Example-Based Video Color Grading
Supplemental material – Curvature estimation

Nicolas Bonneel[1]* Kalyan Sunkavalli[2] Sylvain Paris[2] Hanspeter Pfister[1]
[1]Harvard University [2]Adobe

This supplementatl material derives the exact form of the color tranformation curvature being used in Sec.5 of the paper "Example-based Video Color Grading".

The single-frame color matching technique described in Sec. 4 of the paper estimates a series of color transfer functions, $\mathbf{T}_t, t = 1, 2, \ldots, n$, one for each input video frame. In turn, each transfer function consists of a non-linear transfer function for luminance, $\mathbf{T}_t^l$ and three affine matrices, $\mathbf{T}_t^{c,i}, i = \{1, 2, 3\}$, for chrominance. These color transforms are designed to transfer the color style of the model video to the input video clip. However, as discussed before (see Sec. 3 and Fig. 2 of the paper), directly applying these color transforms to the input video frames leads to temporal artifacts such as flickering and color shifts. The second stage of our pipeline is a novel differential-geometry based temporal smoothing scheme that is designed to remove these artifacts. We treat each per-frame transformation $\mathbf{T}_t$ as a point in a high-dimensional space of color transformations; the entire set of transformations, thus, forms a curve in this space. Regions of this curve with a high curvature correspond exactly to instants in the video with temporal artifacts. With this is mind, we automatically detect points on this curve with low curvature. We treat these points as *keyframes*; interpolating the color transformations at these keyframes produces a set of temporally coherent color transformations that, when applied to the input video, no longer lead to artifacts. This process approximates the notion of curvature flow [Ilmanen 1995] encountered in differential geometry. Note that, contrary to curve simplification methods that aim at preserving the shape of a curve by approximating areas of high curvature more accurately, we try to remove areas of high curvature (see Fig. 8 in the paper) in order to smooth the curve and obtain temporal coherence.

1 Curvature estimation

The transformations $\mathbf{T}_t$ define a curve in the space of all possible color transformations, sampled at each frame, and we use differential geometry principles to analyze this curve. In particular, we would like to filter this curve by detecting and smoothing out regions of high curvature; this requires us to define a notion of curvature for this high-dimensional space. However, one component of each transformation is a non-linear luminance mapping, and analyzing the resulting infinite-dimensional transformation curve is not tractable. Therefore, in order to simplify the curvature computation we approximate the 1-d non-linear luminance transform with a affine transform, i.e., a scale and translation in 1-d. This approximation is used only to make the computation the curvature of the transformation curve tractable, and to find the appropriate keyframes. As will be discussed later, the final color transfer functions themselves are constructed by interpolating the original transfer functions (including the non-linear luminance transfer function).

Each per-frame transformation, $\mathbf{T}_t$, therefore, consists of the luminance scale and translation (two scalars per-segment), the chrominance translations (three 2x1 vectors per-segment) and the chrominance scaling matrices (three 2x2 symmetric positive definite matrices per-segment). Each transformation therefore lies in the Cartesian product of all these individual spaces – a 17-dimensional space (or alternatively, a 34-dimensional space if the videos are segmented into foreground and background). To analyze the transformation curve, we would like to define a notion of curvature in this high-dimensional space. While many curvature operators exist [Kitagawa 2005], we focus on the covariant derivative along the curve's tangent vector, i.e., the vector $\nabla_{\dot{\mathbf{T}}}\dot{\mathbf{T}}$. The covariant derivative of a curve's tangent vector generalizes the traditional notion of the second derivative in Euclidean spaces to arbitrary Riemannian manifolds. We chose this particular form for curvature because it appropriately captures changes in the velocity of color transformations *along* the transformation curve, and consequently, its smoothness. This makes it particularly useful for our application of temporally smoothing the transform curve. This is in contrast to the second fundamental form (that is typically used by standard curvature flows), which captures variations *orthogonal* to the curve. For instance, a non-constant speed geodesic or any curve in a one-dimensional manifold posses a vanishing second fundamental form. In contrast, the covariant derivative of the tangent vector field to a curve only vanishes for constant speed geodesic curves, i.e., constant speed geodesics are defined by $\nabla_{\dot{\mathbf{T}}}\dot{\mathbf{T}} = 0$. The covariant derivative hence appropriately captures the desired changes in velocity of the color transformations, and consequently, its smoothness.

