Hitting sets:

VC-dimension and Multicut

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Informatique
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ECOLE DOCTORALE

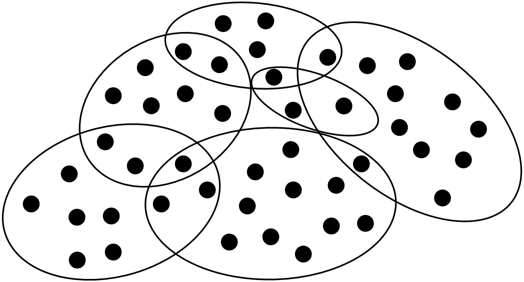
UNIVERSITÉ MONTPELLIER 2
SCIENCES ET TECHNIQUES

Hypergraphs

Definition Hypergraph

A hypergraph is a pair $H = (V, E)$ where :

- V is a set of *vertices*.
- E is a set of subsets of V called *hyperedges*.



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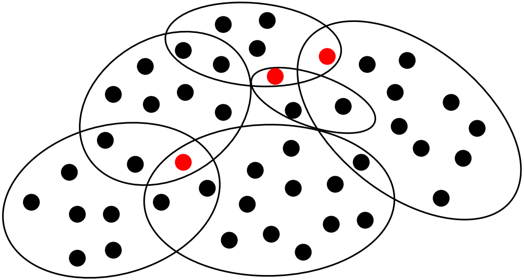
A hitting set is a subset of vertices intersecting all the hyperedges.

τ : minimum size of a hitting set. $(= 3)$



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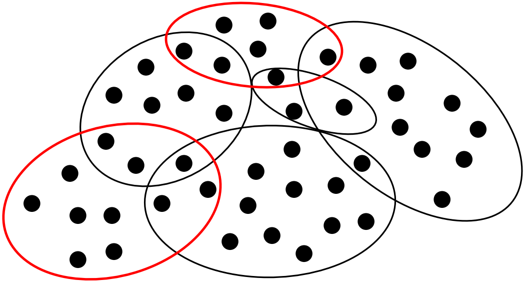

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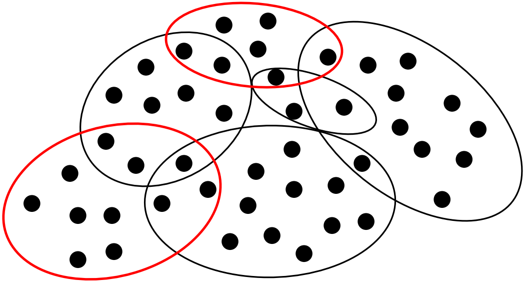

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Erdős-Pósa property

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Bounding τ in function of ν?

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Bounding τ in function of ν? **No.**
Hypergraph where every set of size $\frac{n}{2} + 1$ is a hyperedge.

- $\nu = 1$.
- $\tau = \frac{n}{2}$.

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Definition

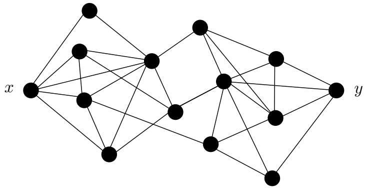
A class $\mathcal{H}$ of hypergraphs has the Erdős-Pósa property if there exists a function f such that for every $H \in \mathcal{H}$

$$\tau \leq f(\nu(H)).$$

Illustration : Menger's theorem

Theorem (Menger)

The maximum number of edge-disjoint *xy*-paths is the size of a minimum *xy*-separator.



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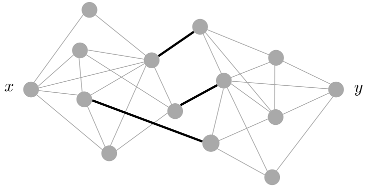

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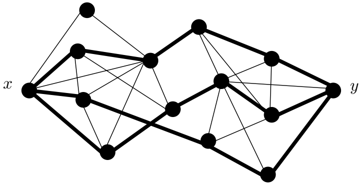

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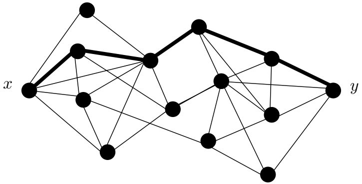

- Vertices : edges of the graph.
- Hyperedges : xy-paths.

