On the chromatic number of wheel-free graphs with no large bipartite graphs

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Abstract

A wheel is an induced cycle C plus a vertex connected to at least three vertices of C. Trotignon [14] asked if the class of wheel-free graphs is χ-bounded, *i.e.* if the chromatic number of every graph with no induced copy of a wheel is bounded by a function of its maximal clique. In this paper, we prove a weaker statement: for every ℓ, the class of graphs with no induced wheel and no induced $K_{\ell,\ell}$ is χ-bounded.

Moreover, we show that the chromatic number of every triangle-free graph with no $K_{\ell,\ell}$ and no k-wheel (a cycle C plus a vertex incident to at least k vertices of C) is bounded. We also give some applications of these results on the chromatic number of graphs with no cycle with a fixed number of chords.

1 Introduction

All along this paper, we consider classes of graphs closed under induced subgraphs. For standard notations and definitions on graphs, the reader is referred to [6]. Let H be a graph. We say that a graph G is H-free is G does not contain H as an induced subgraph. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to color the vertices of G in such a way two incident vertices receive distinct colors (for standard notations and definitions on graphs, the reader is referred to [6]). The *clique number* $\omega(G)$ of G is the maximum size of a clique of G. In this paper, we investigate the gap between the chromatic number of G and clique number of G. The inequality $\omega(G) \geq \chi(G)$ always holds since any pair of vertices of a clique are incident and then receive distinct colors. In general, the converse of this inequality is not satisfied. Erdős proved that there exist triangle-free graphs with arbitrarily large chromatic number [8].

Chudnovsky et *al.* [5] characterized the class of graphs (closed by induced subgraphs) satisfying $\chi(G) = \omega(G)$, a problem open for more than fifty years (this result is known as the Strong Perfect Graph Theorem). In 1987, Gyárfás introduced the concept of χ-bounded classes [9]. A class $\mathcal{C}$ of graphs is χ*-bounded* if there exists a function f such that any graph $G \in \mathcal{C}$ satisfies $\chi(G) \leq f(\omega(G))$. In its seminal paper, Gyárfás proved that the class of graphs with no induced long path is χ-bounded. Since, χ-bounded classes have received a lot of attention.

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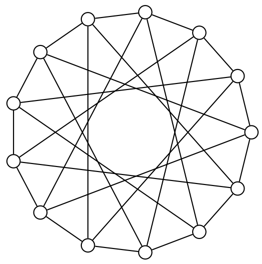

Figure 1: The graph $R(3,5)$. It is a wheel-free graph satisfying $\omega = 2$ and $\chi = 4$. Note that $\chi(R(3,5)) \geq 4$ since $R(3,5)$ has 13 vertices and has no stable set of size 5.

A *k-wheel* is an induced cycle C plus a vertex v connected to at least k vertices of C. Edges between v and C are called the *rays*. A *wheel* is a 3-wheel and a *propeller* is a 2-wheel. Wheels play a central role in the proof of the Strong Perfect Graph Theorem. Actually, more than 50 pages of the proof are devoted to treat wheels. It raises a natural question, first asked by Trotignon (e.g. in [4, 14]): do things become simpler (from a coloring point of view) when we consider wheel-free graphs? In particular, we can ask the following question:

Conjecture 1 (Trotignon [14])**.** *The class of (induced) wheel-free graphs is χ-bounded.*

Conjecture 1 is open even for triangle-free graphs. So far, the best lower bound on the chromatic number of (triangle,wheel)-free graphs is due to Esperet and Stehlík who noted that there exists (triangle, wheel)-free graph of chromatic number is at least 4, using a construction of Zykov [16]. Aboulker et *al.* noticed in [4] that it can also be deduced from the Ramsey graph $R(3,5)$ represented in Figure 1.

While the structure of graphs with no wheel as a subgraph is well-known [13], there are few structural results on induced wheel-free graphs. In this paper, we focus on induced-free graphs. Chudnovsky proved that every non-empty induced wheel-free graph contains a vertex whose neighborhood is a disjoint union of cliques (a proof can be found in [2]). More recently, Aboulker et *al.* proposed a decomposition theorem for 3-connected planar wheel-free graphs [3]. Though, in general the structure of wheel-free graphs is complex since Diot et al. [7] proved that recognizing wheel-free graphs is NP-complete.

Our contribution. In this paper, we prove that every graph of large chromatic number contains either a triangle or a large complete bipartite graph or a wheel as an induced subgraph. *All along this paper, by "contains" we mean "contains as an induced subgraph" and by "free" we mean "induced free".*

Theorem 2. *There exists a function f such that every graph G of chromatic number at least $f(k,\ell)$ contains a triangle, or a k-wheel or a a $K_{\ell,\ell}$ (as induced subgraphs).*

The case $\ell = 2$ can be also stated as follows:

Corollary 3. *For every k, there exists a constant c_k such that every k-wheel-free graph of girth at least 5 has chromatic number at most c_k.*

So k-wheels cannot be avoided if we want to construct graphs with large chromatic number and large girth. In the case of wheels instead of k-wheels, we can extend this result for any clique size. In other words, we show:

Theorem 4. *For every integer ℓ, the class of (wheel, $K_{\ell,\ell}$)-free graphs is χ-bounded (where the function depends on ℓ).*

Related work. Kühn and Osthus proved [11] that every graph of large connectivity contains an induced 1-subdivision of a graph of an arbitrarily large degree or a large complete bipartite graph as a subgraph. In some sense, it means that the complexity of a graph of large degree relies on the existence of complete bipartite graphs or of induced subdivisions of graphs of large degree. Theorem 4 is a result of the same flavor: every graph of large chromatic number number contains either a large complete bipartite graph or a wheel.

The class of k-wheel-free graphs is also related to the class of graphs with no cycle with a fixed number of chords. Let $C = x_1, \dots, x_m$ be a (not necessarily induced) cycle of G. A *chord* of C is an edge $x_i x_j$ such that $|i - j| \neq 1$ modulo m. A *k-cycle* is a cycle with exactly k chords. Trotignon and Vušković proved in [15] that the class of graphs with no 1-cycle is χ-bounded. This result was extended to the class of graphs with no 2-cycle and the class of graphs with no 3-cycle by Aboulker and Bousquet in [1]. For graphs with no 3-cycle, a large (and technical) part of the proof consists in proving that the chromatic number of (3-cycle, triangle)-free graphs is bounded. Let us show that this result can be derived from Theorem 2.

Let G be a triangle-free graph. Theorem 2 with $\ell = 3$ and $k = 5$ asserts that if the chromatic number of G is at least $f(3, 5)$ then the graph contains a triangle or a $K_{3,3}$ or a 5-wheel. Since G is triangle-free, one of the last two cases holds. If G contains a $K_{3,3}$ then G contains a cycle with exactly 3-chords ($K_{3,3}$ is a cycle on six vertices with 3 chords). Now assume that G admits a 5-wheel as an induced subgraph. Let x be the center of the wheel and $y_1, \dots, y_5$ be 5 consecutive neighbors of x on the induced cycle. There exists an induced path P from y_1 to y_5 passing through y_2, y_3 and y_4. Moreover y_1 and y_5 are not incident since otherwise x, y_1, y_5 would be a triangle. Thus xPx is a cycle with precisely 3 chords: xy_2, xy_3 and xy_4. Note nevertheless that the bound obtained in [1] ($\chi \leq 24$) is better than the one provided by Theorem 2.

This argument can be generalized to other cases. The graph $K_{\ell,\ell}$ is a cycle with $\ell(\ell - 2)$ chords. So the previous argument can be immediately generalized to prove the following:

Corollary 5. *Let ℓ be an integer. There exists a constant c_ℓ such that every triangle-free graph with no $\ell(\ell - 2)$-cycle has chromatic number at most c_ℓ.*

So far, this result was known only for $\ell \leq 3$ [1, 15]. Thus Corollary 5 gives the first infinite collection of integers for which this property holds.