To obtain the desired covariant derivative in this high-dimensional Cartesian product, we separate each per-frame transformation $\mathbf{T}_t$ into its individual elements (i.e., the luminance scaling and translation scalars, the chrominance translation vectors, and the chrominance scaling matrices), and independently compute a covariant derivative for each of them [do Carmo 1992]. The final curvature value is computed by combining these individual vectors. We denote one of these elements γ. For each of these elements, the general form of the covariant derivative vector, $\nabla_{\dot{\gamma}}\dot{\gamma}$, is given by:

$$\nabla^i_{\dot{\gamma}}\dot{\gamma} = \ddot{\gamma}^i + \sum_{k,l} \Gamma^i_{k,l}\, \dot{\gamma}^k\, \dot{\gamma}^l, \qquad i = 1..d \tag{1}$$

where $()^i$ denotes the i^{th} component of an n-dimensional vector, $\dot{\gamma}^i$ denotes the i'th component of the derivative wrt time, $\ddot{\gamma}^i$ denotes the second derivative wrt time. In addition, $\Gamma^i_{k,l}$ are real numbers called the Christoffel symbols for a given metric (see section 2 for a derivation of these elements).

Eqn. 1 describes the general form of the covariant derivative vector and can be used to derive the corresponding form for each element of the transformation curve. For example, the luminance and chrominance translation components live in an Euclidean space. Here, the notion of covariant derivative of the tangent vector field coincides with the usual second derivative, i.e., for these quantities, $\nabla_{\dot{\gamma}}\dot{\gamma} = \ddot{\gamma}$.

The luminance and chrominance scaling terms on the other hand lie in a more complex space. Takatsu [Takatsu 2011] describes a Riemannian metric for covariance matrices called the Wasserstein metric, that can be used to compute the corresponding Christoffel symbols (Γ in Eqn. 1). In section 2, we derive the form for these symbols for both the 1-d (luminance scaling) and the 3-d (chrominance scaling) cases. By substituting these values in Eqn. 1, we can

*e-mail:nbonneel@seas.harvard.edu

compute the covariant derivative vector for both the luminance and chrominance scaling terms.

Having computed the covariance derivative vectors for each individual element of the transformation curve, the curvature value for the complete transformation curve is obtained as the norm in the concatenated high-dimensional space, i.e.,

$$\mathbf{K}_t = \left(\sum_i g_i(\nabla_{\dot\gamma_i}\dot\gamma_i, \nabla_{\dot\gamma_i}\dot\gamma_i)\right)^{\frac{1}{2}}, \tag{2}$$

where γ_i denotes the translation and scaling components of the luminance, $\mathbf{T}^\ell$, and chrominance, $\mathbf{T}^c$ transformations, and g denotes a Euclidean metric for translation vectors (i.e., $g(x,y) = \langle x, y\rangle$) and a Wasserstein metric for scalings (see section 2).

2 Curvature computation in the Wasserstein space

In this section, we further detail the curvature computations and provide analytic formulas. We compute the Christoffel symbols Γ^i_{ab} required to evaluate the covariant derivative $\nabla_{\dot\gamma}\dot\gamma$ based on the results of Takatsu [2011]

Takatsu [Takatsu 2011] defines a parameterization of the space of symmetric positive definite matrices – i.e., our chrominance transforms – using a Riemannian submersion $\Pi(M) = M^T M$ from the space of general matrices $\mathcal{G}$ to the space of symmetric positive definite matrices $\mathcal{Sym}^+$. She pushes forward the Euclidean metric in $\mathcal{G}$ to $\mathcal{Sym}^+$ using this submersion, to obtain a metric $g_V(X,Y) = \mathrm{tr}(X\,V\,Y)$ where tr denotes the matrix trace and X and Y are symmetric matrices.