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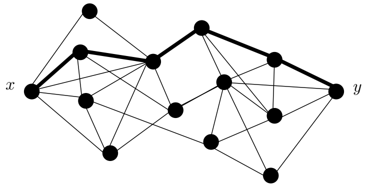


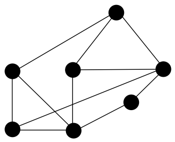
- Vertices : edges of the graph.
- Hyperedges : *xy*-paths.

τ : minimum number of edges whose deletion separate x from y.
ν : maximum number of edge-disjoint *xy*-paths.
Reformulation of Menger's theorem : $\nu = \tau$.

Illustration : Erdős-Pósa theorem

Theorem (Erdős, Pósa)

The minimum size of a feedback vertex set can be bounded in function of the number of maximal vertex-disjoint cycles.



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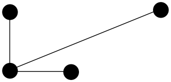

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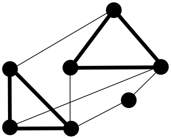

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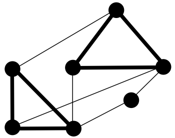


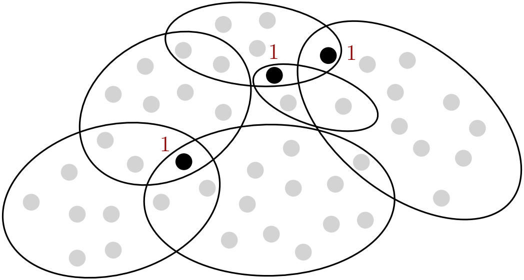
- Vertices : vertices of the graph.
- Hyperedges : cycles of the graph.

τ : minimum feedback vertex set.
ν : maximum number of vertex-disjoint cycles.
Reformulation : $\tau \leq \mathcal{O}(\nu \log \nu)$.

An invariant between τ and ν

Transversal Integer Linear Program

Variables : for each $v_i \in V$, associate x_i a non negative integer.
Constraints : for each $e \in E$, $\sum_{v_i \in e} x_i \geq 1$
Objective function : $\tau = min(\sum_{i=1}^{n} x_i)$



An invariant between τ and ν

Fractional Transversal Linear Program

Variables : for each $v_i \in V$, associate x_i a non negative real.
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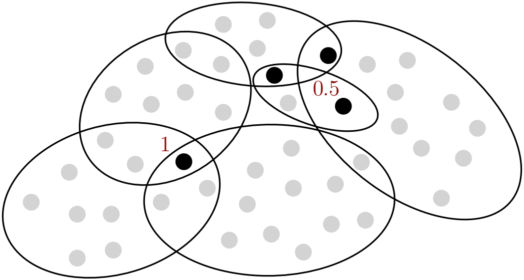

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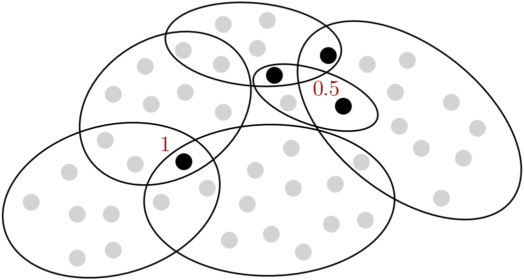

- $\tau \geq \tau^*$.
- τ^* can be computed in polynomial time.

Integrality gap

The integrality gap between τ and τ^* can be arbitrarily large.

Inequalities

$$\nu \leq \tau^* \leq \tau$$

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Structural question : Which conditions ensure :

- $\tau \leq f(\nu)$ (Erdős-Pósa property).
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Complexity question : Computing or approximating τ and ν is hard (under complexity assumptions).

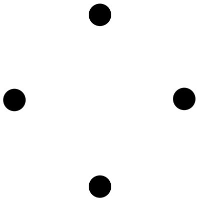
- Find approximation algorithm for τ and ν ?
- Find "efficient" algorithms for these problems ?

VC-dimension

Definition VC-dimension (Vapnik, Chervonenkis '71)

A set $X \subseteq V$ is *shattered* iff for all $Y \subseteq X$, there exists $e \in E$ such that $e \cap X = Y$.
The *VC-dimension* is the maximum size of a shattered set.

