

A Near-Optimal Mechanism for Impartial Selection

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Abstract. We examine strategy-proof elections to select a winner amongst a set of agents, each of whom cares only about winning. This *impartial selection* problem was introduced independently by Holzman and Moulin [5] and Alon et al. [1]. Fisher and Klimm [4] showed that the permutation mechanism is impartial and $\frac{1}{2}$ -optimal, that is, it selects an agent who gains, in expectation, at least half the number of votes of most popular agent. Furthermore, they showed the mechanism is $\frac{7}{12}$ -optimal if agents cannot abstain in the election. We show that a better guarantee is possible, provided the most popular agent receives at least a large enough, but constant, number of votes. Specifically, we prove that, for any $\epsilon > 0$, there is a constant N_ϵ (independent of the number n of voters) such that, if the maximum number of votes of the most popular agent is at least N_ϵ then the permutation mechanism is $(\frac{3}{4} - \epsilon)$ -optimal. This result is tight. Furthermore, in our main result, we prove that near-optimal impartial mechanisms exist. In particular, there is an impartial mechanism that is $(1 - \epsilon)$ -optimal, for any $\epsilon > 0$, provided that the maximum number of votes of the most popular agent is at least a constant M_ϵ .

1 Introduction

Imagine an election where the voters are the candidates and each voter is allowed to vote for as many of the other candidates as she wishes. Now suppose each voter cares only about winning. The goal of the mechanism is to elect the candidate with the maximum support. To achieve this, we desire that the election mechanism be strategy-proof. Thus, we want an *impartial mechanism*, where voting truthfully cannot affect an agent's own chances of election.

This problem, called the *impartial selection problem*, was introduced independently by Holzman and Moulin [5] and Alon et al. [1]. In addition to elections, they were motivated by nomination mechanisms for prestigious prizes and committees, hyperlink formations, and reputation systems in social networks. Fisher and Klimm [4] also proposed the use of such mechanisms for peer review evaluation processes.

The impartial selection problem can be formalized via a directed graph $G = (V, A)$. There is a vertex $v \in V$ for each voter (candidate) v , and there is an arc $(u, v) \in A$ if u votes for v . The aim is to maximize the in-degree of the selected vertex, and we say that an impartial mechanism is α -optimal, for $\alpha \leq 1$, if the

in-degree of the vertex it selects is always at least α times the in-degree of the most popular vertex.

Unfortunately, Moulin [5] and Alon et al. [1] observed that a *deterministic* impartial mechanisms must have an arbitrarily poor approximation guarantee α . Specifically, a deterministic mechanism may have to select a vertex with zero in-degree even when other vertices receive votes; it may even be forced to select a vertex with in-degree one whilst another vertex receives $n - 1$ votes! This negative result motivated Alon et al. [1] to study *randomized* impartial mechanisms. In particular, they examined a simple mechanism dubbed the *2-partition mechanism*. This mechanism independently assigns each vertex to one of two groups $\{V_1, V_2\}$. Then, only the arcs from vertices in V_1 to vertices in V_2 are counted as votes. The vertex with the maximum number of *counted* votes in V_2 is selected (breaking ties arbitrarily). It is straight-forward to verify that this mechanism is impartial and is $\frac{1}{4}$ -optimal. They further conjectured the existence of an $\frac{1}{2}$ -optimal impartial randomized mechanism.

This conjecture was recently proven by Fisher and Klimm [4]. Specifically, they proved that the *permutation mechanism* is impartial and $\frac{1}{2}$ -optimal. This election mechanism examines the vertices in a random order, and can only count the votes of a vertex that go to vertices behind it in the ordering. (See Section 3 for a detailed description of the mechanism and a short proof of Fisher and Klimm's result.) Interestingly, the factor $\frac{1}{2}$ -approximation guarantee is tight. Consider an n -vertex graph containing only a single arc (u, v) . Then, unless u is before v in the random permutation the mechanism will select a vertex with in-degree zero. Thus the expected in-degree of the victor is at most one half.

Observe that this tight example is rather unsatisfactory. It is extremely unnatural and relies on the fact that every vertex bar one abstains from voting. Indeed, Fisher and Klimm [4] showed that without abstentions the permutation mechanism is at least $\frac{7}{12}$ -optimal. They leave open the possibility that the permutation mechanism actually proffers a better approximation guarantee than $\frac{7}{12}$. They do prove, however, that without abstentions the permutation mechanism can be no better than $\frac{2}{3}$ -optimal. Moreover, Fisher and Klimm [4] provide an even stronger inapproximation bound: **no** impartial mechanism can be better than $\frac{3}{4}$ -optimal, even without abstentions.

This appears to severely limit the potential for progress. But, again, the lower bounds are somewhat unsatisfactory. The issue now is not low out-degrees (that is, abstentions) but rather low in-degrees. The lower bounds are all based on instances with extremely small maximum in-degree Δ^- . Specifically, the factor $\frac{1}{2}$ optimal example [1] for the permutation mechanism with abstentions has $\Delta^- = 1$; the factor $\frac{2}{3}$ optimal example [4] for the permutation mechanism without abstentions has $\Delta^- = 3$; the factor $\frac{3}{4}$ optimal example [4] for any randomized mechanism without abstentions has $\Delta^- = 2$. Of course, in applications with a large number n of voters, we would anticipate that the most popular agent receives a moderate number of votes. Do these inapproximability bounds still apply for these more realistic settings? Interestingly, the answer is no, even

for cases where the most popular agent receives only a (large enough) constant number of votes. Specifically, we first prove that the permutation mechanism is nearly $\frac{3}{4}$ -optimal in such instances.

Theorem 1. *For any $\epsilon > 0$, there is a constant N_ϵ such that if $\Delta^- \geq N_\epsilon$ then the permutation mechanism is $(\frac{3}{4} - \epsilon)$ -optimal.*

This result is tight. We show that the permutation mechanism cannot produce a guarantee better than $\frac{3}{4}$ regardless of the magnitude of Δ^- .

This result suggests that it may be possible to find a mechanism that beats the $\frac{3}{4}$ -inapproximability bound of [4], even for constant maximum in-degree. This is indeed the case and spectacularly so. There is an impartial mechanism, which we call the *slicing mechanism*, that produces a near optimal approximation guarantee.

