Sobol' Sequences with Guaranteed-Quality 2D Projections Supplementary Materials: Proofs

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$$K^{(1)} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \to K^{(2)} = \left[\begin{array}{cc|cc} A & B & A+B & A \\ & C & C & 0 \\ \hline & & A & A+B \\ & & & C \end{array}\right] \to K^{(3)} = \left[\begin{array}{cccc|cccc} A & B & A+B & A & B & A+B & A & B \\ & C & C & 0 & C & C & 0 & C \\ & & A & A+B & A & A+B & 0 & 0 \\ & & & C & 0 & C & 0 & 0 \\ \hline & & & & A & B & B & A+B \\ & & & & & C & C & C \\ & & & & & & A & A+B \\ & & & & & & & C \end{array}\right]$$

We assume A, B and C are of size $m \times m$. We need to show that all $(s-1) \times s$ rectangular submatrices T of $K^{(3)}$ anchored on the first row are rank-deficient of at most 1 (i.e., $corank(S) \leq 1$), given that this property $\mathcal{P}$ holds for $K^{(1)}$ and $K^{(2)}$. We also require a property $\mathcal{Q}$ on C: all submatrices T of C obtained by removing $1 \leq t < m$ consecutive columns and the last t rows has $corank(T) \leq 1$.

1 Property $\mathcal{Q}$

It is easy to see that if $\mathcal{Q}$ holds for C, it also holds for $\begin{bmatrix} A & A+B \\ 0 & C \end{bmatrix}$. Indeed, A has full rank, so removing any row and column of A does not change the rank of $\begin{bmatrix} A & A+B \\ 0 & C \end{bmatrix}$.

The base case of this induction is guaranteed by numerical evaluation: property $\mathcal{Q}$ is numerically verified for the initial $e \times e$ block C by checking ranks.

2 Property $\mathcal{P}$

The base case of the induction for property $\mathcal{P}$ is obtained by numerically evaluating ranks on up to $4e \times 4e$ matrices (submatrices of the initial $4e \times 4e$ matrix $K^{(2)}$). We show that if the property $\mathcal{P}$ holds for $K^{(1)}$ and $K^{(2)}$, it also necessarily holds for $K^{(3)}$, and thus, for any number of iterations of the procedure.

The idea is to perform linear combinations of rows and/or columns on matrices to exhibit a block-triangular structure, where diagonal blocks are either full rank, or, for at most one block, satisfy property $\mathcal{P}$ or $\mathcal{Q}$ by hypothesis, which allows to conclude that the corank is at most 1.

Note that matrices that need to be tested can have any size and are of shape $(w-1) \times w$: these matrices are anchored on the top of matrix $K^{(3)}$, and will, in general, contain partial blocks that require particular attention. They may also overlap more blocks horizontally than vertically (e.g., if they slightly overlap block horizontally), which also requires particular attention when considering permutations of rows.

We exhaustively detail below all cases, depending on the size of the tested matrix, and where it is located on the top row.

2.1 $1 \leq w \leq m$: testing $\mathcal{P}$ on all 1x2 blocks and 1x1 blocks

Trivial, by hypothesis

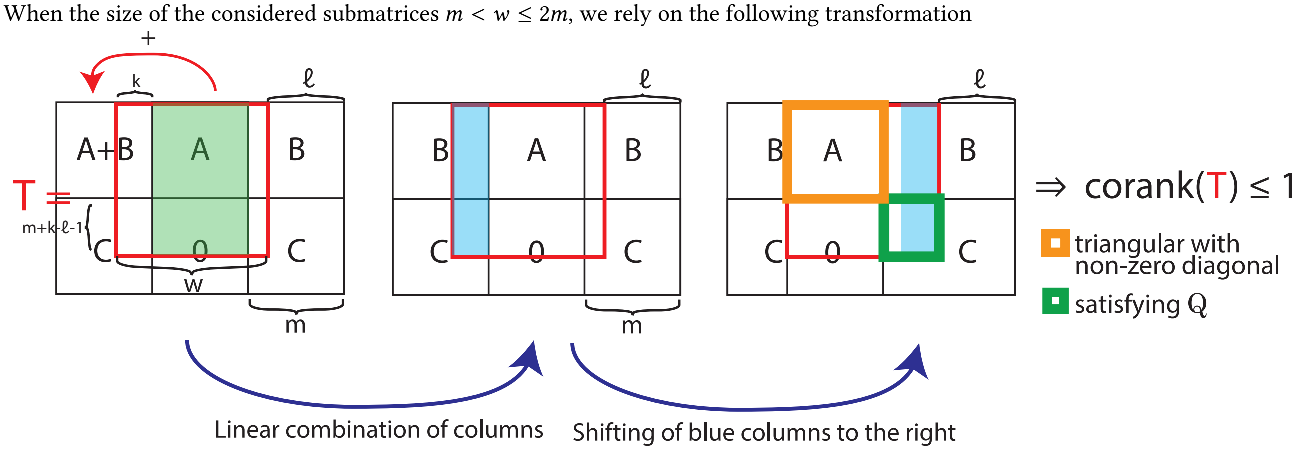
2.2 $m \leq w \leq 2m$: testing $\mathcal{P}$ on all 2x2 blocks (left to right):

- $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \rightarrow$ directly by hypothesis
- $\begin{bmatrix} B & A+B \\ C & C \end{bmatrix} \rightarrow$ directly by hypothesis
- $\begin{bmatrix} A+B & A \\ C & 0 \end{bmatrix} \rightarrow$ directly by hypothesis