Illustration : vertices $= \mathbb{R}^2$ hyperedges $=$ rectangles.



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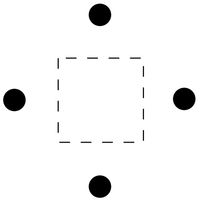

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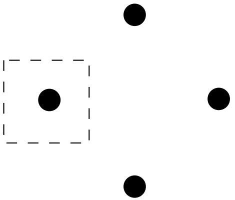

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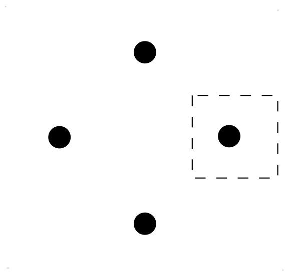

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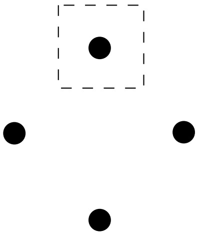

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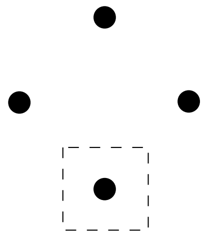

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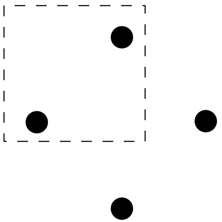

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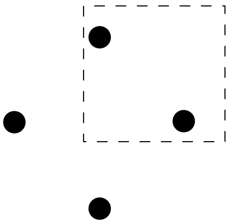

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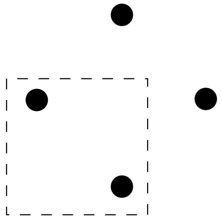

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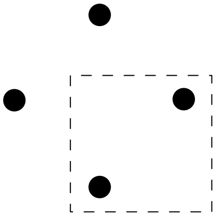

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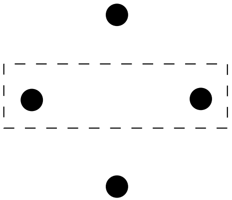

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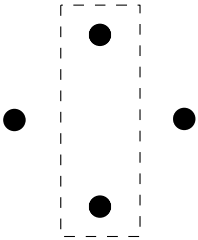

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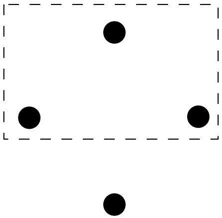

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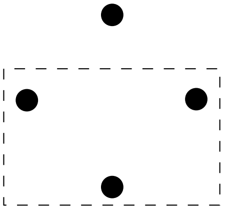

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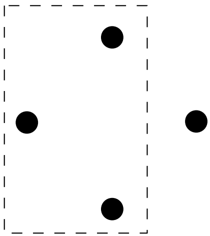

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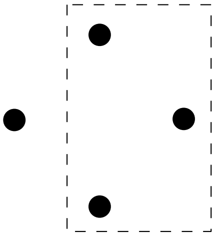

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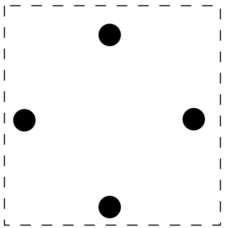

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Bounded VC-dimension gives combinatorial properties :

- Polynomial number of hyperedges.

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- Chromatic number of triangle-free graphs of large degree (Łuczak, Thomassé '10).
- Separating cliques and stable sets in split-free graphs. [1]

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But sometimes a bounded gap between τ and τ^* is not enough...

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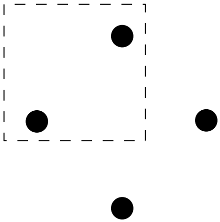

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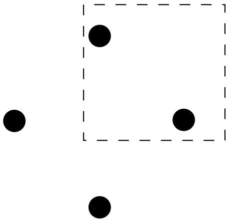

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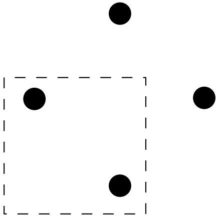

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A set $X \subseteq V$ is 2-*shattered* if for every $Y \subseteq X$ with $|Y| = 2$, there exists $e \in E$, $e \cap X = Y$.
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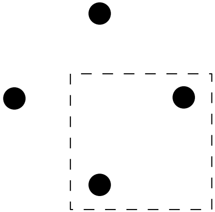

