VC-dimension and Erdős-Pósa property

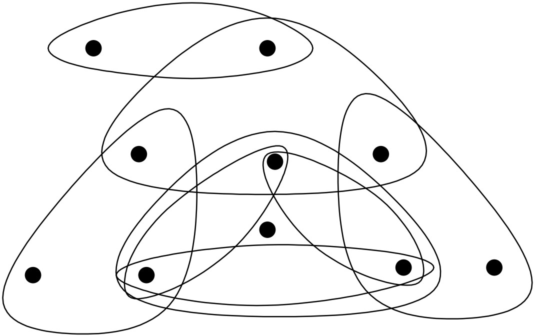
Nicolas Bousquet

LIRMM, University Montpellier II

1. VC-dimension
   - Packing and transversality
   - VC-dimension
   - Applications
2. VC-dimension for graphs
   - Dual VC-dimension
   - VC-dimension of graphs
   - Classes of graphs of bounded VC-dimension
   - Erdős-Pósa property
3. Conclusion

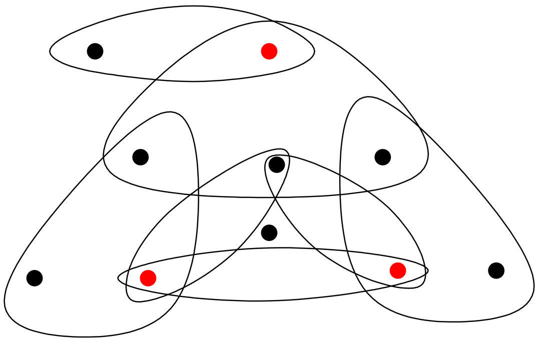
Definition

A *hypergraph* is a pair (V, E) where V is a set of vertices and E is a set of hyperedges (subsets of vertices).



Definition

The *transversality* τ of a hypergraph is the minimum number of vertices which intersect all the hyperedges.



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Linear Programing

Variables : for each $v_i \in V$, associate x_i a non negative integer.
Constraints : for each $e \in E$,

$$\sum_{v_i \in e} x_i \geq 1$$

Objective function :

$$\tau = min(\sum_{i=1}^{n} x_i)$$

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Linear Fractional Relaxation

Variables : for each $v_i \in V$, associate x_i a non negative real number.

Constraints : for each $e \in E$,

$$\sum_{v_i \in e} x_i \geq 1$$

Objective function :

$$\tau^* = min(\sum_{i=1}^{n} x_i)$$

Integrality gap

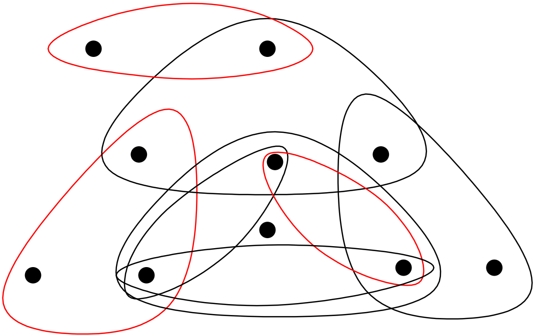
Integrality gap

$V = \{1, ..., 2n\}$
$e \in E$ iff $|e| = n$

- $\tau^* = 2$
- $\tau = n + 1$

Definition

The *packing number* ν *of a hypergraph* is the maximum number of disjoint hyperedges.



Definition

The *packing number* ν *of a hypergraph* is the maximum number of disjoint hyperedges.

Linear Programing

Variables : for each $e_i \in E$, associate x_i a non negative integer.
Constraints : for each $v \in V$,

$$\sum_{e_i / v \in e} x_i \leq 1$$

Objective function :

$$\nu = max(\sum_{i=1}^{|E|} x_i)$$

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Integrality gap

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The vertices of H are the edges of a clique on n vertices.
The hyperedges are the maximum stars of the clique.