Theorem 2. *For any $\epsilon > 0$, there is a constant M_ϵ such that if $\Delta^- \geq M_\epsilon$ then the slicing mechanism is $(1 - \epsilon)$ -optimal.*

The slicing mechanism differs from previous mechanisms in that it adds an initial sampling phase. In this first phase, it samples a small fraction of the vertices. It then uses the votes of these vertices to build a non-random ordering of the other vertices. This specific ordering is exploited in the second phase to elect a vertex with very high expected in-degree.

These results, as in previous works [1,4,5], relate to single-winner elections. Some of the motivating applications, however, involve multiple-winner elections. We remark that our main result can be generalized to multiple-winner elections via small modifications to the mechanism.

2 The Model

We begin by formalizing the impartial selection problem and introducing some necessary notation. An election is represented via a directed graph $G = (V, A)$. The number of vertices of G is denoted by n , and each vertex represents an agent (voter/candidate). An agent can make multiple votes, but cannot vote for herself nor vote more than once for any other agent. Thus, the graph G is loopless and contains no multiple arcs.

A vertex u is an *in-neighbor* of v if there is an arc $uv \in A$. In this case, we say that u votes for v . Given a subset $Y \subseteq V$ and $v \in V$, the *in-degree of v in Y* , denoted by $d_Y^-(v)$, is the number of in-neighbors of v in Y . For simplicity, we denote $d_Y^-(v)$, the *in-degree of v* , by $d^-(v)$. The maximum in-degree of any vertex in G is denoted in by $\Delta^-(G)$, or simply by Δ when there is no ambiguity.

A mechanism is *impartial* if, for every vertex v , the probability of selecting v is not modified when the out-neighborhood of v is modified. That is, if v changes its votes then this does not affect the probability of v being elected. More formally, take any pair of graphs G and G' on the same vertex set V . Let v be a vertex. Then we require that the probability that v is elected in G is equal

to the probability that v is elected in G' , whenever $N_G^+(u) = N_{G'}^+(u)$ for every $u \neq v$.

Given $1 \geq \alpha \geq 0$, an impartial mechanism is α -optimal if for any graph G , the expected degree of the winner differs from the maximum degree by a factor of at most α , that is,

$$\frac{\sum_{v \in V} d^-(v) \cdot \Pr(v \text{ is the winner})}{\Delta} \geq \alpha$$

Finally, given an integer k , the set $\{1, 2, \dots, k\}$ is denoted by $[k]$.

3 The Permutation Mechanism

In this section, we analyze the permutation mechanism of Fisher and Klimm [4]. This election mechanism examines the vertices in a random order $\{\pi_1, \pi_2, \dots, \pi_n\}$. At time t , the mechanism selects a *provisional leader* y_t from the amongst the set $\Pi_t = \{\pi_1, \dots, \pi_t\}$. At time $t + 1$ the mechanism then examines π_{t+1} . If π_{t+1} receives at least as many votes as y_t from $\Pi_t \setminus y_t$ then π_{t+1} is declared the provisional leader y_{t+1} . Otherwise $y_{t+1} := y_t$. The winner of the election is y_n . A formal description of the permutation mechanism is given in Procedure 1.

Procedure 1 The Permutation Mechanism

Input: A directed graph $G = (V, A)$.
Let π be a random permutation of $V = [n]$.
 $y_1 \leftarrow \pi_1$;
for $i = 1$ to $n - 1$ **do**
 if $d_{\Pi_i \setminus \{y_i\}}^-(\pi_{i+1}) \geq d_{\Pi_i \setminus \{y_i\}}^-(y_i)$ **then**
 $y_{i+1} \leftarrow \pi_{i+1}$
 else
 $y_{i+1} \leftarrow y_i$
 end if
end for
output y_n

Observe that the permutation mechanism is impartial because it has the following property: the votes of a vertex are only considered after it has been eliminated. Specifically, the votes of π_t are considered at time $\tau > t$ only if π_t is not the provisional leader at time $\tau - 1$. But, if π_t is not the provisional leader at time $\tau > t$ then it cannot be elected. This ensures that eliminated agents have no interest to lie, *i.e.* the mechanism is impartial. Fisher and Klimm [4] proved this mechanism is $\frac{1}{2}$ -optimal using an intricate analysis based upon viewing the permutation mechanism as a generalization of the 2-partition mechanism. First, we present a simpler proof of their result.

Theorem 3. [4] *The permutation mechanism is $\frac{1}{2}$ -optimal.*

Proof. Let v be a vertex with maximum in-degree Δ . Now suppose exactly j of its neighbors appear before v in the random ordering π . In this case, at the time v is considered it has received at least $j - 1$ valid votes (one of the j votes may not be counted if it comes from the provisional leader).

Suppose v is now declared the provisional leader. Then all j of these votes become valid. (Indeed, if one of the in-neighbors of v was the provisional leader, this is no longer the case.) On the other-hand, suppose v is now declared a loser. Then, because ties are broken in favor of newly examined vertices, the provisional leader must already be receiving at least j valid votes. Thus in either case, the final winner y_n must also receive at least j valid votes (the in-degree of the provisional leader is non-decreasing).

Now with probability $\frac{1}{\Delta+1}$, exactly j of its neighbors appear before v , for any $0 \leq j \leq \Delta$. Thus, in expectation, the winner receives at least $\frac{1}{\Delta+1} \cdot \sum_{j=0}^{\Delta} j = \frac{1}{2} \Delta$ votes. \square

As discussed in the introduction, the factor $\frac{1}{2}$ -approximation guarantee in Theorem 3 is tight. This tightness is slightly misleading, though. Recall that the tight example was a graph with just a single arc. In general the permutation mechanism is $\frac{3}{4}$ -optimal. Specifically,

Theorem 1. *For any $\epsilon > 0$, there is a constant N_ϵ such that if $\Delta^- \geq N_\epsilon$ then the permutation mechanism is $(\frac{3}{4} - \epsilon)$ -optimal.*

The $\frac{3}{4}$ bound in Theorem 1 is tight in a very strong sense. There are tight examples for any choice of Δ , no matter how large; see Theorem 4. The proof of Theorem 1 has two basic components. The first is the basic observation, used above in the proof of Theorem 3, that the mechanism will perform well if the vertex v of highest in-degree has many neighbors before it in the random permutation. The second is the observation that if the mechanism does well when v does not participate then it will do at least as well when v does participate. In order to be able to apply these two observations simultaneously, however, we must show random permutations are "well-behaved". Specifically, we say that a permutation π of $[n]$ is (Δ, ϵ) -balanced if, for every $0 \leq k \leq n$,

$$|[\Delta] \cap \Pi_k| \geq \left(\frac{k}{n} - \epsilon \right) \cdot \Delta$$

and we want to show that a random permutation is typically balanced.