2.1 In 1D

In 1D, $y = \Pi(x) = x^2$. We want to push-forward the Euclidean metric to $\mathcal{Sym}^+$ using Π, i.e., transfer an Euclidean metric for vectors in an Euclidean space to an appropriate metric in the space defined by applying the function Π to all points in the Euclidean space. We have $\mathrm{dy} = 2x\ \mathrm{dx} = 2\sqrt{y}\,\mathrm{dx}$. The metric is hence given by $g_x(\mathrm{dx},\mathrm{dx}) = G(\frac{\mathrm{dy}}{2\sqrt{y}}, \frac{\mathrm{dy}}{2\sqrt{y}}) = \frac{1}{4y}\,\mathrm{dy}^2$ where y is a variance, and dy is a 1D vector in the space of variances (a "variance derivative"), and $G(\mathrm{dx},\mathrm{dx}) = \mathrm{dx}^2$ the Euclidean metric.

In 1D, the unique Christoffel symbol for our curve γ is thus obtained by $\Gamma = \frac{1}{2} g^{-1}\frac{\mathrm{d}\,g}{\mathrm{dy}} = -1/(2\gamma_t)$, where γ_t is the current variance. The covariant derivative can be computed using:

$$\nabla_{\dot\gamma}\dot\gamma = \ddot\gamma - \frac{\dot\gamma^2}{2\gamma}$$

2.2 For 2×2 matrices (3D)

We first define a set of standard basis vectors – for instance, the standard basis $\{\mathbf{x_1} = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \mathbf{x_2} = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], \mathbf{x_3} = \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\}$. Any 2×2 symmetric matrix can be written as a linear combination of these three basis matrices, and the space is hence three-dimensional.

In fact, Takatsu's metric is not given in such a standard basis. As such, to obtain tangent vectors X and Y usable with Takatsu's metric at point $V : g_V(X,Y) = \mathrm{tr}(X\,V\,Y)$, one need to solve a small linear system. If A and B are vectors in a standard basis of symmetric matrices, one can obtain vectors X and Y by solving the system $A = X^T V + V^T X$ and $B = Y^T V + V^T Y$. These X and Y can finally be plugged into Takatsu's metric.

For easier reimplementation of our method, we analytically performed such an inversion. The expression of $g_V(A,B)$ (in Hörner form, where $A = \left[\begin{smallmatrix}a_0&a_1\\a_1&a_2\end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix}b_0&b_1\\b_1&b_2\end{smallmatrix}\right]$, $V = \left[\begin{smallmatrix}v_0&v_1\\v_1&v_2\end{smallmatrix}\right]$) is given by:

$$\begin{aligned} g_V(A,B) = & \frac{1}{4\,(v_0+v_2)\,(v_0 v_2 - {v_1}^2)}(((-{v_1}^2 + (v_0+v_2)\,v_0)b_2 \\ & - 2\,v_1 v_0 b_1 + {v_1}^2 b_0)a_2 + (4\,v_2 v_0 b_1 - 2\,b_2 v_0 v_1 \\ & - 2\,v_2 v_1 b_0)a_1 + (-2\,v_2 v_1 b_1 + {v_1}^2 b_2 + \\ & \left(-{v_1}^2 + {v_2}^2 + v_0 v_2\right) b_0)a_0) \end{aligned} \tag{3}$$

In addition to the metric, we need to compute the Christoffel symbols in order to compute the covariant derivative. In three dimensions, Christoffel symbols are defined by:

$$\Gamma^i_{k,l} = \frac{1}{2}\sum_{m=1}^{3} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right) \tag{4}$$

where g_{ij} is the value of the metric for the tangent vectors, $\frac{\partial g_{ij}}{\partial x^k}$ denotes the directional derivative of $g(\mathbf{x_i}, \mathbf{x_j})$ (for two basis vectors) in the direction of $\mathbf{x_k}$ a third basis vector, and g^{im} denotes the components of the inverse matrix whose coefficients are g_{im}. The 3×3 matrix g^{im} is numerically inverted. The basis need not be orthogonal.

