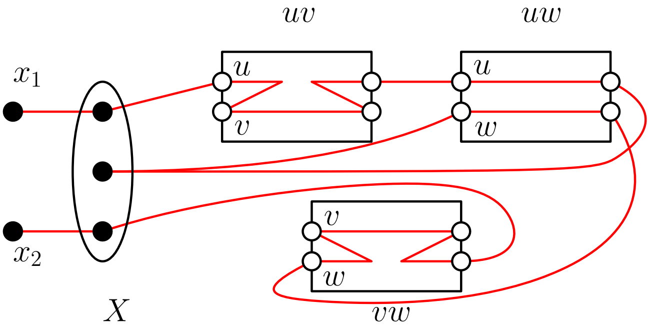

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Theorem [Wrochna]

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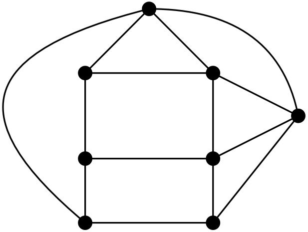
Technical lemmas :

- Show that $|S(T)| \leq k + 1$ for any T with ≤ 3 leaves.
- Show that if T_2 can be obtained from T_1 via an edge flip then it corresponds to a "single step" modication for vertex cover reconfiguration.

STR with many Leaves hardness for planar graphs

Reduction from MINIMUM VERTEX COVER RECONFIGURATION ON PLANAR GRAPHS



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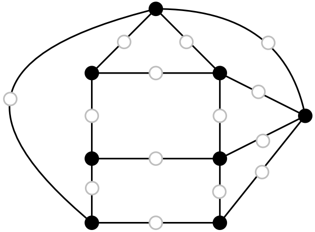

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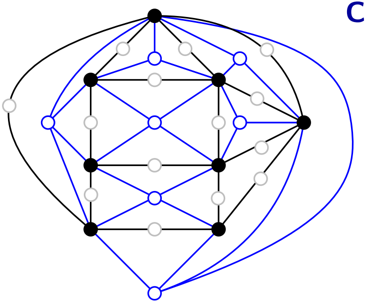

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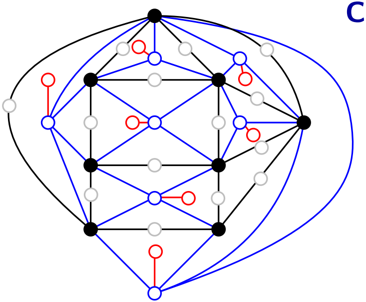

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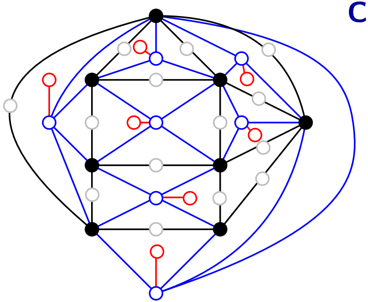

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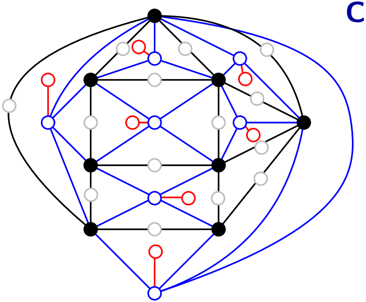

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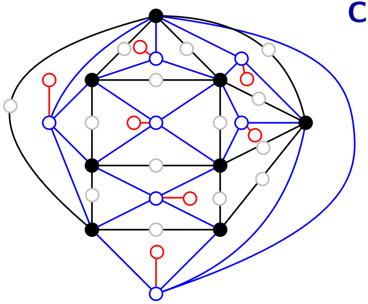

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2. • vertices are leaves.
3. ≥ 1 neighbor of a • vertex is internal.
   $\Rightarrow \geq$ faces + min vertex cover leaves.

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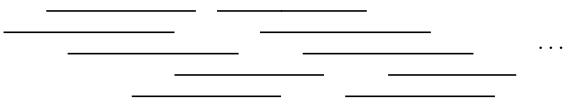
Sketch of the proof :

- All the spanning trees with the same internal nodes are reachable from each other.
- Given two sets X, Y of size 2, we can decide in polytime if there is an edge flip from a tree with internal nodes X to a tree with internal nodes Y.

Polynomial algorithm for interval graphs



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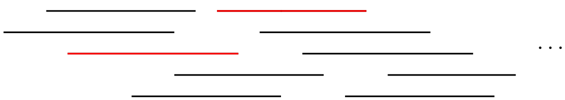

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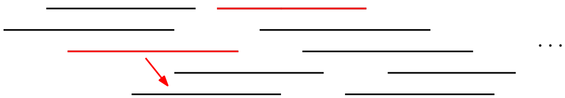

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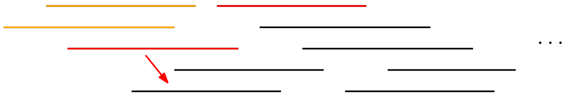

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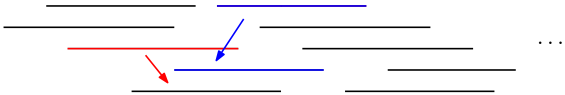

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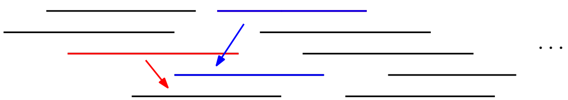

Idea of the polynomial time algorithm :

- Fix the first interval vertex.
- Push left the second interval vertex by "induction". (It is actually more complicated, it is not really an induction since we cannot simply delete the left part, but we can reduce it somehow...)
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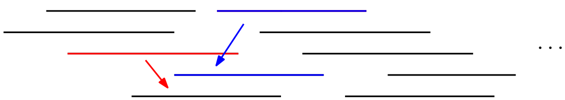

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(Technical) Lemma : The vertex obtained by this algorithm is the rightmost possible position of the first interval vertex in a spanning tree reachable from the initial tree.

Conclusion

Questions :

- Complexity of Spanning Tree Reconfiguration (STR) with Few Leaves on restricted graph classes ? (The reduction do not maintain any parameter...)

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- Lemma : STR with $\geq n-2$ Leaves is in P.
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- Is STR with many leaves polynomial in outerplanar graphs ?

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Questions :

- Complexity of Spanning Tree Reconfiguration (STR) with Few Leaves on restricted graph classes ? (The reduction do not maintain any parameter...)
- Lemma : STR with $\geq n - 2$ Leaves is in P.
  Is STR with $\geq n - \ell$ leaves FPT parameterized by ℓ ?
- Is STR with many leaves polynomial in outerplanar graphs ?

Thanks !