To do this, we need the following result, which provides a large deviation bound for the size of intersection of two sets of fixed cardinalities.

Lemma 1. *For every $\epsilon_1 > 0$, there exists N_1 such that for all positive integers $N_1 < \Delta \leq n$ and $k \leq n$ the following holds. If $X \subseteq [n]$ are chosen uniformly at random subject to $|X| = \Delta$, then*

$$\Pr \left[\left| |X \cap [k]| - \frac{k\Delta}{n} \right| \geq \epsilon_1 \cdot \Delta \right] < \epsilon_1 \quad (1)$$

Proof. See Appendix. □

Lemma 2. *For every $0 < \epsilon_2 < 1$, there exists N_2 such that, for all $n \geq \Delta > N_2$, at least $(1 - \epsilon_2) \cdot n!$ permutations of $[n]$ are (Δ, ϵ_2) -balanced.*

Proof. Let N_1 be chosen to satisfy Lemma 1 with $\epsilon_1 := \frac{\epsilon_2^2}{4}$. We choose $N_2 \geq \max(N_1, \frac{12}{\epsilon_2})$. Let $k_1, k_2, \dots, k_l \in [n]$ be a collection of integers such that for every $k \in [n]$ there exists $i \in [l]$ satisfying $0 \leq k - k_i \leq \epsilon_2 \cdot \frac{n}{3}$. Clearly such a collection can be chosen with $l \leq \frac{n}{\lfloor \frac{\epsilon_2 \cdot n}{3} \rfloor} \leq \frac{4}{\epsilon_2}$, where the last inequality holds as $n \geq \frac{12}{\epsilon_2}$. Let π be a permutation of $[n]$ chosen uniformly at random. By Lemma 1 and the choice of N_2 we have

$$\Pr \left[|[\Delta] \cap \Pi_{k_i}| \leq \left(\frac{k_i}{n} - \epsilon_1 \right) \cdot \Delta \right] < \epsilon_1 \quad (2)$$

for every $1 \leq i \leq l$.

We claim that if $|[\Delta] \cap \Pi_{k_i}| \geq \left(\frac{k_i}{n} - \epsilon_1 \right) \cdot \Delta$ for every $0 \leq i \leq l$ then $|[\Delta] \cap \Pi_k| \geq \left(\frac{k}{n} - \epsilon_2 \right) \cdot \Delta$ for every $0 \leq k \leq n$. Indeed, given k let $i \in [l]$ satisfy $0 \leq k - k_i \leq \epsilon_2 \cdot \frac{n}{3}$. Then

$$\begin{aligned} |[\Delta] \cap \Pi_k| &\geq |[\Delta] \cap \Pi_{k_i}| \\ &\geq \left(\frac{k_i}{n} - \epsilon_1 \right) \cdot \Delta \\ &\geq \left(\frac{k}{n} - \frac{\epsilon_2}{3} - \epsilon_1 \right) \cdot \Delta \\ &= \left(\frac{k}{n} - \frac{\epsilon_2}{3} - \frac{\epsilon_2^2}{4} \right) \cdot \Delta \\ &\geq \left(\frac{k}{n} - \epsilon_2 \right) \cdot \Delta \end{aligned}$$

as claimed. By the union bound applied to (2) we have

$$\begin{aligned} \Pr \left[\forall i : i \leq l : |[\Delta] \cap \Pi_{k_i}| \geq \left(\frac{k_i}{n} - \epsilon_1 \right) \cdot \Delta \right] &\geq 1 - l \cdot \epsilon_1 \\ &\geq 1 - \frac{4}{\epsilon_2} \cdot \frac{\epsilon_2^2}{4} \\ &= 1 - \epsilon_2 \end{aligned} \quad (3)$$

The lemma immediately follows from (3) and the claim above. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let N_2 be chosen to satisfy Lemma 2 with $\epsilon_2 := \frac{\epsilon}{3}$. We show that $N_\epsilon := \max(N_2, \lceil \frac{6}{\epsilon_2} \rceil)$ satisfies the theorem. Let v be a vertex of G with in-degree $\Delta := \Delta^-(G)$. We assume that $V(G) = [n]$, where v is vertex n , and $[\Delta]$ is the set of neighbors of v . For a permutation π , let $d(\pi)$ denote the in-degree of the winner determined by the mechanism.

Let π' be a fixed (Δ, ϵ_2) -balanced permutation of $[n-1]$. We claim that

$$\mathbb{E}[d(\pi) \mid \pi|_{[n-1]} = \pi'] \geq \left(\frac{3}{4} - \epsilon/2\right) \cdot \Delta \quad (4)$$

Note that the theorem follows from (4), as the probability that $\pi|_{[n-1]}$ is not (Δ, ϵ_2) -balanced is at most ϵ_2 by the choice of N_{ϵ} , and thus

$$\mathbb{E}[d(\pi)] \geq (1 - \epsilon_2) \cdot \left(\frac{3}{4} - \epsilon/2\right) \cdot \Delta \geq \left(\frac{3}{4} - \epsilon\right) \cdot \Delta$$

It remains to prove (4). Let w be the winner when the permutation mechanism is applied to $G \setminus v$ and π' , and let x be the number of votes w receives from its left (*i.e.* from vertices before it in the permutation). Let π be a permutation of $[n]$ such that $\pi|_{[n-1]} = \pi'$. It is not hard to check that if at least $x+1$ neighbors of v precede v in π then v wins the election. Moreover, whilst the addition of vertex v can change the winner (and, indeed, produce a less popular winner), it cannot decrease the “left” degree of any vertex. Thus, the (new) winner has still in-degree at least x after the addition of v . So, as π' is (Δ, ϵ_2) -balanced, we have $|\Delta \cap \Pi_{cn}| \geq c\Delta - \epsilon_2\Delta \geq x+1$, whenever $c \geq \frac{x+1}{\Delta} + \epsilon_2$. Furthermore, $\Pr[\pi(x) > cn] \geq 1 - c$. Thus the probability that v wins the election is at least $1 - (x+1)/\Delta - \epsilon_2$. It follows that

$$\begin{aligned} \mathbb{E}[d(\pi) \mid \pi|_{[n-1]} = \pi'] &\geq \left(\frac{x+1}{\Delta} + \epsilon_2\right) \cdot x + \left(1 - \frac{x+1}{\Delta} - \epsilon_2\right) \cdot \Delta \\ &\geq \frac{\Delta^2 - (x+1)\Delta + (x+1)x}{\Delta} - \epsilon_2\Delta \\ &\geq \left(\frac{3}{4} - \epsilon_2 - \frac{1}{\Delta}\right) \cdot \Delta + \frac{(x - \Delta/2)^2}{\Delta} \\ &\geq \left(\frac{3}{4} - \frac{\epsilon}{2}\right) \cdot \Delta \quad \square \end{aligned}$$

The $\frac{3}{4}$ bound provided in Theorem 1 is tight for any Δ .