2.3 $m \leq w \leq 2m$ and $2m \leq w \leq 3m$: testing $\mathcal{P}$ on all 3x3 blocks (left to right):

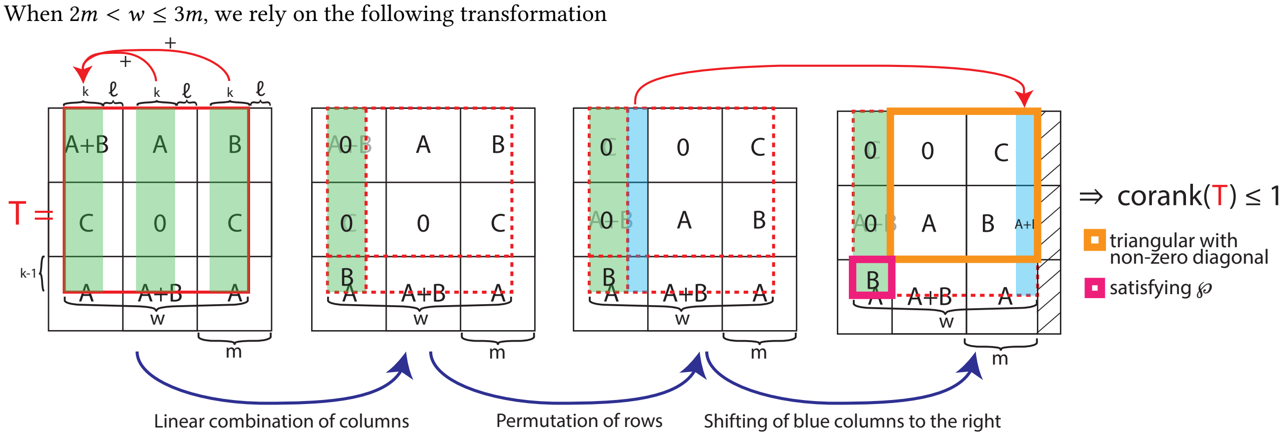
- $\begin{bmatrix} A & B & A+B \\ 0 & C & C \\ 0 & 0 & A \end{bmatrix} \rightarrow$ directly by hypothesis
- $\begin{bmatrix} B & A+B & A \\ C & C & 0 \\ 0 & A & A+B \end{bmatrix} \rightarrow$ directly by hypothesis
- $\begin{bmatrix} A+B & A & B \\ C & 0 & C \\ A & A+B & A \end{bmatrix}$

When the size of the considered submatrices $m < w \leq 2m$, we rely on the following transformation



This results in a block triangular matrix with a diagonal block A and another diagonal block $\widetilde{C}$ consisting of top-leftmost and top-rightmost rows of C. A is invertible by hypothesis, and $corank(\widetilde{C}) \leq 1$ by hypothesis $\mathcal{Q}$.

When $2m < w \leq 3m$, we rely on the following transformation

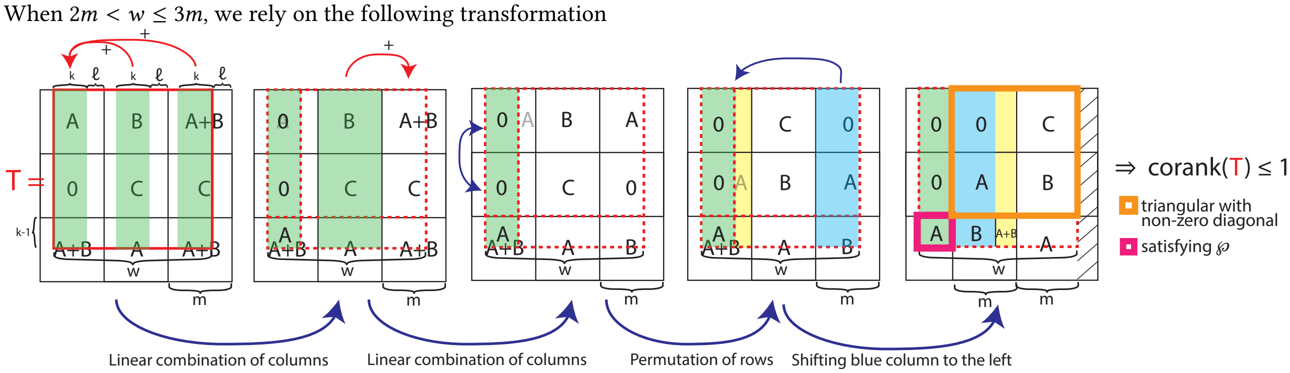


This results in a block triangular matrix with one triangular block which determinant is non-zero, as diagonal elements are non-zero, and one block consisting of a small first-row anchored submatrix of B. Matrix B has property $\mathcal{P}$. The corank of matrix T is thus that of this small submatrix of B [y e], which concludes the proof for this case.

- $\begin{bmatrix} A & B & A+B \\ 0 & C & C \\ A+B & A & A+B \end{bmatrix}$

  When the size of the considered submatrices $m < w \leq 2m$, T only involves the first two (block) rows. They appear in the initial hypotheses, and so $\mathcal{P}$ is satisfied by hypothesis.