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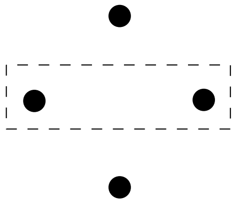

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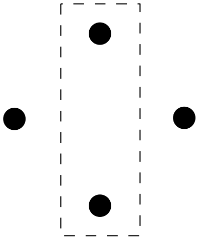

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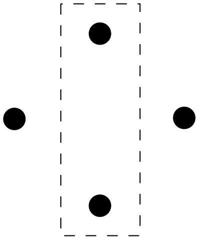

2VC $\geq$ VC since a shattered set is 2-shattered.

Erdős-Pósa property

Theorem (Ding, Seymour, Winkler '91)

Every hypergraph of (dual) 2VC-dimension d satisfies :

$$\tau \leq 11d^2(\nu + d + 3) \cdot \binom{d+\nu}{d}$$

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- Few applications of this theorem.

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Theorem (B., Thomassé)[1]

Every maximal-triangle free graph with no induced subdivision of H has chromatic number at most $f(|H|)$.

1. *Scott's conjecture for maximal triangle-free graphs*, B., Thomassé `CPC'12`

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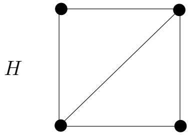

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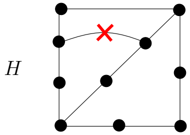

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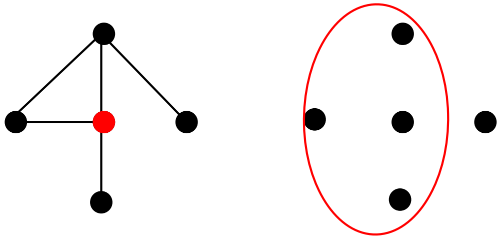
Sketch of the proof

Neighborhood hypergraph Γ of G : hyperedges are closed neighborhoods of G.



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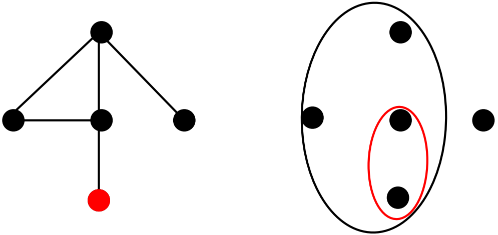

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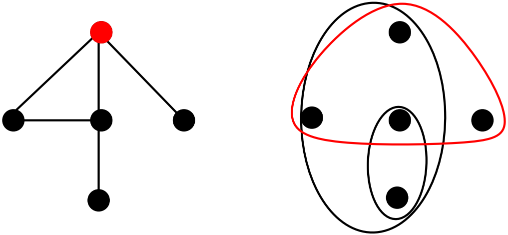

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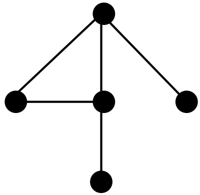

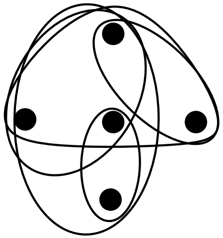

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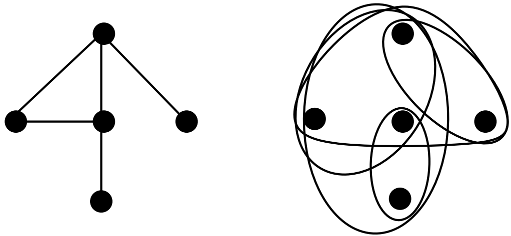

Observations

- A hitting set of Γ is a dominating set of G.
- A triangle-free graph has chromatic number at most 2τ (it is covered by τ induced stars).

Sketch of the proof (II)

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The closed neighborhood hypergraph of a maximal triangle-free graph satisfies $\nu = 1$.

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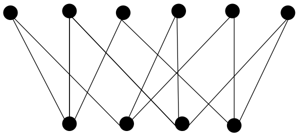
Let us prove that in the latter case, we have an induced subdivision of H.

Intuition



Goal : Extract from this graph a subdivision of H.