Theorem 4. *For every $0 < \epsilon < 1/4$ and every $N > 0$, there exists a directed graph G such that $\Delta^-(G) \geq N$ and the expected degree of the winner selected by the permutation mechanism is at most $(\frac{3}{4} + \epsilon)\Delta^-(G)$.*

Proof. Without loss of generality we assume that $N \geq 1/\epsilon$. Let G' be a directed graph such that $n := |V(G')| \geq (N+1)(N^2 + N + 1) \cdot \log \frac{1}{\epsilon}$, and $d^-(v) = d^+(v) = N$ for every $v \in V(G')$. Let G be obtained from G' by adding a new vertex v_0 and $2N-1$ directed edges from arbitrary vertices in $V(G')$ to v_0 . Thus $\Delta := \Delta^-(G) = d^-(v_0) = 2N-1$.

Now, in G' one can greedily construct a set Z of at least $n/(N^2 + N + 1)$ vertices, such that no two vertices of Z have common in-neighbors and no two vertices of Z are joined by an edge. After a vertex $z \in Z$ is chosen, simply remove z , the in-neighbors and out-neighbors of z , and the out-neighbors of z 's

in-neighbors (the inequality is satisfied since the in-neighbors have a common out-neighbor). Then recurse. Let π be a permutation of $V(G)$ chosen uniformly at random. Let X_v denote the event that a vertex $v \in V(G')$ is preceded by all of its neighbors in π . Clearly $\Pr[X_v] = \frac{1}{N+1}$ for every $v \in V(G')$, and moreover, by construction of Z , the events $\{X_v\}_{v \in Z}$ are mutually independent. Hence

$$\begin{aligned} \Pr[\cup_{v \in V(G')} X_v] &\geq 1 - \left(1 - \frac{1}{N+1}\right)^{\frac{n}{N^2+N}} \\ &\geq 1 - \left(1 - \frac{1}{N+1}\right)^{(N+1) \cdot \log \frac{1}{\epsilon}} \\ &\geq 1 - \epsilon. \end{aligned}$$

Note that if the event $\cup_{v \in V(G')} X_v$ occurs then one of the vertices of G' receives N votes in the permutation mechanism. By symmetry the probability that v_0 is preceded by at most $N-1$ of its neighbors in π is equal to $1/2$. Thus v_0 is not selected as a winner with probability at least $1/2 - \epsilon$. We deduce that the expected in-degree of the winner is at most

$$\begin{aligned} \left(\frac{1}{2} - \epsilon\right) \cdot \frac{\Delta+1}{2} + \left(\frac{1}{2} + \epsilon\right) \cdot \Delta &= \left(\frac{3}{4} + \epsilon\right) \cdot \Delta - \epsilon N + \frac{1}{4} \\ &\leq \left(\frac{3}{4} + \epsilon\right) \cdot \Delta. \quad \square \end{aligned}$$

4 The Slicing Mechanism

In this section, we present the *slicing mechanism* and prove that it outputs a vertex whose expected in-degree is near optimal.

Theorem 2. *For any $\epsilon > 0$, there is a constant M_ϵ such that if $\Delta^- \geq M_\epsilon$ then the slicing mechanism is $(1 - \epsilon)$ -optimal.*

The constant M_ϵ is independent of the number of vertices and is a polynomial function of $\frac{1}{\epsilon}$. We remark that we have made no attempt to optimize this constant.

The *slicing mechanism* is formalized in Procedure 2.

This mechanism consists of three parts which we now informally discuss. In the first part, the *sampling phase*, we independently at random collect a sample \mathcal{X} of the vertices. We use arcs incident to \mathcal{X} to estimate the in-degree of every other vertex in the graph. In the second part, the *slicing phase*, we partition the unsampled vertices into slices, where each slice consists of vertices with roughly the same *estimated-degree*. The third part, the *election phase*, selects the winning vertex. It does this by considering each slice in increasing order (of estimated-degrees). After the i -th slice is examined the mechanism selects as *provisional leader*, y_i , the vertex that has the largest number of in-neighbors amongst the set of vertices \mathcal{R} that have currently been eliminated. The winning vertex is the provisional leader after the final slice has been examined.

Procedure 2 The Slicing Mechanism

SAMPLING PHASE[Sample] Draw a random sample \mathcal{X} , where each vertex is sampled with probability ϵ .**for all** $v \in V \setminus \mathcal{X}$ **do** [Estimated-Degree.] $d_e(v) \leftarrow \frac{1}{\epsilon} \cdot d_{\mathcal{X}}^-(v)$ **end for****SLICING PHASE**[Slices] Create $\tau = \lceil \frac{1}{\epsilon^2} \rceil$ sets $\{S_1, \dots, S_\tau\}$ initialized to empty sets. $\Delta_e \leftarrow \max_{v \in V \setminus \mathcal{X}} (d_e(v))$ **for all** $v \in V \setminus \mathcal{X}$ **do** **for** $i = 1$ to τ **do** **if** $(i-1)\epsilon^2 \cdot \Delta_e \leq d_e(v) \leq i\epsilon^2 \cdot \Delta_e$ **then** $S_i \leftarrow S_i \cup \{v\}$ **end if** **end for****end for****ELECTION PHASE**[Revealed Set] $\mathcal{R} \leftarrow \mathcal{X}$ [Provisional Winner] $y_0 \leftarrow \operatorname{argmax}_{u \in V \setminus \mathcal{R}} (d_{\mathcal{R}}^-(u))$

[Break ties arbitrarily.]

for $i = 1$ to τ **do** **for all** $v \in S_i \setminus \{y_{i-1}\}$ **do** $\mathcal{R} \leftarrow \mathcal{R} \cup \{v\}$ with probability $(1 - \epsilon)$. **end for** $y'_i \leftarrow \operatorname{argmax}_{u \in V \setminus \mathcal{R}} (d_{\mathcal{R}}^-(u))$

[Break ties arbitrarily.]