- $\nu = 1$
- $\nu^* = n/2$

Erdős-Pósa property

Duality Theorem of Linear Programing

$$\tau^* = \nu^*$$

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Inequalities

$$\nu \leq \nu^* = \tau^* \leq \tau$$

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$$\nu \leq \nu^* = \tau^* \leq \tau$$

Erdős-Pósa property

A class $\mathcal{H}$ of hypergraphs has the Erdős-Pósa property iff there exists a function f such that for all $H \in \mathcal{H}$, $\tau \leq f(\nu)$.

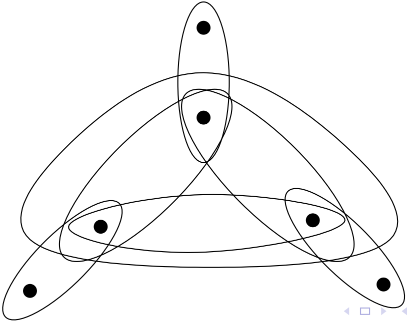
Theorem (Erdős-Pósa)

The cycle hypergraph of a graph has the Erdős-Pósa property.

VC-dimension

Definition

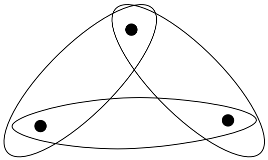
A set $X \subseteq V$ is *shattered* iff for all $Y \subseteq X$, there exists $e \in E$ such that $e \cap X = Y$.
The *VC-dimension* of a hypergraph is the maximum size of a shattered set.



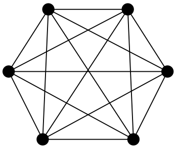
2VC-dimension

Definition

A set $X \subseteq V$ is *2-shattered* iff for all $Y \subseteq X$ with $|Y| = 2$, it exists $e \in E$ such that $e \cap X = Y$.
The 2*VC-dimension* of a hypergraph is the maximum size of a 2-shattered set.



Gap between VC-dimension and 2VC-dimension



Theorem

Theorem (Vapnik, Chervonenkis '72)

For every hypergraph H of VC-dimension d :

$$\tau \leq 20 d \tau^* log(\tau^*)$$

Applications

k-majority tournament

$V = \{1, ..., n\}$. Let P_1, P_{2k-1} be linear orders on V.
The tournament realized by $P_1, ..., P_{2k-1}$ has an edge from i to j iff $i \succ j$ in at least k orders.
A tournament is a *k-majority tournament* iff it can be realized by $2k-1$ linear orders.

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Theorem (Alon, Brightwell, Kierstead, Kotochka, Winkler '04)

Each k-majority tournament has a dominating set of size $O(k \cdot log(k))$.

Proof

- Consider the hypergraph H with hyperedges the in-neighborhoods of the vertices of T : a transversal of H is a dominating set of T.

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- τ^* is bounded (by 2).
- The VC-dimension is bounded (by $O(k \cdot log(k))$).

Theorem (Vapnik, Chervonenkis '72)

For every hypergraph H of VC-dimension d :

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Conjecture

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The partial orders $P_1, ..., P_k$ *cover* a graph G iff $x_i \succ x_j$ for $P_l \iff (x_i, x_j) \in E$.

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Conjecture

A tournament covered by at most k partial orders have a dominating set of size at most $f(k)$.

Dual hypergraph

Bipartite incidence graph

A hypergraph $H = (V, E)$ can be seen as a *bipartite incidence graph* G with vertex set (V, E) where (v, e) in an edge iff $v \in e$.

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The pair (V, E) is oriented : the hypergraph associated to the pair (E, V) is the *dual hypergraph* denoted by H^d.

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Definition

The *dual VC-dimension* is the VC-dimension of the dual hypergraph.

Duality gap

VC-dimension and dual VC-dimension

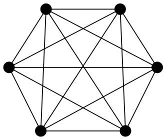
- VC-dimension and dual VC-dimension are linked by an exponential function.

If the VC-dimension is arbitrarily large, you have the bipartite graph you want as an induced subgraph.

Duality gap

VC-dimension and dual VC-dimension

- VC-dimension and dual VC-dimension are linked by an exponential function.
- A arbitrarily large gap is possible between the 2VC-dimension and the dual 2-VC-dimension.