To evaluate $\Gamma^i_{k,l}$, one needs the directional derivative of g in a direction $D = \left[\begin{smallmatrix}d_0&d_1\\d_1&d_2\end{smallmatrix}\right]$. We also analytically derived the formula for the directional derivative of g in a direction D:

$$\begin{aligned} \frac{\partial g_V(A,B)}{\partial D} = & \frac{1}{4\,(v_0+v_2)^2\,(v_0\,v_2 - {v_1}^2)^2}((a_2-a_0)(b_0-b_2) \\ & (d_0+d_2){v_1}^4 + 2\,(a_1\,(b_2-b_0) + (a_2-a_0)b_1) \\ & (d_0\,v_2 - d_2\,v_0){v_1}^3 - ((2\,(a_2\,b_1 + a_1\,b_2)d_1 + \\ & (a_2\,b_0 + 4\,a_1\,b_1 - (a_2-a_0)b_2)d_2 + a_2\,b_2\,d_0){v_0}^2 + \\ & 2\,((a_2\,b_0 - (a_2-a_0)b_2)d_2 + (a_1\,(b_0+b_2) + \\ & (a_0+a_2)b_1)d_1 + ((a_2-a_0)b_0 + a_0\,b_2)d_0)v_2\,v_0 + \\ & (a_0\,b_0\,d_2 + (a_0\,b_2 + 4\,a_1\,b_1 + (a_2-a_0)b_0)d_0 + \\ & 2\,(a_1\,b_0 + a_0\,b_1)d_1){v_2}^2){v_1}^2 + (2\,((a_2\,b_1 + a_1\,b_2) \\ & (2\,d_2 + d_0) + ((a_0+a_2)b_2 + a_2\,b_0 + 4\,a_1\,b_1)d_1) \\ & v_2\,{v_0}^2 + 2\,((a_1\,b_0 + a_0\,b_1)(d_2 + 2\,d_0) + (a_0\,b_2 + \\ & 4\,a_1\,b_1 + (a_0+a_2)b_0)d_1){v_2}^2 v_0 + 2\,(a_2\,b_1 + a_1\,b_2) \\ & d_2\,{v_0}^3 + 2\,(a_0\,b_0\,d_1 + (a_1\,b_0 + a_0\,b_1)d_0){v_2}^3)v_1 + \\ & a_2\,b_2\,{v_0}^3(2\,d_1\,v_1 - d_2\,v_0) - 2\,((a_2\,b_1 + a_1\,b_2)d_1 + \\ & a_2\,b_2\,d_2)v_2\,{v_0}^3 - 2\,((a_1\,b_0 + a_0\,b_1)d_1 + a_0\,b_0\,d_0) \\ & {v_2}^3 v_0 - (2\,(a_1\,(b_0+b_2) + (a_0+a_2)b_1)d_1 + \\ & (4\,a_1\,b_1 + a_2\,b_2)d_2 + (4\,a_1\,b_1 + a_0\,b_0)d_0){v_2}^2{v_0}^2 - \\ & a_0\,{v_2}^4 b_0\,d_0) \end{aligned} \tag{5}$$

The desired covariant derivative can finally be obtained by plugging the expression of Γ^i (Eq. 4) into the main paper's formula:

$$\nabla^i_{\dot\gamma}\dot\gamma = \ddot\gamma^i + \sum_{k,l} \dot\gamma^k \, \Gamma^i_{k,l} \, \dot\gamma^l, \qquad\qquad i = 1..d$$

where the three Christoffel symbols $\Gamma^i_{k,l}$ are expressed using Eq 3 and 5.

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