 $\mathcal{R} \leftarrow (\mathcal{R} \cup S_i \cup \{y_{i-1}\}) \setminus \{y'_i\}$ [Provisional Winner] $y_i \leftarrow \operatorname{argmax}_{u \in V \setminus \mathcal{R}} (d_{\mathcal{R}}^-(u))$

[Break ties arbitrarily.]

 $\mathcal{R} \leftarrow (\mathcal{R} \cup \{y'_i\}) \setminus \{y_i\}$ **end for**The elected vertex is y_τ .

We emphasize, again, that the impartiality of the mechanisms follows from the fact that the votes of a vertex are only revealed when it has been eliminated, that is, added to \mathcal{R} . Observe that at any stage we have one provisional leader; if this leader changes when we examine a slice then the votes of the previous leader are revealed if its slice has already been examined.

4.1 Analysis of the Sampling Phase

Observe that the sampling phase is used to estimate the in-degree of each unsampled vertex v . Since each in-neighbor of v is sampled in \mathcal{X} with probability ϵ , we anticipate that an ϵ -fraction of the in-neighbors of v are sampled. Thus, we have an *estimated in-degree* $d_e(v) := \frac{1}{\epsilon} \cdot d_{\mathcal{X}}^-(v)$, for each vertex $v \in V \setminus \mathcal{X}$. It will be important to know how often these estimates are (roughly) accurate. In particular, we say that a vertex u is $\hat{\epsilon}$ -well-estimated if $|d_e(u) - d(u)| \leq \hat{\epsilon}d(u)$.

We will be interested in the case where $\hat{\epsilon} \ll \epsilon$. (In particular, we will later select $\hat{\epsilon} = \frac{\epsilon^2}{4}$.) Before analyzing the probability that a vertex is $\hat{\epsilon}$ -well-estimated, recall the classical Chernoff bound.

Theorem 5. [Chernoff bound]

Let $(X_i)_{i \leq n}$ be n independent Bernoulli variables each having probability p . Then

$$\Pr\left[\left|\sum X_i - pn\right| \geq \delta pn\right] \leq e^{-\frac{\delta^2 pn}{3}}.$$

Corollary 1. For any vertex v of in-degree at least $\Delta_0 = \max(3000, \frac{9\epsilon^2}{\hat{\epsilon}^4})$, the probability that v is not $\hat{\epsilon}$ -well-estimated is at most $\frac{1}{d(v)^6}$.

Proof. The proof is an application of Theorem 5. For every in-neighbor u_i of v , the vertex u_i is sampled with probability ϵ . Denote by X_i the Bernoulli variable corresponding to “ u_i is in \mathcal{X} ” which has value 1 if $u_i \in \mathcal{X}$ and 0 otherwise. The variables X_i are obviously independent and identically distributed. Note that $\sum X_i = \epsilon \cdot d_e(u)$ and its expectation is $\epsilon \cdot d(u)$.

$$\begin{aligned} \Pr\left(\left|d_e(u) - d(u)\right| \geq \frac{\hat{\epsilon}}{\epsilon} \cdot d(u)\right) &= \Pr\left(\left|\epsilon \cdot d_e(u) - \epsilon \cdot d(u)\right| \geq \hat{\epsilon} \cdot d(u)\right) \\ &\leq e^{-\frac{\hat{\epsilon}^2 d(u)}{3\epsilon}} \\ &\leq e^{-\frac{\hat{\epsilon}^2 \sqrt{\Delta_0}}{3\epsilon} \cdot \sqrt{d(u)}} \\ &\leq e^{-\sqrt{d(u)}} \\ &\leq \frac{1}{d(u)^6} \end{aligned}$$

Here the first inequality is an application of Theorem 5 with $\delta = \frac{\hat{\epsilon}}{\epsilon}$. The second inequality holds because $d(u) \geq \Delta_0$. The third inequality follows as $\Delta_0 = \frac{9\epsilon^2}{\hat{\epsilon}^4}$. Finally, the fourth inequality holds since $\sqrt{d(u)} \geq 6 \log(d(u))$ when $d(u) \geq \Delta_0 \geq 3000$. \square

Let $\hat{\epsilon} := \frac{\epsilon^2}{4}$. We will be interested in the probability that every vertex of high degree in a local region is $\hat{\epsilon}$ -well-estimated. Specifically, let x be a vertex of maximum in-degree Δ . Denote by $N^{-k}(x)$ the set of vertices which can reach x with an oriented path of length at most k . For instance, $N^{-1}(x)$ is the in-neighborhood of x plus x . Applying the union bound with Corollary 1, we obtain:

Corollary 2. Let x be a vertex of in-degree Δ . If $\Delta \geq \Delta_1 = \max(\frac{\Delta_0}{\epsilon^2}, \frac{3}{\epsilon^5})$ then, with probability $(1 - \epsilon)$, any vertex of $N^{-3}(x)$ of in-degree at least $\epsilon^2 \cdot \Delta$ is $\hat{\epsilon}$ -well-estimated.

Proof. Since $\epsilon^2 \cdot \Delta \geq \Delta_0$, Corollary 1 ensures that a vertex of in-degree at least $\epsilon^2 \cdot \Delta$ is not $\hat{\epsilon}$ -well-estimated with probability at most $\frac{1}{(\epsilon^2 \Delta)^6}$. There are at most $1 + \Delta + \Delta^2 + \Delta^3 \leq 3 \cdot \Delta^3$ vertices in $N^{-3}(x)$ since $\Delta \geq 2$. The union bound implies that every vertex in $N^{-3}(x)$ with in-degree at least $\epsilon^2 \Delta$ is $\hat{\epsilon}$ -well-estimated with probability at least $(1 - \frac{3\Delta^3}{\epsilon^{12}\Delta^6})$. As $\Delta \geq \frac{3}{\epsilon^5}$, the conclusion holds. \square

It the rest of this section we will make a set of assumptions. Given these assumptions, we will prove that the mechanism outputs a vertex of high expected in-degree. We will say that the mechanism “fails” if these assumptions do not hold. We will then show that the probability that the mechanism fails is very small. The two assumptions we make are:

(A1) Vertex x is not sampled. This assumption fails with probability ϵ .