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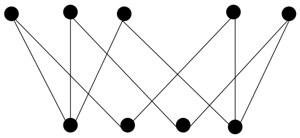

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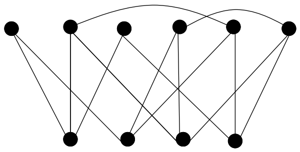

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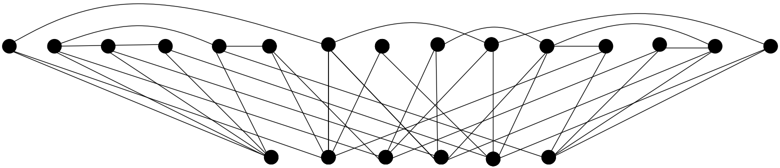

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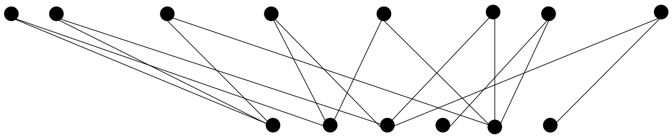

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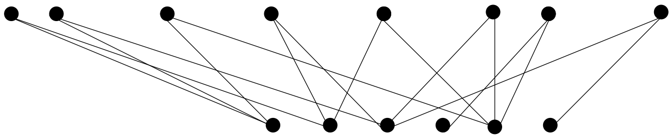

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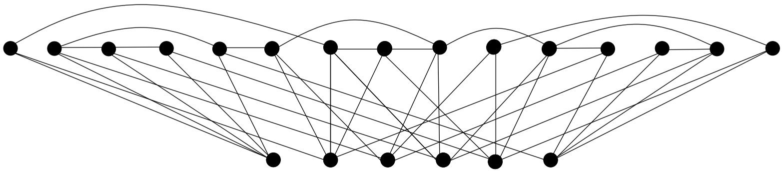
Question : How can we transform this intuition into a proof.

Formal proof



We have $\binom{k}{2}$ vertices on the top.

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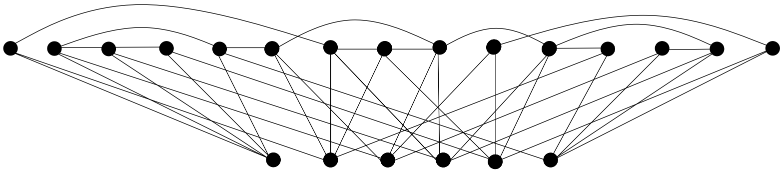

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Theorem (Kim '95)

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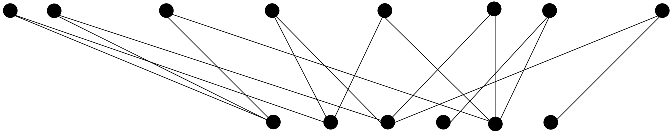

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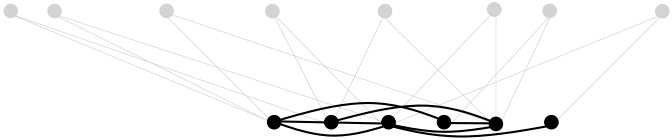

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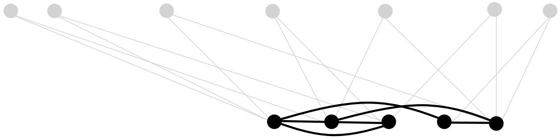

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- G' has $\mathcal{O}(k\sqrt{\log k})$ edges $\Rightarrow$ G' has a subdivision of H.

Theorem (Mader '67)

Every graph on k vertices with more than $512 \cdot |H| \cdot k$ edges has a (non induced) subdivision of H.

A second application

Theorem (Chepoi, Estellon, Vaxès '07)

Every planar graph of diameter 2ℓ can be covered by c balls of radius ℓ (where c is independent from ℓ).

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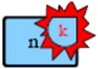

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Non FPT problems (under complexity assumptions) :

- Dominating set.
- Independent set.

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Input : A graph G, a set of pairs (x_i, y_i), an integer k.
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τ : minimum size of a multicut.