  When $2m < w \leq 3m$, we rely on the following transformation

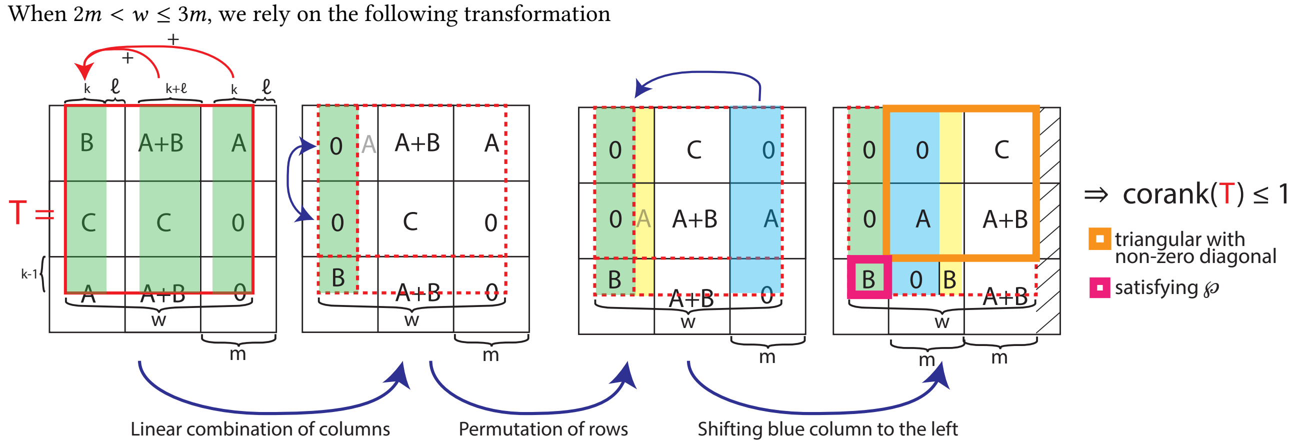


This results in a block triangular matrix with one triangular block which determinant is non-zero, as diagonal elements are non-zero, and one block consisting of a small first-row anchored submatrix of A. Matrix A has property $\mathcal{P}$. The corank of matrix T is thus that of this small submatrix of A [y e], which concludes the proof for this case.

- $\begin{bmatrix} B & A+B & A \\ C & C & 0 \\ A & A+B & 0 \end{bmatrix}$

  When the size of the considered submatrices $w \leq 2m$, T only involves the first two (block) rows. They appear in the initial hypotheses, and so $\mathcal{P}$ is satisfied by hypothesis.

  When $2m < w \leq 3m$, we rely on the following transformation

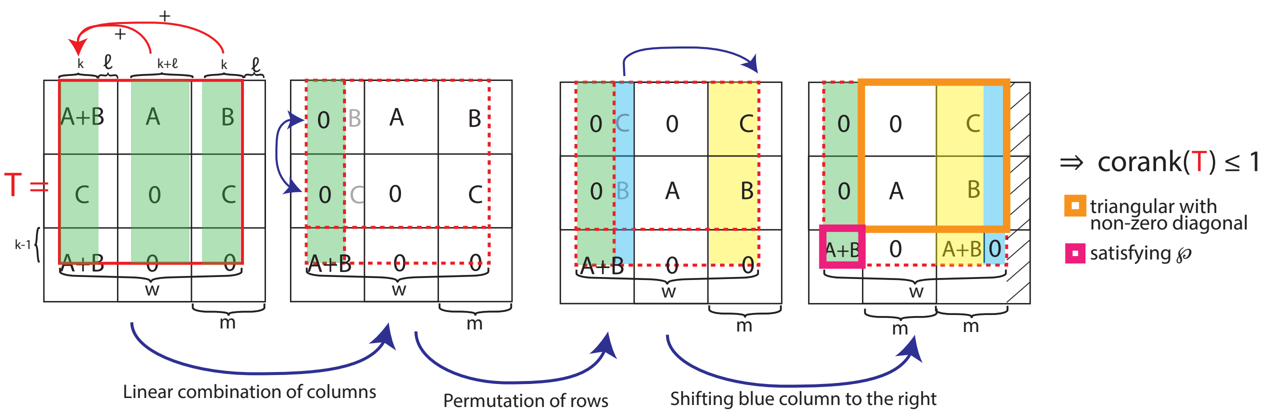


This results in a block triangular matrix with one triangular block which determinant is non-zero as diagonal elements are non-zero, and one block consisting of a small first-row anchored submatrix of B. Matrix B has property $\mathcal{P}$. The corank of matrix T is thus that of this small submatrix of B [y e], which concludes the proof for this case.

- $\begin{bmatrix} A+B & A & B \\ C & 0 & C \\ A+B & 0 & 0 \end{bmatrix}$

  When $m < w \leq 2m$, T only involves the two first (block) rows. We have already encountered this case for these two rows and provided a proof (see above for matrix $\begin{bmatrix} A+B & A & B \\ C & 0 & C \\ A & A+B & A \end{bmatrix}$).