(A2) Every vertex in $N^{-2}(x)$ with in-degree at least $\epsilon^2 \cdot \Delta$ is regionally well-estimated. Here, we say a vertex is *regionally well-estimated* if its degree is $\hat{\epsilon}$ -well-estimated and all its in-neighbors of in-degree at least $\epsilon^2 \Delta$ are also $\hat{\epsilon}$ -well-estimated. Corollary 2 ensures that all the vertices of $N^{-2}(x)$ of degree at least $\epsilon^2 \cdot \Delta$ are regionally well-estimated with probability $(1 - \epsilon)$. Thus, this assumption also fails with probability at most ϵ .

4.2 Analysis of the Slicing Phase

Now we consider the slicing phase. In this phase we partition the unsampled vertices into groups (slices) according to their estimated degrees. The *width* of a slice is the difference between the upper and lower estimated-degree requirements for vertices in that group. We will need the following bounds on the width of a slice.

Lemma 3. *The width of any slice is at least $(1 - \hat{\epsilon})\epsilon^2 \cdot \Delta$ and at most $\epsilon \cdot \Delta$.*

Proof. By assumption, the vertex x of maximum degree is $\hat{\epsilon}$ -well-estimated. Thus, $\Delta_e \geq (1 - \hat{\epsilon}) \cdot \Delta$. Therefore the width of any slice is at least $(1 - \hat{\epsilon})\epsilon^2 \cdot \Delta$.

On the other-hand, take any vertex u . At at most Δ of u 's in-neighbors can be sampled because it has degree at most Δ . It follows that $d_e(u) \leq \frac{\Delta}{\epsilon}$. Thus, $\Delta_e \leq \frac{\Delta}{\epsilon}$, and the width of any slice is at most $\epsilon \cdot \Delta$. \square

4.3 Analysis of the Election Phase

We are now ready to analyze the election phase. Initially we reveal every vertex in the sample \mathcal{X} . The vertex y_0 with largest estimated-degree is then the provisional winner. We then treat the slices in increasing order of estimated-degree. When we consider slice S_i , we will reveal every vertex in S_i except one (if it is the provisional winner y_i). For technical reasons, we will denote by S_0 the set \mathcal{X} . Observe that the set \mathcal{R} is the set of already revealed (eliminated) vertices.

Now let $S_{\leq \ell} = \cup_{j=0}^{\ell} S_j$, and denote by $d_{\ell}(u) = |\{v \in S_{\leq \ell} : vu \in A\}|$ the number of in-neighbors of v that are in S_j , for $j \leq \ell$. Then we begin by proving two lemmas. The first, Lemma 4, states that if a vertex $u \in S_{\ell}$ has a large $d_{\ell-1}(u)$ then the elected vertex has large in-degree. The second, Lemma 5, guarantees that the elected vertex has a large in-degree (with high probability) if there are many regionally well-estimated vertices in S_{ℓ} with large $d_{\ell}(u)$. These lemmas will be applied to a vertex x of in-degree Δ : either many in-neighbors of x are

in slices before x and Lemma 4 will apply, or many in-neighbors of x are in its slice and we will apply Lemma 5 to this set of in-neighbors.

Lemma 4. *Take $u \in S_{\ell+1}$. If $d_\ell(u) = d$, the elected vertex has in-degree at least $d - 1$.*

Proof. When we select the provisional winner y_ℓ all vertices of $S_{\leq \ell} = \cup_{j=0}^{\ell} S_j$ (but at most one, y'_ℓ , if it is in this set) have been revealed. Now $u \in S_{\ell+1}$ is an eligible candidate for y_ℓ . Thus, at that time, $d_{\mathcal{R}}^-(y_\ell) \geq d_{\mathcal{R}}^-(u) \geq d_\ell(u) - 1 = d - 1$. Since the in-degrees of the provisional winners can only increase, the elected vertex y_τ has in-degree at least $d - 1$. Note that the minus one comes from the fact that y'_ℓ can be an in-neighbor of u . \square

Lemma 5. *Let $\Delta \geq \Delta_1 = \frac{12^2}{\epsilon^4}$. If there exists an integer ℓ and a set $Z \subseteq S_\ell$ of size at least $\epsilon\Delta$ of regionally well-estimated vertices with $d_\ell(z) \geq (1 - 3\epsilon)\Delta + 1$ for every $z \in Z$, then with probability at least $(1 - \epsilon)$ we have $d(y'_\ell) \geq (1 - 5\epsilon)\Delta$.*

Proof. First, by selecting a subset of Z if necessary, we may assume that $Z = \lceil \epsilon\Delta \rceil$. Now we define a collection of bad events and show that $d(y_\ell) \geq (1 - 5\epsilon)\Delta$ if none of these events arise. We then show the probability that any of these bad events occurs is small.

Let \mathbf{B}_0 be the event that every vertex in Z is placed in \mathcal{R} when we sample the vertices of slice ℓ . We may assume the provisional leader $y_{\ell-1}$ is not in Z . Thus, since $|Z| \geq \epsilon\Delta$, the probability of event \mathbf{B}_0 is at most

$$(1 - \epsilon)^{\epsilon\Delta} = e^{\epsilon\Delta \cdot \log(1 - \epsilon)} \leq e^{-\epsilon^2\Delta} \leq \frac{\epsilon}{2}$$

Here the first inequality holds since $\log(1 - \epsilon) \leq -\epsilon$. The second inequality holds as $\Delta \geq \frac{2}{\epsilon^4} \geq \frac{12^2}{\epsilon^4}$.

Now take any $z \in Z$ and let U_z be the set of in-neighbors of z in $S_{\leq \ell} \setminus \{y_{\ell-1}\}$. We have $|U_z| \geq (1 - 3\epsilon) \cdot \Delta$. Let \mathbf{B}_z be the event that less than $(1 - 5\epsilon) \cdot \Delta$ vertices of $|U_z|$ are in \mathcal{R} at the time we sample the vertices of slice ℓ .

