Welfare and Rationality guarantees for the SMRA

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Spectrum auctions

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Specificities:

- The bidders discover their own valuations' functions.
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Two main auctions used worldwide:

- The SMRA (Simultaneous Multi-Round Auction).
- The CCA (Combinatorial Clock Auction).

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Return the "best possible" allocation.

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Item vs package bidding:

- Item bidding in the SMRA: a bid for S at price $p(S)$ is the "union" of the bids for s at price $p(s)$ for $s \in S$.
- Package bidding in the CCA: all or nothing bid at price $p(S)$.

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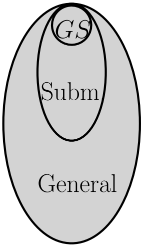

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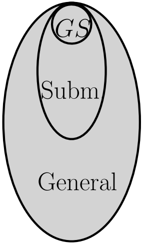

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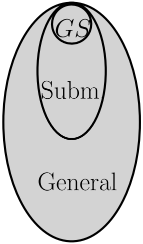

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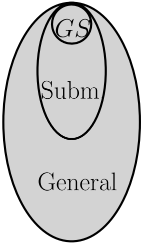

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For the CCA: Any valuation function and truthful bidding ⇒ $Polylog(n, m)$·size of a max. bid guarantee [B., Cai, Hunkenschröder, Vetta '16].

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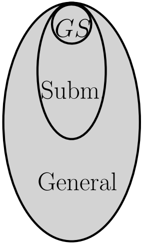

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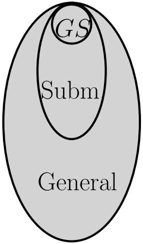

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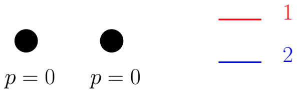
Question: Similar guarantee for the SMRA? **NO !**

No similar guarantee for the SMRA

2 bidders $\{1, 2\}$.
2 items $\{a, b\}$.

$$v_i(S) = \begin{cases} 1 \text{ if } |S| = 1 \\ N \text{ if } |S| = 2 \end{cases}$$



Definition (truthful bidders)

Bidder i is truthful if, at each round, she bids on a set S:

- containing all the items provisionally won by i,
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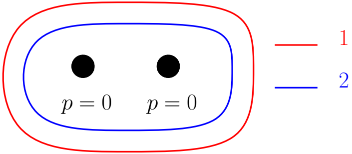

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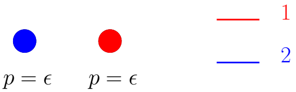

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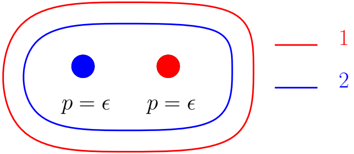

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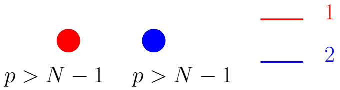

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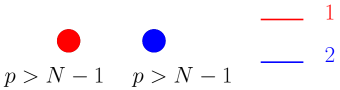

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- The final allocation is not necessarily individually rational.
- The allocation welfare may be 2 while optimal welfare is N.

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> Complementarities in spectrum auctions are "limited".

Question: Any guarantee if valuation functions have "bounded" complementarities?

α-near submodularity

Definition (α-near submodular)

A valuation function v is α-near submodular if for every $A \subseteq B$ and $x \notin B$

$$v(B \cup x) - v(B) \leq \alpha \cdot \Big(v(A \cup x) - v(A) \Big)$$

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- $\alpha = 1 \Leftrightarrow$ the valuation function is submodular.
- $\alpha = 2$: the marginal value of any item in B is at most twice its value in A.

Guarantee for truthful bidders

Theorem (B., Cai, Vetta)

Under truthful bidding, if valuation functions are α-near submodular then the allocation of the SMRA is:

- α-individually rational.
- $(\alpha + 1)$-optimal.

where:

- α-individually rational means: if bidder i is allocated S then $\alpha \cdot v_i(S) \geq p(S)$ where $p(S)$ is the price paid by i for S.
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Proof: generalization of the [Fu, Kleinberg, Lavi] proof for submodular valuation functions.

Almost tightness

Theorem (B., Cai, Vetta)

Truthful bidding + α-near submodular valuations functions $\Rightarrow$ Allocation that is :

- Not $(\alpha - \epsilon)$-individually rational (for every $\epsilon > 0$).
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A lot of bidders.

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Problem: Usually bidders want to be individually rational. So truthful bidding might not be a realistic assumption...

Conservative strategies

Definition (secure)

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Theorem (B., Cai, Vetta)

The strategy of bidder i is individually rational if and only if i always makes secure bids (even if we assume that other bidders are truthful / secure truthful / unit demand bidders).

where secure truthful bidding means bidding on the secure set S:

- containing the items provisionally allocated to her
- maximizing the utility $v_i(S) - p(S)$.

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Theorem (B., Cai, Vetta)

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Sketch:

Let $\mathcal{S} = (S_1, \ldots, S_k)$ be the allocation of the SMRA and let $\mathcal{S}^* = (S_1^*, \ldots, S_k^*)$ be the optimal allocation.

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$$p(s) \geq v_i(Q \cup s) - v(Q).$$

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- The sum over i gives the conclusion.

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Thanks for your attention !