  When $2m < w \leq 3m$, we rely on the following transformation



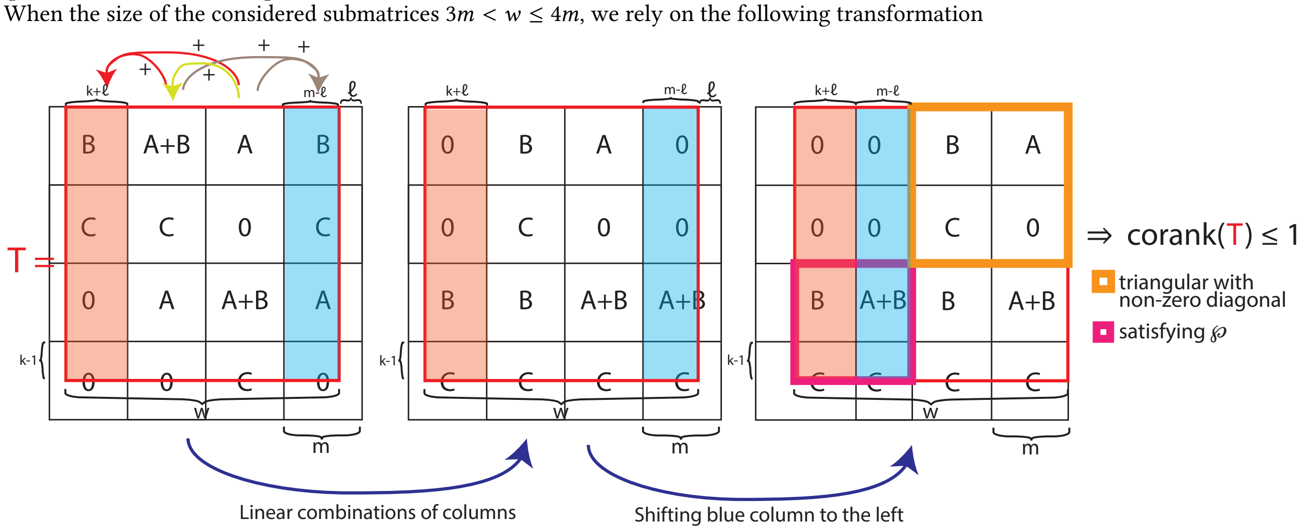
This results in a block triangular matrix with one triangular block which determinant is non-zero as diagonal elements are non-zero, and one block consisting of a small first-row anchored submatrix of $A + B$. Matrix $A + B$ has property $\mathcal{P}$. The corank of matrix T is thus that of this small submatrix of $A + B$ [y e], which concludes the proof for this case.

2.4 $2m \leq w \leq 3m$ and $3m \leq w \leq 4m$: testing $\mathcal{P}$ on all 4x4 blocks (left to right):

From now on, all our transformations only involve simple (block) column-wise operations. When the size of the considered submatrices $2m < w \leq 3m$, T only involves the first three blocks of rows. We apply the same column transformations as for the case $3m \leq w \leq 4m$, and conclude similarly.

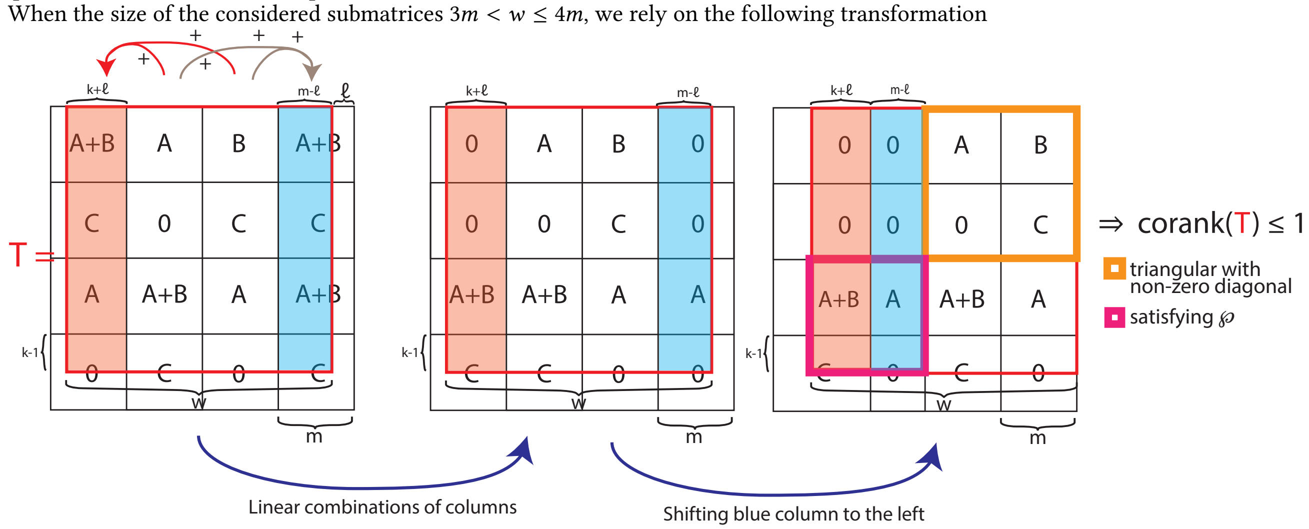
- $\begin{bmatrix} A & B & A+B & A \\ 0 & C & C & 0 \\ 0 & 0 & A & A+B \\ 0 & 0 & 0 & C \end{bmatrix} \longrightarrow$ directly by hypothesis
- $\begin{bmatrix} B & A+B & A & B \\ C & C & 0 & C \\ 0 & A & A+B & A \\ 0 & 0 & C & 0 \end{bmatrix}$

When the size of the considered submatrices $3m < w \leq 4m$, we rely on the following transformation