To analyze the probability of this event consider any $u_i \in U_z$. Now, if $u_i \in S_j$ for $j < \ell$ then u_i is already in \mathcal{R} . Otherwise, if $u_i \in S_\ell$ then it is now added to \mathcal{R} with probability $(1 - \epsilon)$. So consider a random variable Y_i which has value 1 if $u_i \in \mathcal{R}$ after this sampling and 0 otherwise. Note the Y_i are not identically distributed. So let X_i be variables which are independent and identically distributed and such that $X_i = Y_i$ if $u_i \in S_\ell$ and X_i has value 1 with probability $(1 - \epsilon)$ otherwise. We have

$$\begin{aligned} \Pr\left[\sum_i Y_i \leq (1 - \epsilon - \hat{\epsilon}) \cdot |U_z|\right] &\leq \Pr\left[\sum_i X_i \leq (1 - \epsilon - \hat{\epsilon}) \cdot |U_z|\right] \\ &\leq e^{-\hat{\epsilon}^2(1 - \epsilon) \cdot |U_z|/3} \\ &\leq e^{-\frac{1}{3}\hat{\epsilon}^2(1 - \epsilon)(1 - 3\epsilon)\Delta} \\ &\leq e^{-2\sqrt{\Delta}} \\ &\leq \frac{1}{3\Delta} \end{aligned}$$

Here the first inequality follows from Theorem 5. The second inequality holds as $|U_z| \geq (1 - 3\epsilon)\Delta$. The third inequality holds by the choice $\sqrt{\Delta} \geq \frac{12}{\epsilon^2} \geq \frac{6}{\epsilon^2(1-\epsilon)(1-3\epsilon)}$. The fourth one is satisfied since $\Delta \geq 100$.

We now apply the union bound to the events $\mathbf{B}_0 \cup \bigcup_{z \in Z} \mathbf{B}_z$. Since $Z = \lceil \epsilon\Delta \rceil$, none of these events occur with probability at least $1 - \frac{\epsilon\Delta+1}{3\Delta} - \frac{\epsilon}{2} \geq 1 - \epsilon$. Thus, with probability at least $1 - \epsilon$, after the sampling of the slice S_ℓ , there is a vertex $z \in Z$ that is not in \mathcal{R} but that has at least $(1 - 5\epsilon) \cdot \Delta$ in-neighbors in \mathcal{R} . The new provisional leader y'_ℓ must then satisfy $d_{\mathcal{R}}^-(y'_\ell) \geq d_{\mathcal{R}}^-(z) \geq (1 - 5\epsilon)\Delta$, as required, since $(1 - 5\epsilon) \cdot \Delta \leq (1 - \epsilon - \hat{\epsilon}) \cdot |U_z|$. \square

Proof of Theorem 2. We may now prove that the slicing mechanism is nearly optimal. We assume that $\Delta \geq M_\epsilon = \max(\Delta_1, \Delta_2)$ and that $\epsilon \leq \frac{1}{8}$. Let x be a vertex of in-degree Δ . We assume that x is not selected in \mathcal{X} during the sampling phase and that all the vertices of $N^{-2}(x)$ with in-degree at least $\epsilon^2\Delta$ are regionally well-evaluated. We need the following claim, where k denotes the integer such that the vertex x of maximum in-degree is in S_k .

Claim. Let u be a vertex in $N^{-3}(x)$ that is not in $S_{\leq k}$. Then $u \in S_{k+1}$.

Proof. Take a vertex $u \in N^{-3}(x)$. If $d^-(u) \leq \epsilon^2\Delta$, then its estimated degree is at most $\epsilon\Delta \leq d_e(x)$. Thus now we can assume that $d^-(u) \geq \epsilon^2\Delta$ and then u is $\hat{\epsilon}$ -well-estimated by assumption on x . First observe that the set of possible estimated degrees of u intersects at most two slices. To see this note that the range of $d_e(u)$ is less than $2\hat{\epsilon} \cdot \Delta$ as u is $\hat{\epsilon}$ -well-estimated. On the other-hand, by Lemma 3, the width of a slice is at least $(1 - \hat{\epsilon})\epsilon^2 \cdot \Delta$. Since $\hat{\epsilon} = \frac{\epsilon^2}{4}$ we have

$$2\hat{\epsilon}\Delta \leq \frac{1}{2}\epsilon^2\Delta < (1 - \hat{\epsilon})\epsilon^2\Delta$$

Since the range is less than the width, the observation follows.

The vertex x of maximum degree is $\hat{\epsilon}$ -well-estimated and is in the slice k . Therefore, because u is $\hat{\epsilon}$ -well-estimated (and necessarily $d(u) \leq d(x)$), there must be a slice smaller than or equal to k in its range of u . Thus u cannot be in a slice with index exceeding $k + 1$. \square

Assume first that there exists a set Z_1 of at least $\epsilon\Delta$ vertices of $N^{-2}(x)$ such that $Z_1 \cap S_i = \emptyset$ for $i \leq k$. The claim ensures that $Z_1 \subseteq S_{k+1}$. By considering a subset of Z_1 , we may assume that $|Z_1| = \epsilon\Delta$. (We assume that $\epsilon\Delta$ is an integer, for simplicity.) Then, for any $z \in Z_1$, we have $d_e(z) \geq d_e(x)$ since z is in the slice after x . Moreover, as x and z are $\hat{\epsilon}$ -well-estimated, the facts that $|d_e(z) - d(z)| \leq \hat{\epsilon}\Delta$ and $|d_e(x) - d(x)| \leq \hat{\epsilon}\Delta$ imply that

$$d(z) \geq (1 - 2\hat{\epsilon})\Delta \geq (1 - \epsilon)\Delta$$

Furthermore, by the above claim, we must have that $d_{k+1}(z) = d(z)$. Consequently, we may apply Lemma 5 to the set Z_1 . This ensures that the in-degree of the elected vertex is at least $(1 - 5\epsilon)\Delta$ with probability at least $(1 - \frac{1}{\epsilon})$.

On the other hand, assume that now less than $\epsilon\Delta$ vertices of $N^{-2}(x)$ are in S_{k+1} . In particular, we have $d_k(x) \geq (1 - \epsilon)\Delta$. If $d_{k-1}(x) \geq (1 - 4\epsilon) \cdot \Delta + 1$ then the elected vertex has degree at least $(1 - 4\epsilon)\Delta$ by Lemma 4.

So, assume that $d_{k-1}(x) \leq (1 - 4\epsilon)\Delta$. Then at least $3\epsilon\Delta$ in-neighbors of x are in S_k . Denote by Z_2 a set of $\epsilon\Delta$ in-neighbors of x in S_k . Every vertex $z \in Z_2$ has in-degree at least $(1 - \epsilon - 2\hat{\epsilon})\Delta \geq (1 - 2\epsilon)\Delta + 1$; this follows because both x and z are $\hat{\epsilon}$ -well-estimated and because the width of a slice is at most $\epsilon\Delta$ (Lemma 3). For any $z \in Z_2$, since at most $\epsilon\Delta$ of the in-neighbors of z are in S_{k+1} , we have $d_k(z) \geq (1 - 3\epsilon)\Delta + 1$. Moreover, by assumption all the vertices of Z_2 are regionally well-estimated. Hence, by Lemma 5, with probability at least $(1 - \epsilon)$, the degree of the elected vertex is at least $(1 - 5\epsilon)\Delta$, as desired.