Organization of the paper. In Section 2, we give notations and definitions related to extractions and subdivisions of a graph. In Section 3, we present the main ingredients of the proof of Theorems 2 and 4. Section 4 is devoted to prove the technical lemmas stated in Section 3.

2 Preliminaries

Let $G = (V, E)$ be a graph and let A, B be two disjoint subsets of vertices of G. The sets A and B are *incident* if there exists an edge with one endpoint in A and one endpoint in B. Two sets which are not incident are *independent*.

Extractions. A *path* P is a sequence of vertices $x_0, \dots, x_k$ such that for every $i \leq k - 1$, $x_i x_{i+1}$ is an edge. The *endpoints* of P are the vertices x_0 and x_k. The other vertices are called *internal vertices*. The *length* of a path is its number of edges. The *interior* of P is the path $x_1 x_2 \dots x_{k-1}$, *i.e.* the path P without its endpoints. Given two vertices x and y, the *distance between x and y* is the length of a minimum path between x and y.

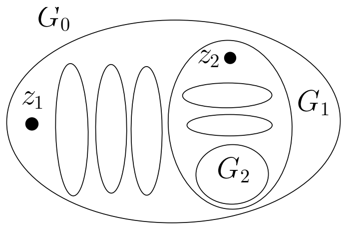

Figure 2: The graph G_2 is a 2-extracted graph of the graph G_0.

Let z be a vertex. The set of vertices at distance exactly ℓ from z is called the *ℓ-th neighborhood of* z and is denoted by $N_\ell(z, G)$. We denote by $G[N_\ell(z)]$ the subgraph of G induced by the vertices $N_\ell(z, G)$. A *father* (for z) of a vertex $x \in N_\ell(z, G)$ is a vertex of $N_{\ell-1}(z, G)$ incident to x (it is a "father" of x if G is rooted in z). For every pair x, y of vertices in $N_\ell(z, G)$, it is easy to see that there exists an induced xy-path Q with internal vertices in $z \cup N_1(z, G) \cup \cdots \cup N_{\ell-1}(z, G)$ such that only the endpoints of the interior of Q are in $N_{\ell-1}(z, G)$ (it suffices to take the concatenation of a minimum xz-path and a zy-path and shortcut it if it is not induced). Since there is no edge between non-consecutive levels, vertices of Q with neighbors in $N_\ell(z, G)$ are either endpoints of Q or endpoints of the interior path of Q. Such paths are called *unimodal paths* and are essential to find subdivided induced structures (see [1] for instance). Note that the interior of a unimodal path may contain only one vertex. In the remaining of the paper, we will not justify anymore the existence of unimodal paths between vertices of $N_\ell(z, G)$ and the capital letter Q is devoted to denote unimodal paths.

Observation 6. *Let G be a graph and z be an integer. There exists an integer ℓ such that $\chi(G[N_\ell(z)]) \geq \lceil \frac{\chi(G)}{2} \rceil$.*

Proof. There is no edge between vertices of non consecutive neighborhoods of z. Indeed if there is an edge between a vertex u of $N_i(z, G)$ and a vertex v of $N_j(z, G)$ with $i < j$, then there is a path of length at most $(i + 1)$ from z to v, *i.e.* $j = i + 1$. So neighborhoods at even distance from z can be colored with the same set of colors. By symmetry, the same holds for vertices at odd distance from z. Thus if for every ℓ the ℓ-th neighborhood of z can be colored with at most $(\lceil \frac{\chi}{2} \rceil - 1)$ colors, then G can be colored with at most $\chi - 1$ colors, a contradiction. $\square$

A *(1-)extracted graph G'* is an induced subgraph of G of chromatic number at least $\frac{\chi(G)}{2}$ such that there exists a vertex z and an integer ℓ satisfying $G' = G[N_\ell(z)]$. Observation 6 ensures that every graph has an extracted graph. For every $p \geq 2$, a *p-extracted graph G'* of G is an extracted graph of a $(p-1)$-extracted graph of G (see Figure 2). In the following we denote by $G_0, G_1, G_2, \ldots, G_p$ a sequence of extracted graphs where $G_0 = G$ and for every $i \geq 1$, the graph G_i is an extracted graph of G_{i-1}. We denote by (z_i, ℓ_i) a pair satisfying $G_i = G_{i-1}[N_{\ell_i}(z_i)]$. Observation 6 ensures that $\chi(G_i) \geq \lceil \frac{\chi}{2^i} \rceil$ for every $i \leq p$. An *i-father* of $x \in G_p$ is a father of x (for z_i) in G_{i-1}. An *i-unimodal path between x and y in G_i* is a unimodal path between x and y in G_{i-1} for z_i. The integer i is the *index* of the unimodal path.

Induced and non-induced bipartite graphs. In this paper, we consider forbidden induced structures. However, for some particular graphs, prohibiting induced and non induced structures are essentially equivalent. For instance, if G does not contain a clique K_p as a subgraph, then G does not contain K_p as an induced subgraph. The same kind of results can be obtained for complete bipartite graphs.

The *Ramsey number* $R(k_1, k_2)$ is the maximum integer such that there exists a graph on $R(k_1, k_2)$ vertices with no clique of size k_1 nor stable set of size k_2. Ramsey numbers can be

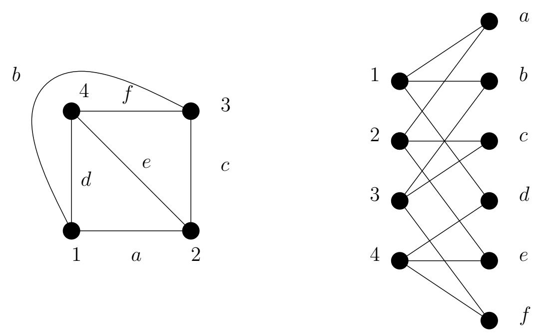

Figure 3: At the right, a 1-subdivision of K_4. Important vertices are vertices of the left hand side and subdivided vertices are vertices of the right hand side.

generalized as follows: $R(k_1, \dots, k_p)$ is the maximum integer such that there exists an edge p-coloring of the clique on $R(k_1, \dots, k_p)$ vertices with no monochromatic clique of size k_i and of color i for every $i \leq p$.[1] Let us first prove the following observation.

Observation 7. *Every $(K_\omega, K_{\ell,\ell})$-free graph has no $K_{R(\omega,\ell)+1,R(\omega,\ell)+1}$ as a subgraph.*

Proof. By contradiction. Let (A, B) be a complete bipartite graph where $|A| = |B| = R(\omega, \ell)+1$. By definition of $R(\omega, \ell)$, the set A contains either a clique on ω vertices or a stable set on ℓ vertices. The same holds for B. If A or B have a clique of size ω, the graph contains a clique of size $\omega + 1$ (this clique plus any vertex of the other part), a contradiction. If both A and B have stable sets of size ℓ, their union induces a $K_{\ell,\ell}$, a contradiction. □

In the following we will denote by L the integer $R(\omega, \ell) + 1$. We have to keep in mind that if G is $K_{\ell,\ell}$-free then G does not contain $K_{L,L}$ as a subgraph.[2]

Subdivisions. A 1-*subdivision* of a graph $H = (A, B)$ is a graph H_s with vertex set $A \cup B$ where there is an edge between $a \in A$ and $b \in B$ if a is an endpoint of the edge b in H (see Figure 3). Equivalently the graph H_s is the adjacency bipartite graph of H. Vertices of A are *important vertices* and they “represent” the vertices of H while vertices of B are *subdivided vertices* and they “represent” the edges of H.

The following statement is a direct corollary of a result of Kühn and Osthus [11].

Theorem 8 (Kühn, Osthus [11])**.** *Let ℓ, k be two integers. Every $K_{L,L}$-free graph with chromatic number at least $d(k, L)$ has an induced 1-subdivision of a k-connected graph.*

Proof. Kühn and Osthus proved in [11] that for every k, L, there exists a function $d'(4k, L)$ such that every graph of average degree degree at least $d'(4k, L)$ with no $K_{L,L}$ as a subgraph contains an induced 1-subdivision of a graph H of average degree at least $4k$.