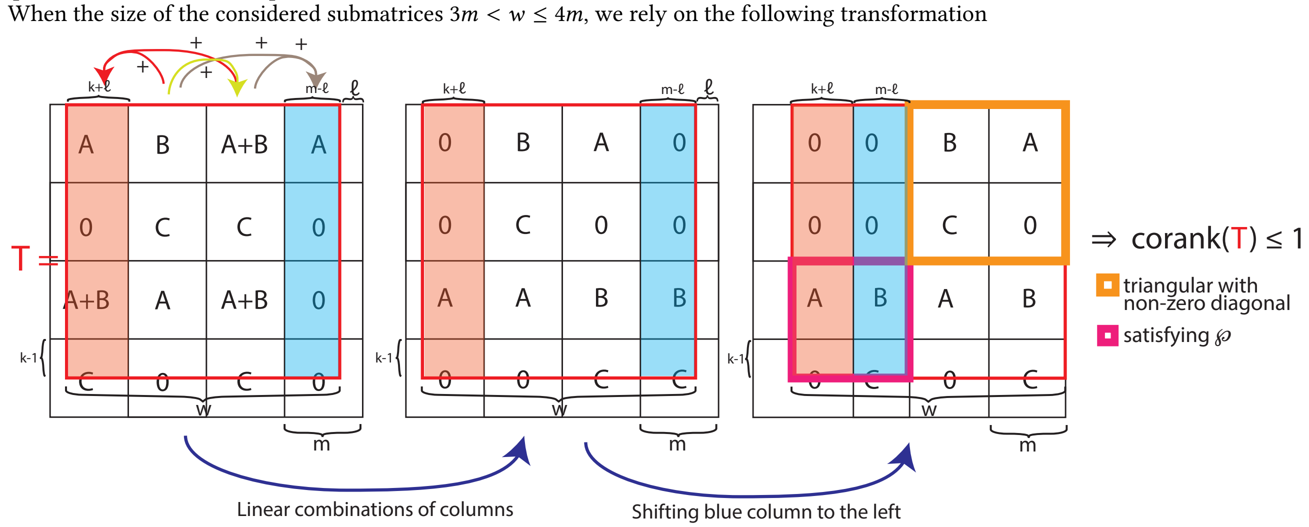
This results in a block triangular matrix with a diagonal block $\begin{bmatrix} B & A \\ C & 0 \end{bmatrix}$ highlighted in bold orange that has a non-zero determinant, and another diagonal block consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} B & A+B \\ C & C \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

- $\begin{bmatrix} A+B & A & B & A+B \\ C & 0 & C & C \\ A & A+B & A & A+B \\ 0 & C & 0 & C \end{bmatrix}$
  When the size of the considered submatrices $3m < w \leq 4m$, we rely on the following transformation



This results in a block triangular matrix with a diagonal block $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ highlighted in bold orange that has non-zero determinant, and another diagonal block consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} A+B & A \\ C & 0 \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

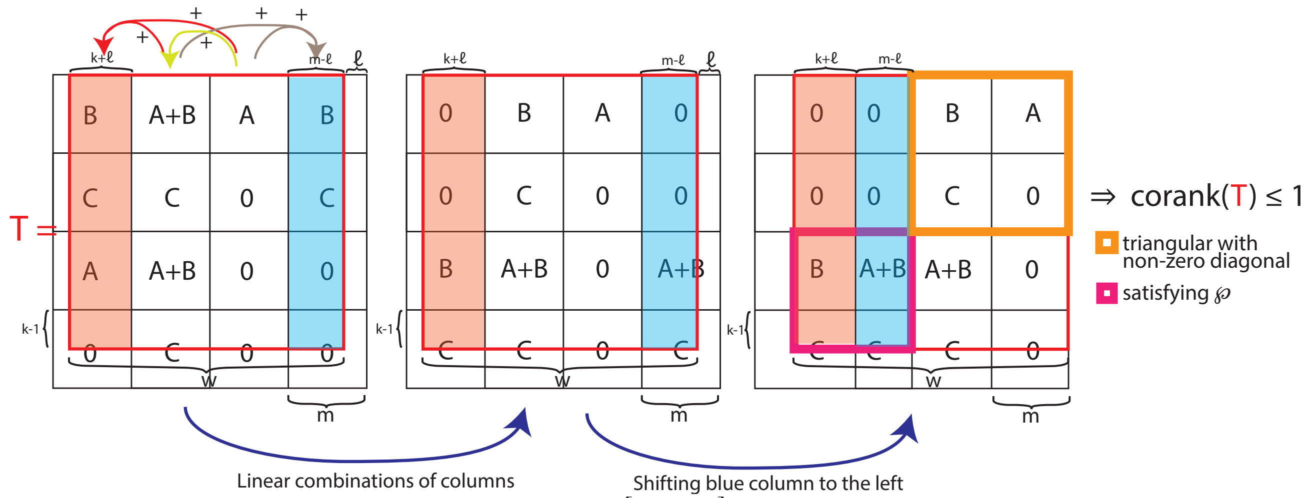
- $\begin{bmatrix} A & B & A+B & A \\ 0 & C & C & 0 \\ A+B & A & A+B & 0 \\ C & 0 & C & 0 \end{bmatrix}$
  When the size of the considered submatrices $3m < w \leq 4m$, we rely on the following transformation



This results in a block triangular matrix with a diagonal block $\begin{bmatrix} B & A \\ C & 0 \end{bmatrix}$ highlighted in bold orange that has a non-zero determinant, and another diagonal block consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

- $\begin{bmatrix} B & A+B & A & B \\ C & C & 0 & C \\ A & A+B & 0 & 0 \\ 0 & C & 0 & 0 \end{bmatrix}$

When the size of the considered submatrices $3m < w \leq 4m$, we rely on the following transformation