Theorem

Theorem (Ding, Seymour, Winkler '91)

Let H be a hypergraph of dual 2VC-dimension d then :

$$\tau \leq 11d^2(\nu + d + 3) \cdot \binom{d+\nu}{d}$$

Covering of a planar graph

Theorem (Chepoi, Estellon, Vaxès '07)

There exists a constant m such that, for every planar graph of diameter $2l$, there are m balls of radius l which cover the graph.

Proof

We consider the hypergraph of balls of radius l.

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Observation

H and H^d are the same : $x \in B(y, l)$ iff $y \in B(x, l)$.

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(p, q)-property

A hypergraph has the (p, q)-property iff for each set of p hyperedges, q of them share a vertex.

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Theorem (Matousek)

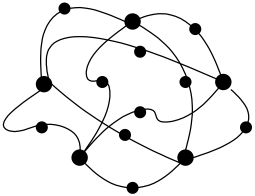
Let H be a hypergraph of dual VC-dimension d. There exists a function f such that if H has the $(p, d+1)$ property then $\tau \leq f(p, d)$.

Proof

Proof

- Verify that the dual VC-dimension of H is bounded (by 4).
- Verify that H has the $(p, 6)$-property for a constant p.

Assume that the dual VC-dimension is at least 5.



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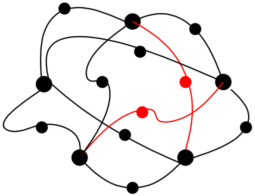

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Theorem (Matousek)

There exists a function f such that if H has the $(p, 6)$ property then $\tau \leq f(p, 6)$.

Hence there exists a fixed number of balls which cover all the vertices of the graph.

Another proof

- The dual 2VC-dimension is bounded by 4.
- The packing number is equal to 1.

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Ding, Seymour, Winkler

Let H be a hypergraph of dual 2VC-dimension d then :

$$\tau \leq 11d^2(\nu + d + 3) \cdot \binom{d+\nu}{d}$$

Then τ is bounded by a constant (880 000).

Conjecture

Conjecture (Chepoi, Estellon, Vaxès)

There exists a constant c such that the hypergraph of the balls of radius l of a planar graph satisfies :

$$\tau \leq c \cdot \nu$$

Definition

Definition

We denote by $B(x,l)$ the set of the vertices at distance at most l from x in the graph.
The *hypergraph of iterated neighborhoods* of G is the hypergraph on V with hyperedges $B(x,l)$ for all x and l.

Definition

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The VC-dimension of a graph is equal to the VC-dimension of the hypergraph of iterated neighborhoods.

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The VC-dimension of a graph is equal to the maximum VC-dimension of the hypergraph of iterated neighborhoods for all induced subgraph.

Graphs of bounded VC-dimension

Theorem (B.,Thomassé)

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- The K_n minor free graphs have VC-dimension at most $n-1$.

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- The graphs of rankwidth k have VC-dimension at most $3 \cdot 2^{2k+1} + 2$.

Erdős-Pósa property

Theorem

There exists a function f such that if G has VC-dimension d then, the hypergraph of the balls of radius l satisfies :

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Theorem (Matousek)

Let H be a hypergraph of dual VC-dimension d. There exists a function f such that if H has the $(p, d+1)$ property then $\tau \leq f(p, d)$.

- The VC-dimension is bounded.
- We have to verify that the $(p, d+1)$ property is verified for p which depends only of ν and d.

Open problem

Circle graphs without triangle have bounded VC-dimension.

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Open problems

- A class of graphs of bounded VC-dimension is χ-bounded.

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- A class of graphs of bounded 2VC-dimension is χ-bounded.

Open problem

Circle graphs without triangle have bounded VC-dimension.

Open problems

- A class of graphs of bounded VC-dimension is χ-bounded.
- A class of graphs of bounded 2VC-dimension is χ-bounded.
- A class of triangle free graphs of bounded 2VC-dimension has chromatic number at most c.