Now, let G be a graph satisfying $\chi(G) \geq d'(4k, L)$. The deletion of the vertices of degree at most $d'(4k, L) - 1$ does not modify the chromatic number. So the graph G has an induced subgraph of average degree at least $d'(4k, L)$. By [11], G contains a 1-subdivision of a graph of average degree at least $4k$. Mader proved that every graph with average degree at least $4k$ has a k-connected subgraph (see [6] for a proof). So G contains a 1-subdivision of a k-connected graph. □

[1] When there are two colors, we usually consider that color 1 corresponds to an edge and color 2 to a non-edge.

[2] Note that, for triangle-free graphs, we have $R(2, \ell) = \ell - 1$. Thus for Theorem 2 and Corollary 3, we have $L = \ell$.

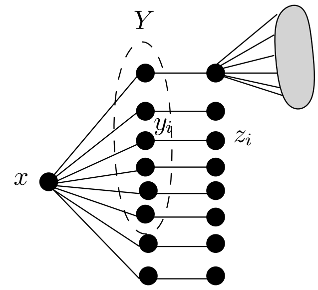

Figure 4: The vertex x. The set Y is a subset of vertices incident to x. The extended neighbor of y_i is the unique neighbor of y_i distinct from x in H_s. A collection of α-good unimodal path see at most an α fraction of the gray vertices.

3 Structure of the proofs of Theorems 2 and 4

In this section, we give the main steps of the proofs of Theorems 2 and 4. The proofs of technical lemmas are deferred to Section 4. Since both proofs follow the same scheme, we present simultaneously a proof of both results. We will mention the places where slight modifications have to be considered.

Let f, ext be two large enough functions[3]. Assume by contradiction that there exists a ($K_{L,L}$, k-wheel, triangle)-free graph G (resp. ($K_{L,L}$, wheel)-free graph of clique number ω) of chromatic number at least $2^{\mathrm{ext}(k,L)} \cdot d(4 \cdot f(k,L), L)$. Remind that L depends on ω.

Let $G_{\mathrm{ext}(k,L)}$ be an $\mathrm{ext}(k,L)$-extracted graph of G. Observation 6 ensures that the chromatic number of $G_{\mathrm{ext}(k,L)}$ is at least $\frac{\chi(G)}{2^{\mathrm{ext}(k,L)}}$ and then we have $\chi(G_{\mathrm{ext}(k,L)}) \geq d(4 \cdot f(k,L), L)$. Theorem 8 ensures that $G_{\mathrm{ext}(k,L)}$ contains as an induced subgraph a 1-subdivision H_s of a $f(k,L)$-connected graph H.

Let x be an important vertex of H_s and $Y = \{y_1, \ldots, y_{f(k,L)}\}$ be a subset of neighbors of x in H_s. The set Y exists since the degree of any important vertex of H_s is at least the connectivity of H. The *extended neighbor* z_i of y_i is the unique neighbor of y_i distinct from x in H_s (informally y_i is "the edge" xz_i) (see Figure 4).

Our goal consists in building a wheel of center x. Let us briefly explain how we will proceed. Since H has large connectivity, we can find vertex-disjoint paths $\mathcal{Q}$ between the extended neighbors of Y. Since H_s is a 1-subdivision of H, paths of $\mathcal{Q}$ are independent in H_s.[4] The paths of $\mathcal{Q}$ can be completed into independent paths between vertices of Y using edges between the vertices of Y and their extended neighbors. Since x is incident to all the vertices of Y, if we can transform this collection of paths into an induced cycle, we obtain a wheel. However, vertices of Y only have two neighbors in H_s; one of them is x (we want x to be the center of the wheel, so it cannot be on the cycle) and the other neighbor already is in a path of $\mathcal{Q}$. Thus we need to leave H_s in order to transform $\mathcal{Q}$ into a cycle. Actually, we will close this collection of paths into a cycle using unimodal paths. These unimodal paths have to be pairwise independent and "independent enough" with H_s to obtain an induced cycle. Actually, we need the following.

Let α be a positive constant. A collection of unimodal paths $\mathcal{Q}$ with endpoints in $Y' \subseteq Y$ is *α-good* if the following holds:

[3] We will not explicit functions f and ext. Though, in the statement of each lemma, we give the conditions on f and ext we need to prove it. Thus, if the reader is interested on these functions, he can compute them by combining these conditions. Note however that f and ext are exponential functions.

[4] By abuse of notation, we will consider that paths of H are paths of H_s since there is a natural function transforming paths of H into paths in H_s.

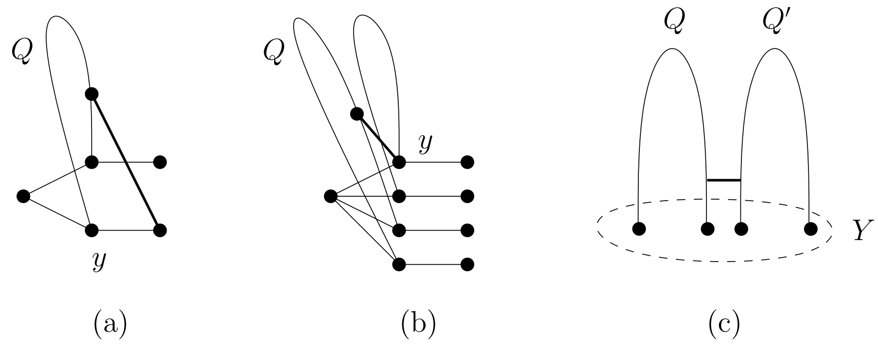

Figure 5: (a) The path Q does not satisfy (1). (b) The path Q does not satisfy (2). (c) Since both endpoints of the bold edge are fathers of endpoints of y, (3) is not satisfied.

(1) For every $y \in Y'$, the unimodal path $Q \in \mathcal{Q}$ with endpoint y is the unique path of $\mathcal{Q}$ incident to y.

(2) For every $y \in Y'$, no vertex of $Q \in \mathcal{Q}$ is incident to the extended neighbor of y (except y).

(3) For every $Q, Q' \in \mathcal{Q}$, any father of an endpoint of Q is not incident to any father of an endpoint of Q'.

(4) For every $y \in Y'$ and every path $Q \in \mathcal{Q}$, the set Q is incident to at most $\alpha \cdot |N(z, H_s)|$ vertices of $N(z, H_s)$ where z is the extended neighbor of y.

Points (1), (2) and (3) are illustrated in Figure 5. A collection $\mathcal{Q}$ of unimodal α-good paths is *independent* if for every $Q, Q' \in \mathcal{Q}$, the sets Q and Q' are independent. To avoid cumbersome notations, we will omit H_s in the definition of the neighborhoods when no confusion is possible. The most technical part of this paper consists in proving the following lemma.

Lemma 9. *If $f(k, L)$ and $\text{ext}(k, L)$ are large enough, then every (triangle, k-wheel)-graph* ***and*** *every wheel-free graph contains*

- *A complete bipartite graph $K_{L,L}$ as a subgraph.*
- *Or a collection of $\lceil \frac{k}{2} \rceil$ independent $\frac{1}{4k}$-good unimodal paths with endpoints in Y.*

Section 4 is devoted to prove Lemma 9. Actually, in Section 4.1, we show that we can extract a large collection of $\frac{1}{8k}$-good unimodal paths and in Section 4.2, we will show that we can transform it into a collection of k $\frac{1}{4k}$-good independent unimodal paths.

Two vertices u, v of H_s are *locally disjoint* if $N(u) \cap N(v)$ does not contain an important vertex. Informally it means that if u is an important vertex then v is not a subdivided vertex incident to u and if both u and v are subdivided vertices then the "edges" u and v of H do not share an endpoint. A vertex $u \in V(G) \setminus H_s$ has *k locally disjoint neighbors* if there are k pairwise locally disjoint vertices in $N(u) \cap H_s$. Let us now recall the Linkage Theorem.