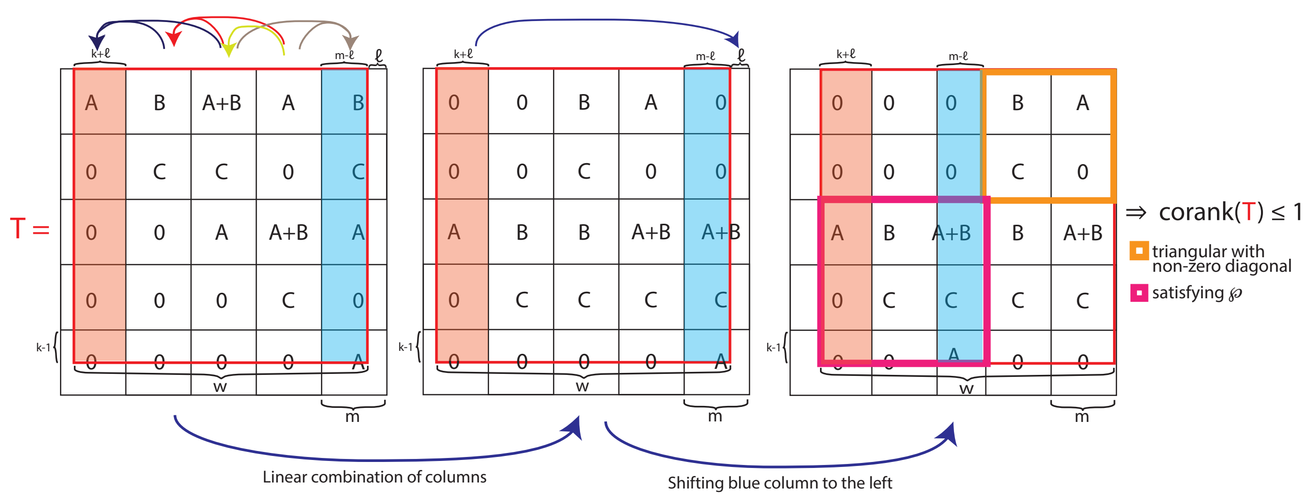
This results in a block triangular matrix with a diagonal block $\begin{bmatrix} B & A \\ C & 0 \end{bmatrix}$ highlighted in bold orange that has non-zero determinant, and another diagonal block consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} B & A+B \\ C & C \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

2.5 $3m \leq w \leq 4m$ and $4m \leq w \leq 5m$: testing $\mathcal{P}$ on all 5x5 blocks (left to right):

Similarly, all our transformations only involve simple (block) column-wise operations. When the size of the considered submatrices $3m < w \leq 4m$, T only involves the first four blocks of rows. We apply the same column transformations as for the case $4m \leq w \leq 5m$, and conclude similarly.

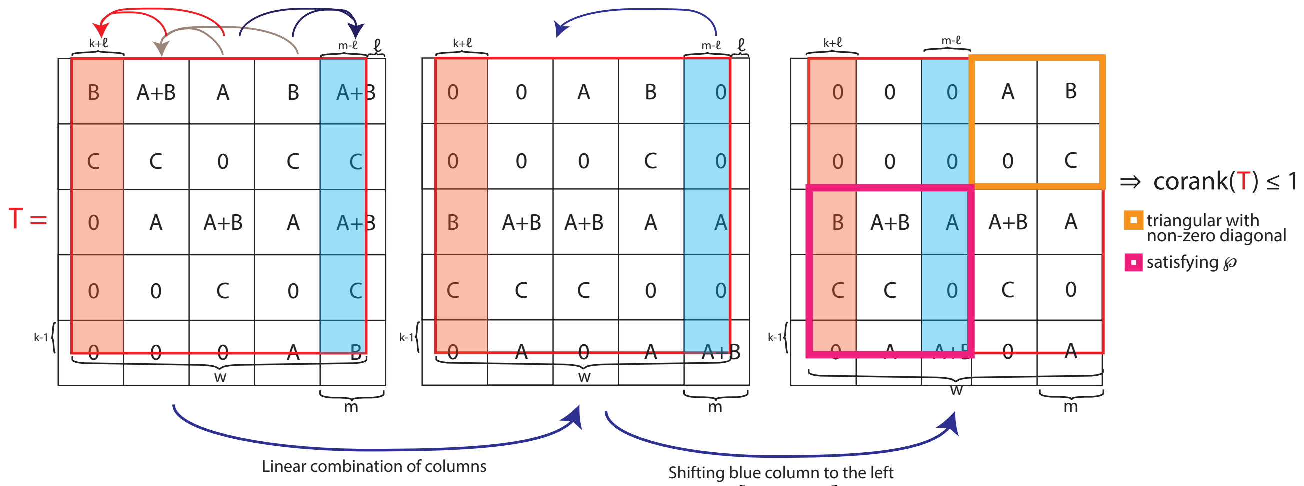
- $\begin{bmatrix} A & B & A+B & A & B \\ 0 & C & C & 0 & C \\ 0 & 0 & A & A+B & A \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & A \end{bmatrix}$

When the size of the considered submatrices $4m < w \leq 5m$, we rely on the following transformation