Recent results

- Polynomial kernel for trees. [1]

1. *A polynomial kernel for Multicut in trees*, B., Daligault, Thomassé, Yeo STACS'09

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Theorem (Marx and Razgon / B., Daligault, Thomassé)[2]

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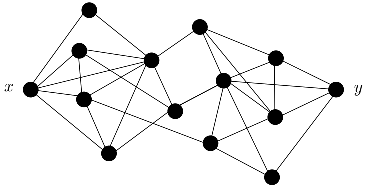

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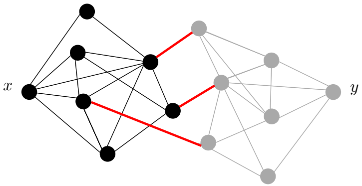

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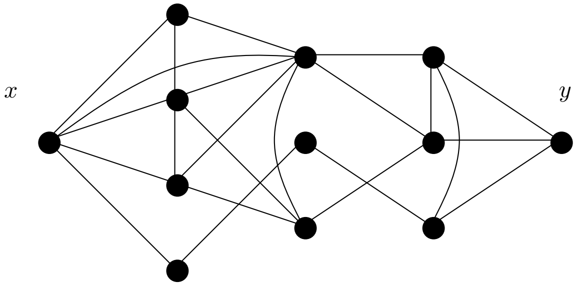

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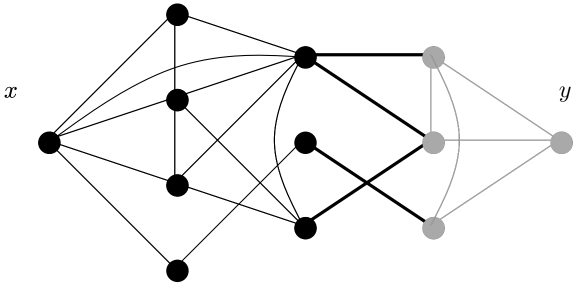

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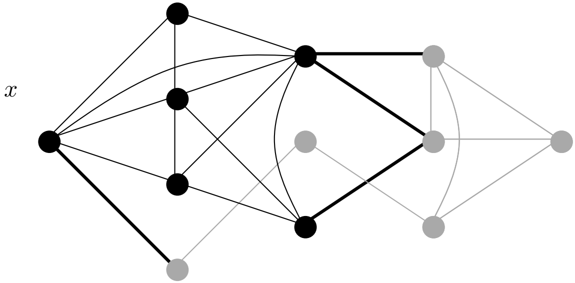


Important separators

Theorem (Marx '06, Chen *et al.* '08)

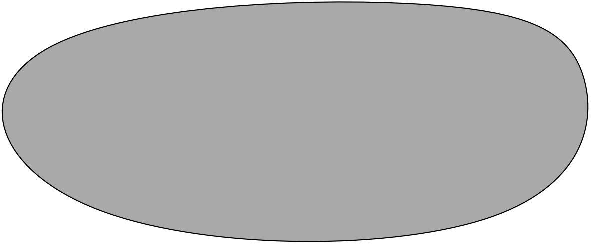
The number of important separators of size k is at most 4^k.

Many applications :

- Multiway Cut (Marx '06).
- Directed Feedback Vertex set (Chen *et al.*, Marx, Razgon '08).
- Almost 2-SAT (Marx, Razgon '09).
- Directed Multiway Cut (Chitnis, Hajiaghayi, Marx '13).

Multicut is FPT

Sketch of the proof :



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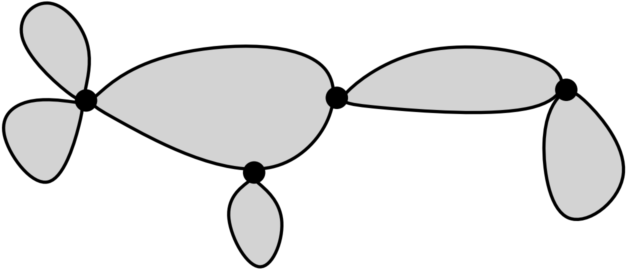
Sketch of the proof :

- Find a vertex multicut.