Theorem 10 (Linkage [12])**.** *Let $k \geq 0$. Let G be a $10k$-connected graph. For every set of (non necessarily disjoint) vertices $x_1, \ldots, x_k, y_1, \ldots y_k$, there exist interior vertex-disjoint paths $P_1, \ldots, P_k$ such that for every $i \leq k$, P_i is a path from x_i to y_i.*

Lemma 11. *Assume that H is $10k$-connected. If a vertex $u \notin G \setminus H_s$ has k locally disjoint neighbors in H_s then G has a k-wheel.*

Proof. Assume that a vertex $u \notin H_s$ has k locally disjoint neighbors $u_1, \dots, u_k$. Let X and Y be two subsets of vertices of H defined as follows: if u_i is an important vertex, we set $x_i = u_i$ and $y_{i-1} = u_i$ (in order to simplify notations we denote in the same way vertices of H and important vertices of H_s). If u_i is a subdivided vertex then x_i and y_{i-1} are the two important vertices incident to u_i in H_s. Indices have to be understood modulo k.

Since H is $10k$-connected, Theorem 10 ensures that there exists a collection P_i of interior vertex-disjoint paths in H between x_i and y_i for every $i \leq k$. The concatenation of these paths can be transformed into a cycle of H (we just have to add the edges $y_{i-1}x_i$ if $y_{i-1} \neq x_i$). Any cycle of H can be transformed into an induced cycle of H_s (where each edge is subdivided once). So there exists an induced cycle of H_s passing through all the vertices $u_1, \dots, u_k$. The vertex u has at least k vertices on that cycle, which provides a k-wheel. $\square$

We now have all the ingredients to conclude.

Lemma 12. *Assume that H is $(4k^2 + 10k)$-connected and $k \geq 3$. Let x be an important vertex of H_s. If there are $\lceil \frac{k}{2} \rceil$ independent $\frac{1}{2k}$-good unimodal paths $\mathcal{Q}$ with endpoints in $N(x)$, the graph G has a k-wheel.*

Proof. Let $\mathcal{Q} = \{Q_1, \cdots, Q_{\lceil \frac{k}{2} \rceil}\}$ be a collection of independent $\frac{1}{2k}$-good unimodal paths with endpoints in $N(x)$. For every $i \leq \lceil \frac{k}{2} \rceil$, we denote by y_{2i-1} and y_{2i} the endpoints of Q_i and by f_{2i-1} and f_{2i} their respective fathers in Q_i.

Since Q_i is unimodal, if $u \in Q_i$ is incident to H_s then u is either f_{2i-1} or f_{2i}. Thus at most $2\lceil \frac{k}{2} \rceil \leq k+1$ vertices of $\mathcal{Q}$ have neighbors in H_s. Lemma 11 ensures that if there exists i such that f_i has at least k locally disjoint neighbors in H_s, then G contains a k-wheel. So we can assume that, for every i, the vertex f_i has less than k locally disjoint neighbors in H_s. Let F_i be a set of locally disjoint neighbors of f_i of maximum size. Let W_i defined as follows: if $z \in F_i$ is an important vertex, add z in W_i. If z is a subdivided vertex, add both neighbors of z in W_i. Let $W = \bigcup_{i=1}^{2\lceil \frac{k}{2} \rceil} W_i$. Since $|W_i| < 2k$, we have $|W| < 2k^2 + 2k$. Let H^1 be the subgraph of H where the vertices of W have been deleted (since W only contain important vertices, it corresponds to vertices of H). We denote by H_s^1 the subgraph of H_s induced by the vertices and the edges of H^1.

Claim 1. *No vertex of Q_i is incident to H_s^1.*

Proof. Assume by contradiction that f_i is incident to $u \in H_s^1$. Let $u_i \in F_i$. Let us prove that u is not incident to u_i. Indeed, if u_i is a subdivided vertex, both neighbors of u_i have been deleted in H^1 and then in H_s^1. If u_i is an important vertex, then u_i has been deleted from H, and then also are the edges incident to u_i, *i.e.* the neighbors of u_i in H_s. Thus $N(u) \cap N(u_i)$ is empty for every i, and then u can be added in F_i, contradicting the maximality of F_i. $\square$

For every i, let z_i be the extended neighbor of y_i. Since $\mathcal{Q}$ is $\frac{1}{2k}$-good, at most $\frac{|N(z_i)|}{2k}$ vertices of $N(z_i)$ are in the neighborhood of Q_j for every $j \leq \lceil \frac{k}{2} \rceil$. Let u be a vertex of $N_2(z_i)$. The vertex u is *hit* if there exists j such that the subdivided vertex between z and u is in $N(Q_j)$. Otherwise u is *safe*. The number of vertices of $N_2(z_i)$ hit by Q_j is at most $\frac{|N(z_i)|}{2k}$ since $\mathcal{Q}$ is $\frac{1}{2k}$-good. Since $\mathcal{Q}$ contains less than k paths, at least a half of the vertices of $N_2(z_i)$ are safe. Since H is $4k^2 + 10k$ connected, there are at least $2k^2 + 5k$ safe vertices in $N_2(z_i)$ denoted by Z_i.

Delete from H^1 the vertices $Y \cup \{x\}$ and denote by H' the resulting graph. Since at most $(2k^2 + 3k + 3)$ vertices of H have been deleted, at least $2k - 3$ vertices of Z_i are in H'. For every i, let s_i be a vertex of $Z_i \cap H'$ such that $s_i \neq s_j$ for every $i \neq j$ (since $i \leq k+2$, such a collection of vertices exists).

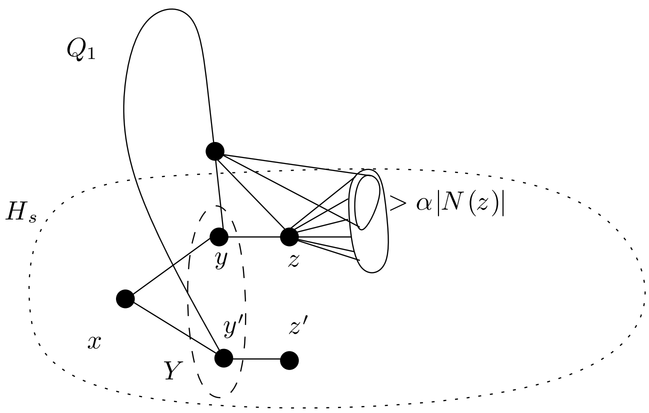

Figure 6: The path Q_1 is not self α-good for two reasons. First a vertex of Q_1 is incident to z (the bold edge). Secondly, the vertices of Q_1 are incident to more than $\alpha|N(z)|$ neighbors of z.

Since at most $(2k^2 + 3k + 3)$ vertices of H have been deleted in H', the graph H' is at least $(2k^2 + 7k - 3)$-connected. By Theorem 10, there is a collection of interior vertex-disjoint paths P_i between s_{2i} and s_{2i+1} (modulo $2\lceil \frac{k}{2} \rceil$) for every $i \leq \lceil \frac{k}{2} \rceil$. The path P_i can be completed into a $y_{2i}y_{2i+1}$-path R_i. Indeed, for every $j \leq 2\lceil \frac{k}{2} \rceil$, we just add the vertices y_j, z_j and the subdivided vertex incident to both z_j and s_j to obtain the desired paths.