Consequently, the slicing mechanism typically outputs a near-optimal vertex. So what is the probability that assumptions made during the proof fail to hold? Recall that Assumptions (A1) and (A2) fail to hold with probability at most 2ϵ . Given these two assumptions, Lemma 5 fails to output a provisional leader with in-degree at least $(1 - 5\epsilon) \cdot \Delta$ with probability at most ϵ . Thus the total failure probability is at most 3ϵ . Consequently, the expected in-degree of the elected vertex is at least $(1 - 3\epsilon)(1 - 5\epsilon) \cdot \Delta \geq (1 - 8\epsilon)\Delta$, which concludes the proof of Theorem 2. \square

We conclude with some remarks. Here $M_\epsilon = \mathcal{O}(\frac{1}{\epsilon^8})$; the degree of this polynomial can certainly be improved as we did not attempt to optimize it.

The slicing mechanism can be adapted to select a fixed number c of winners rather than one. Let us briefly explain how. Instead of selecting only one provisional winner y during each iteration of the election phase, we can select a set of size c containing unrevealed vertices maximizing $d_{\mathcal{R}}^-$. Let x_1, \dots, x_c be the c vertices of highest in-degree. With high probability all the vertices of $N^{-2}(x_i)$ are regionally well-estimated for every $i \leq c$ and with high probability none of them are selected during the sampling phase. Now consider two cases: either $N^{-2}(x_1)$ contains many vertices of degree almost Δ , and then an adaptation of Lemma 5 ensures that with high probability c vertices are not sampled during the sampling of the election phase and the c elected vertices have large degree. Or $N^{-2}(x_1)$ has few vertices of degree almost Δ and then when the slice of x_1 is considered, if x_1 is not selected, then all the selected vertices have degree almost Δ by Lemma 4. A similar argument can be repeated for every vertex x_i .

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Appendix

In this appendix, we include the proof omitted from the main text due to space constraints.

Lemma 1. *For every $\epsilon_1 > 0$ there exists N_1 such that for all positive integers $N_1 < \Delta \leq n$ and $k \leq n$ the following holds. If $X \subseteq [n]$ are chosen uniformly at random subject to $|X| = \Delta$, then $\Pr [||X \cap [k]| - \frac{k\Delta}{n}| \geq \epsilon_1 \Delta] < \epsilon_1$.*

Proof. Let $k \leq n$. Clearly, the lemma holds for $\Delta = n$, and so we assume $\Delta < n$. Let $p = \Delta/n$, and let Y be a random subset of $[n]$ obtained by choosing every element of $[n]$ independently with probability p . Our first goal is to lower bound $\Pr[|Y| = \Delta]$. We start by deriving the following estimate:

$$\frac{n!}{(n-\Delta)!} (n-\Delta)^{n-\Delta} \geq n^n e^{-\Delta} \quad (5)$$

which holds for all non-negative integers Δ . We prove (5) by induction on Δ . The base case $\Delta = 0$ is trivial for any non-negative n . For the induction step, we have

$$\begin{aligned} \frac{n!}{(n-\Delta)!} (n-\Delta)^{n-\Delta} &= n \frac{(n-1)!}{(n-\Delta)!} (n-\Delta)^{n-\Delta} \\ &\geq n(n-1)^{n-1} e^{-(\Delta-1)} \\ &\geq n(n-1)^{n-1} \left(1 + \frac{1}{n-1}\right)^{n-1} e^{-\Delta} \\ &= n^n e^{-\Delta} \end{aligned}$$

Here the first inequality follows by induction, the second inequality follows from the fact that $(1 + \frac{1}{n})^n \leq e$ for every n . Thus (5) holds for all Δ . We then

have

$$\begin{aligned}
\Pr[|Y| = \Delta] &= \binom{n}{\Delta} p^\Delta (1-p)^{n-\Delta} \\
&= \frac{n!}{(n-\Delta)! \Delta!} \left(\frac{\Delta}{n}\right)^\Delta \left(1 - \frac{\Delta}{n}\right)^{n-\Delta} \\
&= \frac{\Delta^\Delta}{\Delta!} \left(\frac{n!}{(n-\Delta)!} \frac{(n-\Delta)^{n-\Delta}}{n^n}\right) \\
&\geq \frac{\Delta^\Delta}{\Delta!} e^{-\Delta} \\
&\geq e^{-\Delta} \frac{\Delta^\Delta}{3\sqrt{\Delta}(\Delta/e)^\Delta} \\
&= \frac{1}{3\sqrt{\Delta}}
\end{aligned}$$

Here the first inequality follows by (5). The second inequality applies when Δ is sufficiently large. Indeed Stirling's formula ensures that $n! \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{n}{e}\right)^n$; thus, since $\sqrt{2\pi} \leq 3$, we have $\Delta! \leq 3\sqrt{\Delta}(\frac{\Delta}{e})^\Delta$. Therefore

$$\Pr \left[\left| |X \cap [k]| - \frac{k\Delta}{n} \right| \geq \epsilon_1 \Delta \right] \leq 3\sqrt{\Delta} \cdot \Pr \left[\left| |Y \cap [k]| - \frac{k\Delta}{n} \right| \geq \epsilon_1 \Delta \right] \quad (6)$$

By Theorem 5, applied with $\delta = \epsilon_1$ and $\Delta = pn$, we have

$$\Pr \left[\left| |Y \cap [k]| - \frac{k\Delta}{n} \right| \geq \epsilon_1 \Delta \right] \leq e^{-\frac{\epsilon_1^2 \Delta}{3}} \quad (7)$$

Combining (6) and (7), we deduce that

$$\Pr \left[\left| |X \cap [k]| - \frac{k\Delta}{n} \right| \geq \epsilon_1 \Delta \right] \leq 3\sqrt{\Delta} e^{-\frac{\epsilon_1^2 \Delta}{3}} \quad (8)$$

Clearly, for Δ sufficiently large with respect to ϵ_1 we have $3\sqrt{\Delta} e^{-\frac{\epsilon_1^2 \Delta}{3}} \leq \epsilon_1$. It follows that (8) gives the lemma for such Δ , as desired. \square