Geometric construction of Wasserstein Barycenters

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Optimal Transport theory consists in finding a map T pushing forward an input measure μ to a target measure ν, which minimizes a "transport cost". This cost often consists in the sum of the squared distances travelled by all particles during their motion. Formally, the Monge problem consists in minimizing :

$$\min C^2(\mu, \nu) = \min_T \int_X \|x - T(x)\|^2 \mathrm{d}\mu(x)$$

such that $T\#\mu = \nu$. This, for instance, allows to define the "displacement interpolation" between probability measures. This theory has seen many applications in computer graphics, but more generally in machine learning, medical and astro imaging, simulation (from fluids to the early stages of the universe), and operational research.

Recently, methods have been developed to efficiently compute the optimal transport map (and displacement interpolation) in the case where ν is a (weighted) sum of Diracs, and μ is absolutely continuous [2, 1]. This is performed by computing and optimizing for a Power Diagram of the target measure (that is, a weighted Voronoi Diagram) for which effective algorithms exist. It is often referred to as semi-discrete optimal transport. A symmetrized semi-discrete transport is currently under development by a collaborator of the project, allowing for continuous-continuous transport based on the semi-discrete approach and Centroidal Power Diagrams [3].

At the same time, the notion of Wasserstein barycenter has been developed. Given a set of input measures $\{\mu_i\}_{i=1..N}$ along with weights $\{w_i\}_{i=1..N}$, this consists in finding a "barycenter" measure $\bar{\mu}$ defined similarly to a Euclidean barycenter :

$$\bar{\mu} = \mathrm{argmin}_\mu \sum_{i=1}^{N} w_i C^2(\mu_i, \mu)$$

In this framework, the displacement interpolation of two measures is exactly the barycenter between these two measures, with weights $w_1 = 1 - t$ and

$w_2 = t$. However, contrary to displacement interpolation, very few numerical techniques exist for the barycenter problem and the geometric method described above does not hold.

The goal of this internship is to investigate how Wasserstein barycenters can be characterized geometrically. Specifically, the student will need to study how the symmetrized semi-discrete algorithm extends to more than two input measures and if time permits, implements this algorithm in 2-d or 3-d.

The project will be supervised by Nicolas Bonneel and Julie Digne (computer science junior researchers at LIRIS), and will be performed in collaboration with Bruno Levy (computer science senior researcher at INRIA) and Jean-Marie Mirebeau (applied math junior researcher at Orsay mathematics laboratory).

Références

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[2] Quentin Mérigot. A multiscale approach to optimal transport. *Computer Graphics Forum*, 30(5) :1583–1592, 2011.

[3] S. Xin, B. Lévy, Z. Chen, L. Chu, Y. Yue, and W. Wang. Centroidal power diagrams. *ACM Transactions on Graphics (SIGGRAPH Asia)*, 2016.