Note that $C = Q_1R_1Q_2 \ldots R_{\lceil \frac{k}{2} \rceil}$ is a cycle. Let us prove that this cycle is induced. First note that, by assumption, the paths Q_i are independent since $\mathcal{Q}$ is a collection of independent paths. Since H_s is a 1-subdivision of H, the paths P_i are independent. Since $s_i \neq s_j$ for $i \neq j$, the paths R_i are also vertex-disjoint and independent. So we just have to show that there is no edge between Q_i and R_j. Since $\mathcal{Q}$ is $\frac{1}{2k}$-good, no vertex of Q_i is incident to y_j for any j by (1) (except the path with endpoint y_j). Moreover, no Q_i is incident to z_j for any j by (2). Since s_j is safe, no vertex of Q_i is incident to the subdivided vertex incident to z_j and s_j. Since P_j is a path included in H_s^1, Claim 1 ensures that no vertex of Q_i is incident to P_j. Thus C is an induced cycle and the vertex x has at least k neighbors on C, *i.e.*, the graph contains a k-wheel. □

4 Extracting good unimodal paths

This section is devoted to prove Lemma 9. Before proving it, let us first recall a well-known result of extremal graph theory.

Theorem 13 (Kővári, Sós, Turán [10]). *Let ℓ be an integer.*

- [Bipartite version] *There exists a function T_1 such that for every ℓ, ϵ, every $K_{\ell,\ell}$-free bipartite graph $((A, B), E)$ where both A, B have size at least $T_1(\ell, \epsilon)$ has less than $\epsilon \cdot |A| \cdot |B|$ edges.*

- [Graph version] *There exists a function T_2 such that for every ℓ, ϵ and every graph with $|V| = n \geq T_2(\ell, \epsilon)$ with no $K_{\ell,\ell}$ as a subgraph has less than $\epsilon \cdot \binom{n}{2}$ edges.*

4.1 Good unimodal paths

Let $f(k, \ell)$ and $\text{ext}(k, \ell)$ be two functions detailed later. All along this section we assume that we are in the following setting:

Let G be a graph. In an $\mathrm{ext}(k, L)$-extracted subgraph of G, there exists a subgraph H_s such that H_s is an induced 1-subdivision of a $f(k, L)$-connected graph H. Let x be an important vertex of H_s and Y be a subset of $N(x) \cap H_s$ of size $f(k, L)$.

A path Q is *self α-good* if $\mathcal{Q} = \{Q\}$ satisfies points (1), (2) and (4) of the definition of α-good paths. A collection of paths $\mathcal{Q}$ is *self α-good* if every path of $\mathcal{Q}$ is self α-good. Note that any collection of α-good paths is self α-good.

Lemma 14. *Let $i \leq \mathrm{ext}(k, L)$. Let $f(k, L) \geq 20$. If G is a wheel-free graph, at most one vertex y of Y has a father y^i incident to its extended neighbor z.*

Proof. Let y be a vertex of Y and z be the extended neighbor of y. Assume that a ith father y^i of y is incident to both y and z. First assume that y^i has another neighbor w in H_s. If w is a subdivided vertex, we denote by w_1 and w_2 the two neighbors of w. Otherwise we set $w_1 = w$ and $w_2 = w$. Since H is at least 20-connected, Theorem 10 ensures that there are two vertex-disjoint paths P_1, P_2 in H such that P_1 is an xw_1-path and P_2 is a w_2z-path. Thus P_1wP_2 provides an induced xz-path in H_s. The addition of y to this path provides an induced cycle in H_s. Since y^i is incident to at least three vertices on it, the graph contains a wheel. So we can assume that $N(y^i) \cap H_s = \{y, z\}$.

Assume now that $y_2 \in Y$ with $y_2 \neq y$ has a father y_2^i incident to its extended neighbor z_2. The previous paragraph ensures that $N(y_2^i) \cap H_s = \{y_2, z_2\}$. Let Q be a unimodal path path between z and y_2 passing through y^i and y_2^i. Remind that, by definition of unimodal paths, y^i and y_2^i are the only vertices of Q incident to H_s. Let P be a path in $H_s \setminus \{z_2, y\}$ between x and z (it exists since $H \setminus \{z_2, y\}$ is 18-connected). Since $N(y^i) \cap H_s = \{y, z\}$ and $N(y_2^i) \cap H_s = \{y_2, z_2\}$, the concatenation of P and Q (plus the edge y_2x) provides an induced cycle. Moreover, the vertex y is incident to 3 vertices on it, *i.e.*, G contains a wheel, a contradiction. □

Note that, for triangle-free graphs, the vertex y^i cannot be incident to both y and z and then Lemma 14 can be avoided. Lemma 14 is the unique lemma where we specifically need to forbid (3)-wheels and not k-wheels. In the remaining steps, we can prove the existence of a $K_{L,L}$ as a subgraph or of an induced k-wheel except in the following lemma where we need to apply Lemma 14.

Lemma 15. *Let G be a (triangle, $K_{\ell,\ell}$)-free or a (wheel, $K_{\ell,\ell}$)-free graph and p an integer. Let*

$$f(k, L) \geq 2\Big(R\Big(\frac{1}{3} \cdot T_2(L, \frac{1}{10}), 2 \cdot T_1(L, \alpha)\Big) + 2p + 2\Big) \qquad \text{and} \qquad \mathrm{ext}(k, L) \geq p + 2 \cdot T_1(L, \alpha)$$

then there exists a collection of p endpoint-disjoint self 2α-good unimodal paths of distinct indices with endpoints in Y.

Proof. Let $i \leq \mathrm{ext}(k, L)$. Let y^i be a ith father of $y \in Y$ and z be the extended neighbor of y. The vertex y is a *candidate for i* if y^i is incident to at most $\alpha \cdot |N(z)|$ vertices of $N(z)$ (for simplicity we omit $\cap H_s$).

Claim 2. *For every $y \in Y$, there are at most $T_1(L, \alpha)$ integers i for which y is not a candidate.*

Proof. Let $y \in Y$. Assume by contradiction that there are $T_1(L, \alpha) + 1$ integers i for which y is not a candidate. Let $F = \{y^i \mid y \text{ is not a candidate for } i\}$ and let Z_y be the set of neighbors of the extended neighbor of y. For every i such that y is not a candidate for i, the vertex y^i has at least $\alpha|Z_y|$ neighbors in Z_y. Thus the bipartite graph on vertex set (Z_y, F) has at least $\alpha \cdot |F| \cdot |Z_y|$ edges. Since $|Z_y| \geq f(k, L) > T_1(L, \alpha)$ and $|F| > T_1(L, \alpha)$, Theorem 13 ensures that G has a $K_{L,L}$, a contradiction. □

By Claim 2, the number of indices i such that at least half of the vertices of Y are not candidate for i is at most $2 \cdot T_1(L, \alpha)$. Since $\text{ext}(k, L) \geq 2 \cdot T_1(L, \alpha) + p$, it means that there are at least p integers I such that, for every $i \in I$, at least half of the vertices of Y are candidates for i. Let $i \in I$. Let Y_i be the subset of Y such that $y \in Y_i$ if y is a candidate for i. We have

$$|Y_i| \geq \frac{f(k, L)}{2} \geq R\Big(\frac{1}{3} \cdot T_2(L, \frac{1}{10}), 2 \cdot T_1(L, \alpha)\Big) + 2p + 2.$$

Claim 3. *Let $i \in I$. Let Y' be a subset of Y_i where at most $2p$ vertices have been deleted. There exists a self 2α-good unimodal path of index i with endpoints in Y'.*

Proof. If G is a triangle-free graph then no vertex if incident to both y and its extended neighbor. If G is a wheel-free graph then Lemma 14 ensures that there exists at most one vertex $y \in Y'$ such that the ith father y^i of y is incident to the extended neighbor of y. So, for both assumptions of Lemma 15, free to delete at most one vertex of Y', we can assume that for every $y \in Y'$, the vertex y^i is not incident to the extended neighbor of y. Note that

$$|Y'| \geq R\Big(\frac{1}{3} \cdot T_2(L, \frac{1}{10}), 2 \cdot T_1(L, \alpha)\Big) + 1.$$

For every $y \in Y'$, we denote by z the extended neighbor of y. We create an auxiliary graph H on vertex set Y'. We color the edges of H with 3 colors. Let $a, b \in Y'$:

- If y_a^i is incident to $\{y_b, z_b\}$ or y_b^i is incident to $\{y_a, z_a\}$ then we color ab with color 1.
- If y_a^i is incident to at least $\alpha|N(z_b)|$ vertices of $N(z_b)$ or y_b^i is incident to at least $\alpha|N(z_a)|$ vertices of $N(z_a)$, we color ab with color 2.
- Otherwise, we color ab with color 3.