Multicut is FPT

Sketch of the proof :

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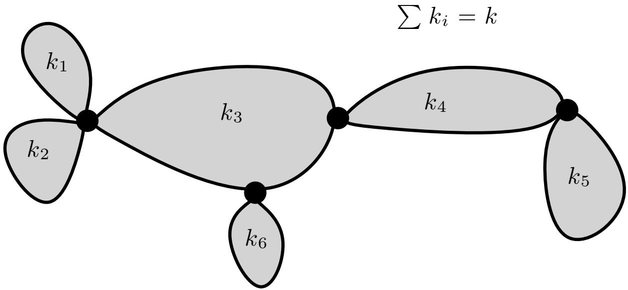
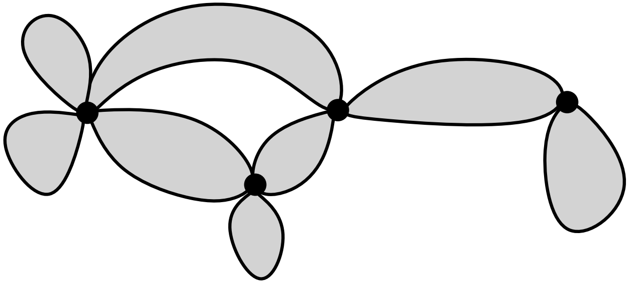
- Guess the number of edges in each component.



Multicut is FPT

Sketch of the proof :

- Find a vertex multicut.
- Guess the number of edges in each component.
- Reduce components with at least 3 attachment vertices.



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- Guess the number of edges in each component.
- Reduce components with at least 3 attachment vertices.
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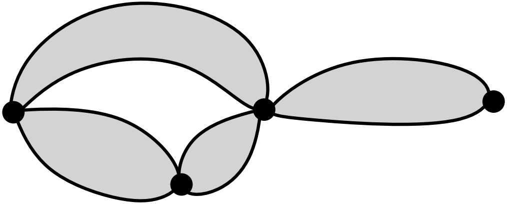
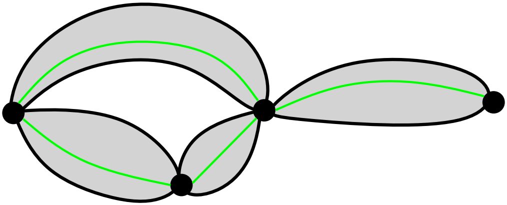


Multicut is FPT

Sketch of the proof :

- Find a vertex multicut.
- Guess the number of edges in each component.
- Reduce components with at least 3 attachment vertices.
- Treat components with one attachment vertex.
- Find an important path on 2-components.



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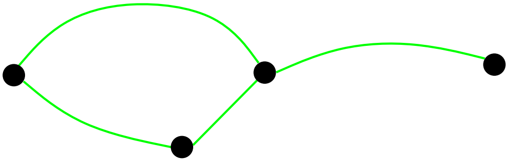
- Reduce each component to a path.



Multicut is FPT

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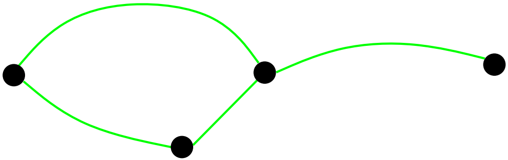
- Reduce each component to a path.
- Solve a 2-SAT instance.



Further work

VC-dimension.

- **Combinatorics.** Close gaps between upper and lower bounds.
- **Application.** Adapt and apply the methods to many other problems (domination-like problems may work).
- **Algorithm.** Find NP-complete problems which become polynomial when the VC-dimension is bounded.

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Graph separation problems.

- **Combinatorics.** Find the exact upper bound on the number of important separators.
- **Application.** Find a simpler FPT-algorithm for Multicut.
- **Algorithm.** Many open problems for Multicut in Directed graphs.

Other Contributions

Algorithmic and parameterized complexity :

- Parameterized results for domination on circle graphs.
- Sparsest Subgraph is FPT on chordal graphs.

Graph structure and coloring :

- χ-boundedness for graphs with no cycle with 2/3 chords.
- Erdős-Hajnal conjecture for paths and antipaths.
- Rainbow colorings of 3-chromatic graphs.
- Recoloring graphs via tree-decompositions.
- AVD-colorings for graphs of large Δ.

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joint work with Pierre Aboulker, Stéphane Bessy, Marthe Bonamy, Marin Bougeret, Jean Daligault, Rodolphe Giroudeau, Daniel Gonçalves, Hervé Hocquard, Aurélie Lagoutte, George Mertzios, Christophe Paul, Ignasi Sau, Stéphan Thomassé, Rémi Watrigant, Anders Yeo.