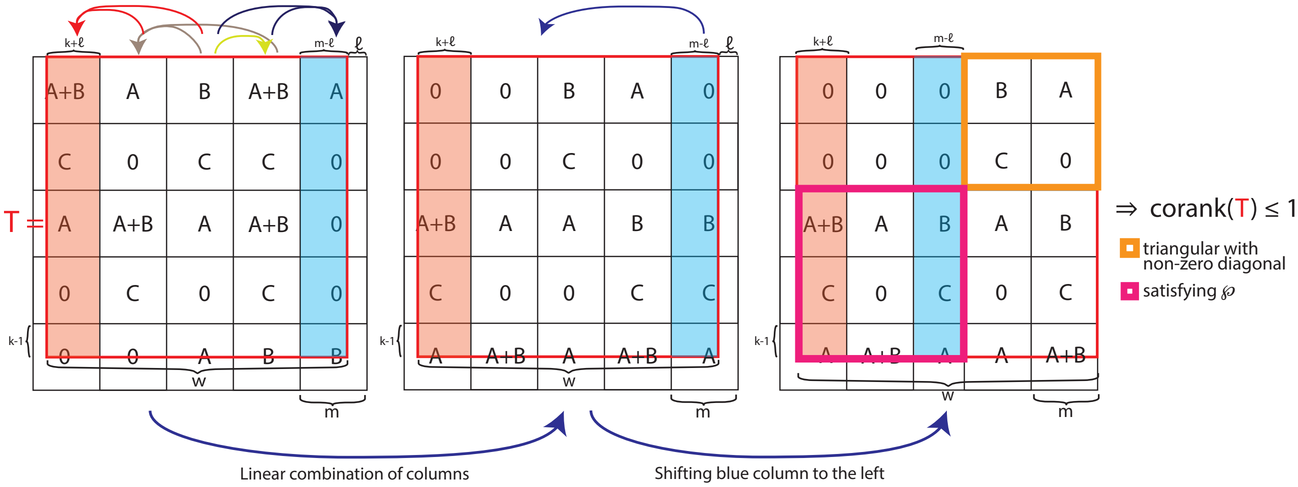
This results in a block triangular matrix with a diagonal block of the form $\begin{bmatrix} B & A \\ C & 0 \end{bmatrix}$ highlighted in bold orange that has a non-zero determinant, and another diagonal block consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} A & B & A+B \\ 0 & C & C \\ 0 & 0 & A \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

- $\begin{bmatrix} B & A+B & A & B & A+B \\ C & C & 0 & C & C \\ 0 & A & A+B & A & A+B \\ 0 & 0 & C & 0 & C \\ 0 & 0 & 0 & A & B \end{bmatrix}$
When the size of the considered submatrices $4m < w \leq 5m$, we rely on the following transformation



This results in a block triangular matrix with a diagonal block of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ highlighted in bold orange that has a non-zero determinant, and another diagonal block consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} B & A+B & A \\ C & C & 0 \\ 0 & A & A+B \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

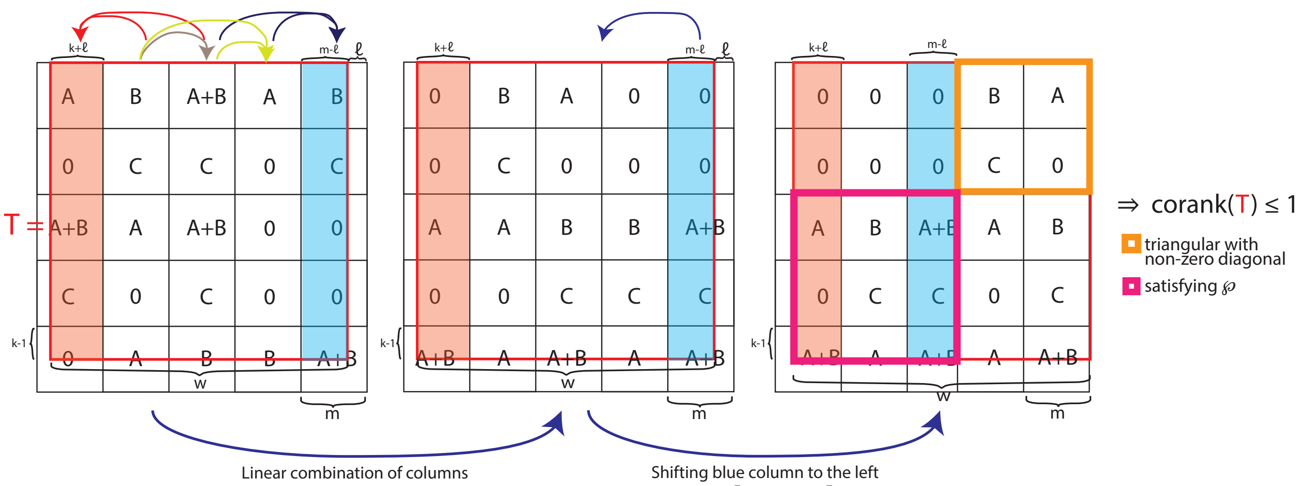
- $\begin{bmatrix} A+B & A & B & A+B & A \\ C & 0 & C & C & 0 \\ A & A+B & A & A+B & 0 \\ 0 & C & 0 & C & 0 \\ 0 & 0 & A & B & B \end{bmatrix}$
When the size of the considered submatrices $4m < w \leq 5m$, we rely on the following transformation



This results in a block triangular matrix with a diagonal block of the form $\begin{bmatrix} B & A \\ C & 0 \end{bmatrix}$ highlighted in bold orange that has a non-zero determinant, and another diagonal block consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} A+B & A & B \\ C & 0 & C \\ A & A+B & A \end{bmatrix}$ highlighted in bold magenta, for which we demonstrated above that $\mathcal{P}$ holds, and has thus a corank lower or equal to 1. This concludes the proof for this case.