(remind that two sets are incident if there is at least one edge between them). Our goal consists in showing that there is an edge of color 3. Since $|Y'| > R(\frac{1}{3} \cdot T_2(L, \frac{1}{10}), 2 \cdot T_1(L, \alpha))$, Ramsey's theorem ensures that there is an edge of color 3 or there is a clique of color 1 of size $\frac{1}{3} \cdot T_2(L, \frac{1}{10}) + 1$ or there is a clique of color 2 of size $T_1(L, \alpha) + 1$.

Assume first that H has a monochromatic clique K of color 1 of size $\frac{1}{3} \cdot T_2(L, \frac{1}{10}) + 1$. Consider the restriction of G to the vertices $\cup_{b \in K}\{y_b, z_b, y_b^i\}$. By construction, for every $a, b \in K$, there is an edge between $\{y_a, z_a, y_a^i\}$ and $\{y_b, z_b, y_b^i\}$. Thus this graph contains $3|K|$ vertices and at least $\binom{|K|}{2}$ edges (plus $2|K|$ edges since $\{y_b, z_b, y_b^i\}$ induces 2 edges for every $b \in K$). Thus there are at least $\frac{1}{9} \cdot \binom{3|K|}{2}$ edges. Since $3|K| > T_2(L, \frac{1}{9})$, the graph contains a $K_{L,L}$ as a subgraph by Theorem 13, a contradiction.

Assume now that H has a monochromatic clique K of color 2 of size $2 \cdot T_1(L, \alpha) + 1$. For every $a, b \in K$, either y_b^i is incident to at least $\alpha|N(z_a)|$ vertices of $N(z_a)$ or y_a^i is incident to at least $\alpha|N(z_b)|$ vertices of $N(z_b)$. Thus there exist $a \in K$ and $K' \subseteq K$ of size at least $\frac{|K|-1}{2}$ such that for every $b \in K'$, the vertex y_b^i is incident to at least $\alpha|N(z_a)|$ vertices of $N(z_a)$. Let $F = \{y_b^i \text{ with } b \in K'\}$ and $Z_a = N(z_a)$. The bipartite graph induced by $F \cup Z_a$ has at least $\alpha|F| \cdot |N(z_a)|$ edges. Moreover, $|K'| \geq T_1(L, \alpha)$ and $|Z_a| \geq f(k, L) > T_1(L, \alpha)$. Thus Theorem 13 ensures that G contains a $K_{L,L}$ as a subgraph, a contradiction.

So there is an edge ab of color 3. Let Q_i be an i-unimodal path between y_a and y_b passing though y_a^i and y_b^i. Remind that only y_a^i and y_b^i can be incident to vertices in H_s. Since the edge has color 3, y_a^i is not incident to $\{y_b, z_b\}$. Moreover y_b^i is not incident to z_b (at most one vertex of Y_i can satisfy this by Lemma 14 and it has been deleted from Y'). Moreover the vertex y_a^i is incident to at most $\alpha \cdot |N(z_b)|$ vertices of $N(z_b)$ since the edge has color 3. Since y_b is a candidate for i, y_b^i is incident to at most $\alpha \cdot |N(z_b)|$ vertices of $N(z_b)$. Thus there are at most $2\alpha \cdot |N(z_b)|$ edges from Q_i to $N(z_b)$. By symmetry, the same holds for z_a. So Q_i is self 2α-good. $\square$

To conclude, we apply iteratively Claim 3. Initially we set $Y_i' = Y_i$ for every $i \in I$. When we find a self 2α-good unimodal path Q_i of index i, we delete the endpoints of Q_i from Y_j' for every $j \neq i$. We repeat this operation p times for p distinct values of $i \in I$. At any step, the size of Y_i' is at least $|Y_i| - 2p$ and then Claim 3 ensures that there is a self 2α-good path of index i, which concludes the proof. $\square$

Let us now show that we can extract from any collection of self α-good paths a collection α-good paths. The proof is similar to the one of Lemma 15 but is slightly simpler.

Lemma 16. *Assume that $f(k, L) > T_1(L, \frac{\alpha}{2})$. Let G be a $K_{\ell,\ell}$-free graph. Any collection $\mathcal{Q}$ of $h(p) := R\Big(\frac{1}{6} \cdot T_2(L, \frac{1}{36}), 2 \cdot T_1(L, \frac{\alpha}{2}), p-1\Big) + 1$ self α-good-unimodal paths has a sub-collection of p α-good increasing unimodal paths.*

Proof. Let us denote by $Q_1, \ldots, Q_{h(p)}$ the paths of $\mathcal{Q}$. For every $i \leq h(p)$, we denote by $y_{i,1}$ and $y_{i,2}$ the endpoints of Q_i, by $y_{i,1}^i$ and $y_{i,2}^i$ their respective fathers in Q_i and by $z_{i,1}$ and $z_{i,2}$ their respective extended neighbors. We denote by F_i the set $\{y_{i,1}, y_{i,2}, y_{i,1}^i, y_{i,2}^i, z_{i,1}, z_{i,2}\}$. We create an auxiliary graph H on vertex set $\{1, \ldots, h(p)\}$. We color the edges of H with 3 colors. For every a, b:

- If there is an edge between the sets F_a and F_b, then we color ab with color 1.
- If Q_a is incident to at least $\alpha|N(z_{b,j})|$ vertices of $N(z_{b,j})$ for some $j \in \{1, 2\}$ or Q_b is incident to at least $\alpha|N(z_{a,j})|$ vertices of $N(z_{a,j})$ for some $j \in \{1, 2\}$, then we color ab with color 2.
- Otherwise, we color ab with color 3.

Since $h(p) > R(\frac{1}{6} \cdot T_2(L, \frac{1}{36}), 2 \cdot T_1(L, \frac{\alpha}{2}), p-1)$, Ramsey's theorem ensures that H contains a clique of color 1 of size $\frac{1}{6} \cdot T_2(L, \frac{1}{36}) + 1$ or a clique of color 2 of size $2 \cdot T_1(L, \frac{\alpha}{2}) + 1$ or a clique of color 3 of size p.

Assume that H has a monochromatic clique K of color 1 of size $\frac{1}{6} \cdot T_2(L, \frac{1}{36}) + 1$. The proof holds as in Lemma 15. Let G' be the subgraph of G induced by $\cup_{a \in K} F_a$. The graph G' has $6|K| > T_2(L, \frac{1}{36})$ vertices and at least $\binom{|K|}{2} + 4|K|$ edges. Thus the density of edges is at least $\frac{1}{36}$, and then the graph contains a $K_{L,L}$ as a subgraph by Theorem 13, a contradiction.

Assume now that H has a monochromatic clique K of color 2 of size $2 \cdot T_1(L, \frac{\alpha}{2}) + 1$. Let $j \in \{1, 2\}$ and $a, b \in K$. The vertex $b \in H$ is (a, j)-*dense* if Q_b is incident to at least $\alpha|N(z_{a,j})|$ vertices of $N(z_{a,j})$. The vertex b is a-*dense* if it is $(a, 1)$-dense or $(a, 2)$-dense. For every pair $a, b \in K$, a is b-dense or b is a-dense since ab is colored with 2. Thus there exist $a \in K$ and $K' \subseteq K$ of size at least $\frac{|K|-1}{2} \geq T_1(L, \frac{\alpha}{2})$ such that for every $b \in K$, b is a-dense. So there are an integer $j \in \{1, 2\}$ and a subset K^* of size $|K'|/2$ such that, for every $b \in K^*$, b is (a, j)-dense. Let $F = \{y_{b,d}^b | b \in K^*, d \in \{1, 2\}\}$. Note that $|F| = 2|K^*| \geq T_1(L, \frac{\alpha}{2})$. Moreover $N(z_{a,j})$ has size at least $f(k, L) > T_1(L, \frac{\alpha}{2})$. Consider the bipartite graph with vertex set $F \cup N(z_{a,j})$. It induces at least $\alpha \cdot |K^*| \cdot |N(z_{a,j})| = \alpha/2 \cdot |F| \cdot |N(z_{a,j})|$ edges. Thus G contains a $K_{L,L}$ as a subgraph by Theorem 13, a contradiction.

Thus there is a clique K of color 3 of size p. Since $\mathcal{Q}$ is self α-good, (1), (2) and (4) are satisfied for the paths of K themselves. Since there is no edge of color 1 in K, (1), (2) and (3) are satisfied. And since there is no edge of color 2 in K, point (4) is satisfied. Thus the collection $\{Q_i \mid i \in K\}$ is a collection of α-good paths, which concludes the proof. $\square$

4.2 Independent unimodal paths

We set $p := 8k^3(32k+2)$.

Lemma 17. *Let G be a ($K_{\ell,\ell}$, k-wheel)-free graph. If there is a collection $\mathcal{Q}$ of size at least $R\Big(2 \cdot T_1(L, \frac{1}{2p}), k-1\Big) + 1$ of α-good unimodal paths of pairwise distinct indices with endpoints in Y, there are k independent 2α-good unimodal paths with endpoints in Y.*

Proof. Construct an auxiliary graph H on $|\mathcal{Q}|$ vertices where ij is an edge if the paths Q_i and Q_j are incident. Since $|\mathcal{Q}| > R(2 \cdot T_1(L, \frac{1}{2p}), k-1)$, Ramsey's theorem ensures that H contains a clique of size $2 \cdot T_1(L, \frac{1}{2p}) + 1$ or a stable set of size k. If there is a stable set S of size k, then the set $\{Q_i \mid i \in S\}$ provides a collection of k independent α-good unimodal paths and then the conclusion holds. Thus we may assume that H contains a clique K of size $2 \cdot T_1(L, \frac{1}{2p}) + 1$, *i.e.*, there exist $2 \cdot T_1(L, \frac{1}{2p}) + 1$ paths of $\mathcal{Q}$ which are pairwise incident. We denote them $Q_1, \ldots, Q_{2 \cdot T_1(L, \frac{1}{2p})+1}$. Without loss of generality, we may assume that Q_i is a unimodal path with index i.

By definition of unimodal paths of index i, the path Q_i is contained in the $(i-1)$th extracted graph. Moreover only endpoints of Q_i and their fathers may be incident to vertices contained in the ith extracted graph. Thus, for every $i < j$, any edge between Q_i and Q_j intersects the endpoints of Q_i or their fathers. Since $\mathcal{Q}$ is α-good, there is no edge between an endpoint of Q_i and a vertex of Q_j by (1). Thus every edge between Q_i and Q_j intersects the father of an endpoint of Q_i.

We set $\mathcal{Q}_1 = \{Q_1, \ldots, Q_{T_1(L, \frac{1}{2p})}\}$ and $\mathcal{Q}_2 = \{Q_{T_1(L, \frac{1}{2p})+1}, \ldots, Q_{2 \cdot T_1(L, \frac{1}{2p})+1}\}$. The *exterior vertices* of $Q_j \in \mathcal{Q}_2$ are the vertices v of Q_j incident to a vertex u of $Q' \in \mathcal{Q}_1$. We denote by F_j the set of exterior vertices of $Q_j \in \mathcal{Q}_2$.

Claim 4. *At most $T_1(L, \frac{1}{2p})$ paths of $\mathcal{Q}_2$ have at most p exterior vertices.*

Proof. Let B the union of the F_j for every $j \in \mathcal{Q}_2$ satisfying $|F_j| \leq p$. Assume by contradiction the size of B is at least $T_1(L, \frac{1}{2p})$ (note that the size of B is at least the number of paths of $\mathcal{Q}_2$ satisfying $|F_j| \leq p$). Let $A = \{f \ : \ f$ is a father of an endpoint of $Q \in \mathcal{Q}_1\}$. We have $|A| = 2|\mathcal{Q}_1| > T_1(L, \frac{1}{2p})$. Remind that, as we already noticed, every edge from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ intersects a vertex of A. Since there is an edge between every pair of paths, the number of edges of the bipartite graph on vertex set $A \cup B$ is at least $\frac{1}{2p}|A| \cdot |B|$. Theorem 13 ensures that G contains a $K_{L,L}$ as subgyraph, a contradiction. $\square$

Since $\mathcal{Q}_2$ contains $T_1(L, \frac{1}{2p}) + 1$ paths, Claim 4 ensures that there exists $Q \in \mathcal{Q}_2$ such that Q has more than p exterior vertices. We set an arbitrary order on Q (from an endpoint to the other). Let $X = \{x_1, \ldots, x_p\}$ be the first p exterior vertices of Q in increasing order. Note that x_1 and x_p are not endpoints of Q. Indeed, by point (1), no vertex of $Q_1 \in \mathcal{Q}_1$ is incident to an endpoint y of Q, *i.e.* y is not an exterior vertex. Moreover x_1 and x_p are not fathers of endpoints of Q. Indeed, by point (3), no vertex of $Q_1 \in \mathcal{Q}_1$ is incident to the father of an endpoint of Q.

Every vertex $v \notin (Q \cup \{x\})$ has less than k neighbors in Q. Indeed, the path Q plus the vertex x is an induced cycle (if a father of an endpoint of Q is incident to x, we shortcut the cycle in order to obtain an induced cycle). Since G does not contain a k-wheel, v is incident to at most $k-1$ vertices of Q. Let $Q_1 \in \mathcal{Q}_1$. Since at most 2 vertices of Q_1 may be incident to Q (the fathers of the endpoints of Q_1), there are at most $2k-2$ edges from Q_1 to X.

We create the following auxiliary graph H'. The vertex set of H' is $A \cup B$ where $A = |\mathcal{Q}_1|$ and $|B| = \frac{p}{2k} \geq 4k^2(32k+2)$. We put an edge between a_i and b_j if the path $Q_i \in \mathcal{Q}_1$ is incident

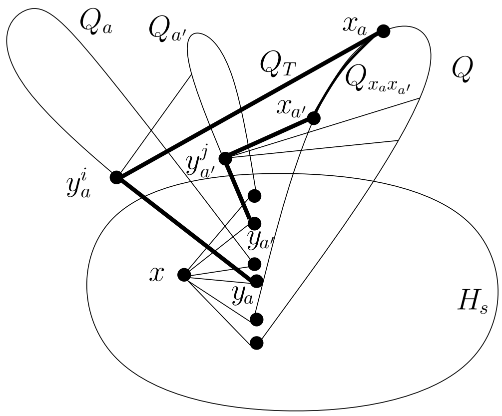

Figure 7: Let $T = (a, b, a')$ be a triplet of $\mathcal{T}$. The bold path is the path Q_T.

to $X_j = \{x_{2kj+1}, \dots, x_{2k(j+1)-1}\}$. Since every path $Q_i \in \mathcal{Q}_1$ is incident to at most $2k-2$ vertices of X, every vertex of A has degree at most $2k - 2$ in B. Moreover, since X_j has size $2k - 1$ and every vertex of X is incident to some $Q_i \in \mathcal{Q}_1$, every vertex of B is incident to at least 2 vertices of A. Indeed, all the vertices of X_j cannot be incident to the same Q_i since Q_i is incident to at most $2k - 2$ vertices of X. Let us prove the following claim.

Claim 5. *There are k pairwise disjoint triplets $(a_1, b, a_2) \in A \times B \times A$ of H' which are independent in H'.*

Proof. Remind that $|B| \geq 4k^2(32k + 2)$. Let $\mathcal{T}_0$ be a collection of triplets (not necessarily independent) initialized to the empty set. Let b be a vertex of B. Since b has degree at least 2, let a_1, a_2 be two neighbors of b. We add (a_1, b, a_2) in $\mathcal{T}_0$. Then, we delete a_1 and a_2 from A and $N(a_1) \cup N(a_2)$ from B. Since the degree of both a_1 and a_2 is at most $2k$, at most $4k$ vertices of B have been deleted. We repeat this procedure until B is empty. Since at most $4k$ vertices are deleted from B at each step, the final size of $\mathcal{T}_0$ is at least $32k^2 + 2k$.

By construction, the triplets of $\mathcal{T}_0$ are pairwise disjoint. Let us show that $\mathcal{T}_0$ admits k independent triplets. We construct a directed graph D on vertex set $\mathcal{T}_0$ where there is an arc from T_1 to T_2 if a vertex of $T_1 \cap A$ is incident to the vertex of $T_2 \cap B$. Every vertex of $T \cap A$ is incident to at most $2k$ vertices of B. Thus each vertex of $T \cap A$ "creates" at most $2k$ out-arcs. Since $|T \cap A| = 2$, the out-degree of T is at most $4k$. Thus D has at most $4k|\mathcal{T}_0|$ arcs.

We now need the following remark: every graph on n vertices with βn edges has at least $n/2$ vertices of degree at most 4β. Indeed, assume by contradiction that more than $n/2$ vertices have degree at least 4β. Then the sum of the degrees of the vertices of the graph is more than $(4\beta) \cdot \frac{n}{2} = 2\beta n$. Thus the graph has more than βn edges, a contradiction.

Applying this remark to D ensures that at least $\frac{|\mathcal{T}_0|}{2} \geq 16k^2 + k$ vertices of $\mathcal{T}_0$ have degree at most $16k$ in D. Thus D has a stable set of size at least $\frac{16k^2+k}{16k+1} \geq k$ (we just peel the graph by selecting any vertex of degree at most $16k$ and deleting its neighborhood). This stable set gives the desired collection of triplets. $\square$

Let $\mathcal{T}$ be a collection of triplets of the bipartite graph H' satisfying the conclusion of Claim 5. Let $T = (a, b, a')$ be a triplet of $\mathcal{T}$. Let us define the path Q_T as follows (see Figure 7 for an illustration). By definition of H', there exist vertices y^i_a and $y^j_{a'}$ which are respectively fathers of endpoints of Q_a and $Q_{a'}$, denoted by y_a and $y_{a'}$, such that both y^i_a and $y^j_{a'}$ are incident to at least one vertex in $\{x_{2kb+1}, \dots, x_{2k(b+1)-1}\}$. Let x_a be a neighbor of y^i_a in $\{x_{2kb+1}, \dots, x_{2k(b+1)-1}\}$. And let $x_{a'}$ be a neighbor of $y^j_{a'}$ in $\{x_{2kb+1}, \dots, x_{2k(b+1)-1}\}$. Assume moreover that none of them is incident to a vertex of the $x_a x_{a'}$-subpath of Q (*i.e.*, x_a and x'_a are chosen as close as possible). Note that we may have $x_a = x_{a'}$. We denote by $Q_{x_a x_{a'}}$ the $x_a x_{a'}$-subpath of Q. Let Q_T be the following path between y_a and $y_{a'}$: $y_a y_a^i x_a Q_{x_a x_{a'}} x_{a'} y_{a'}^j y_{a'}$.

Note that any endpoint y of Q_T is an endpoint of a path $Q_i \in \mathcal{Q}_1$. Moreover the father of y in Q_T is the father of y in Q_i. Note moreover that if y, y' are endpoints of respectively Q_T and $Q_{T'}$ with $y \neq y'$, then y and y' are in distinct paths of $\mathcal{Q}_1$. Let us now prove that the collection $\mathcal{P}$ of paths Q_T for every triplet $T \in \mathcal{T}$ is a collection of independent 2α-good unimodal paths.

Unimodal paths. For every triplet $T = (a, b, a')$, the path Q_T is unimodal. To prove it, we just have to show that $Q_{x_a x_{a'}}$ is not incident to H_s. As we already observed, x_1 and x_p (and then all the vertices of X) are neither endpoints of Q or fathers of endpoints of Q. Thus, $Q_{x_a x_{a'}}$ is not incident to any vertex of H_s since Q is a unimodal path. Thus Q_T is unimodal.

Points (1), (2) and (3). Let $Q_{T_1} \in \mathcal{P}$. Let y be an endpoint of Q_{T_1} and y^i be its father in Q_{T_1} and z be the extended neighbor of y. Now let f be the father of an endpoint of Q_{T_2} with $T_2 \in \mathcal{P}$ with $f \neq y^i$. Let us prove that f is not incident to $\{y, z, y^i\}$ which will prove (1), (2) and (3). There exists $Q_1, Q_2 \in \mathcal{Q}_1$ with $Q_1 \neq Q_2$ such that f is a father of an endpoint of Q_1 and y is an endpoint of Q_2. Remind moreover that y^i is the father of y in Q_2. By (1) for $\mathcal{Q}$, fy is not an edge. By (2) for $\mathcal{Q}$, fz is not an edge and by (3) for $\mathcal{Q}$, fy^i is not an edge. Since it holds for any endpoint y and and father f of an endpoint of $Q_T \in \mathcal{Q}$, all of (1), (2) and (3) hold.

Point (4). Let $Q_T \in \mathcal{P}$. Let y be an endpoint of a path of $\mathcal{P}$ and z be its extended neighbor. As we already observed, the path Q_T is unimodal, thus at most two vertices of Q_T can be incident to $N(z)$, the two fathers f_1, f_2 of the endpoints of Q_T. Moreover, f_1, f_2 are fathers of endpoints of respectively $Q_1, Q_2 \in \mathcal{Q}$ and z is an extended neighbor of an endpoint y of $Q_3 \in \mathcal{Q}$. Thus, by (4) for $\mathcal{Q}$, f_1 is incident to at most $\alpha|N(z)|$ vertices of $N(z)$ and the same holds for f_2. Thus the path Q_T is incident to at most $2\alpha|N(z)|$ vertices of $N(z)$, which proves (4).

Independence of paths of $\mathcal{P}$. Let us finally show that for every $T_1, T_2 \in \mathcal{T}$, the paths Q_{T_1} and Q_{T_2} are independent. Let $T = (a_1, b_1, a'_1)$ and $T_2 = (a_2, b_2, a'_2)$. Remind that $b_1 \neq b_2$ by Claim 5. Without loss of generality we have $b_1 < b_2$. Let us denote by Q_1 and Q_2 the subpath of Q included in respectively Q_T and $Q_{T'}$.

Let us first show that there is no edge between Q_1 and Q_2. Since $b_1 \neq b_2$, Q_1 and Q_2 do not intersect. Since Q_1 is a $x_{2kb_1+1} x_{2k(b_1+1)-1}$-subpath of Q and Q_2 is a $x_{2kb_2+1} x_{2k(b_2+1)-1}$-subpath of Q (and since Q is induced), there is no edge between Q_1 and Q_2. Indeed, the vertex $x_{2k(b_1+1)}$ is between Q_1 and Q_2 in the path Q, so these paths are not adjacent.

Since (1) and (3) hold, an edge between Q_{T_1} and Q_{T_2} must have one endpoint in Q_1 or in Q_2 (or both). By symmetry, we may assume that an edge has an endpoint in Q_1. Since T_1 and T_2 are not adjacent in H', there is no edge between between the fathers of the endpoints of T_2 and the set $\{x_{2kb_1+1}, \ldots, x_{2k(b_1+1)-1}\}$. By definition of X, it means that the fathers of the endpoints of T_2 have no neighbors on Q_1. Thus the paths Q_{T_1} and Q_{T_2} are independent.

□

The combination of Lemmas 15, 16 and 17 ensures that if $f(k, L)$ and $\mathrm{ext}(k, L)$ are large enough, then there exists a collection of k $\frac{1}{4k}$-good independent unimodal paths with endpoints in $N(x)$. This precisely provides the conclusion of Lemma 9.

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