- $\begin{bmatrix} A & B & A+B & A & B \\ 0 & C & C & 0 & C \\ A+B & A & A+B & 0 & 0 \\ C & 0 & C & 0 & 0 \\ 0 & A & B & B & A+B \end{bmatrix}$

When the size of the considered submatrices $4m < w \leq 5m$, we rely on the following transformation



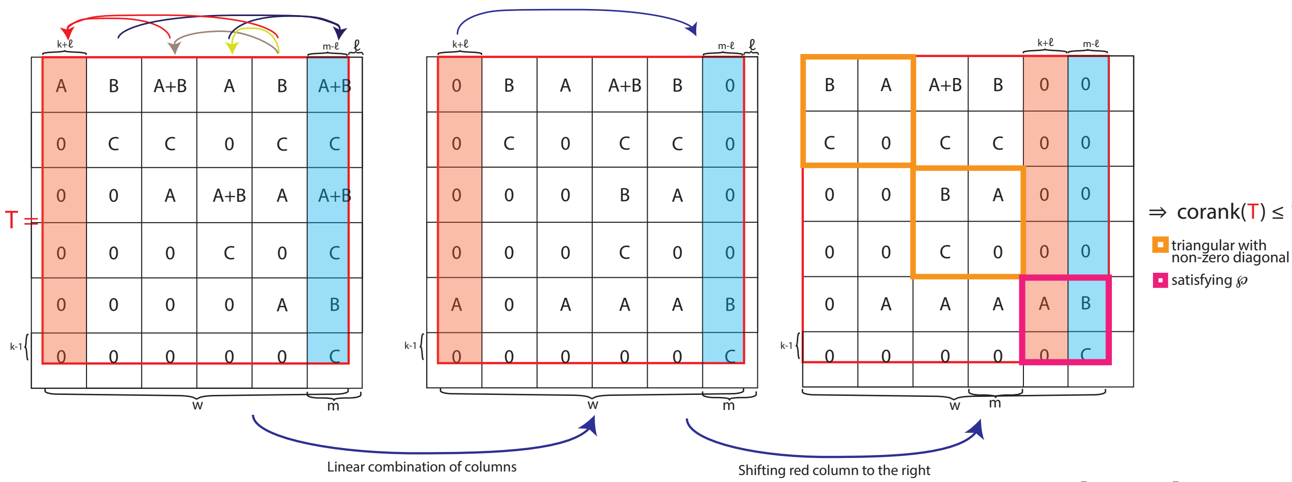
This results in a block triangular matrix with a diagonal block of the form $\begin{bmatrix} B & A \\ C & 0 \end{bmatrix}$ highlighted in bold orange that has a non-zero determinant, and another diagonal block consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} A+B & A & B \\ C & 0 & C \\ A & A+B & A \end{bmatrix}$ highlighted in bold magenta, for which we demonstrated above that $\mathcal{P}$ holds, and has thus a corank lower or equal to 1. This concludes the proof for this case.

When the size of the considered submatrices $3m < w \leq 4m$, T only involves the first four rows. The same transformation can be applied and results A block triangular matrix involving the same non-zero determinant block, and a top-anchored square submatrix of $\begin{bmatrix} A+B & A & B \\ C & 0 & C \end{bmatrix}$ for which $\mathcal{P}$ holds.

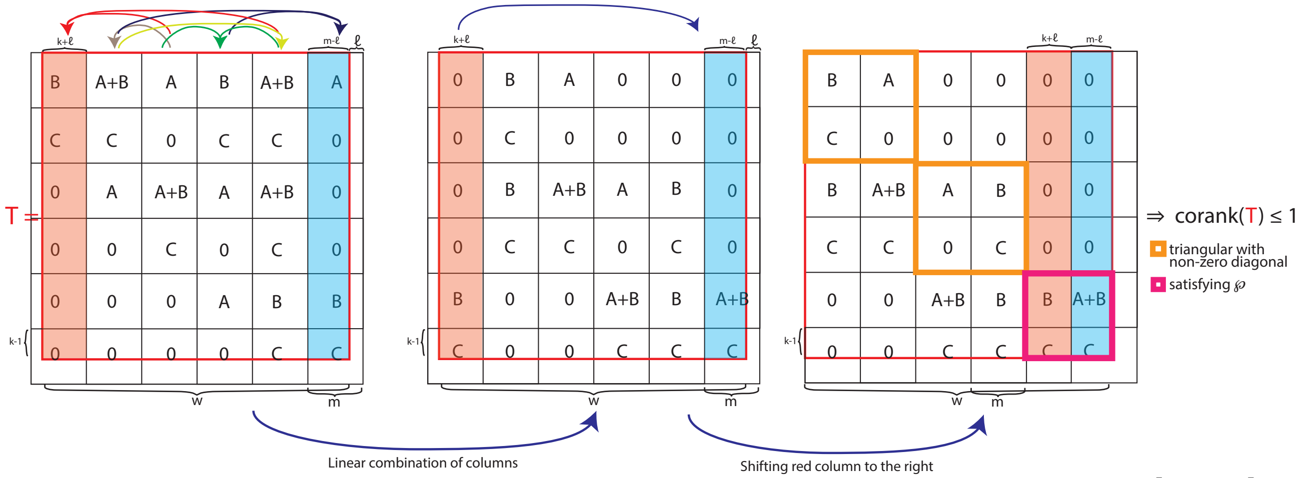
2.6 $4m \leq w \leq 5m$ and $5m \leq w \leq 6m$: testing $\mathcal{P}$ on all 6x6 blocks (left to right):

Similarly, all our transformations only involve simple (block) column-wise operations. When the size of the considered submatrices $4m < w \leq 5m$, T only involves the first five blocks of rows. We apply the same column transformations as for the case $5m \leq w \leq 6m$, and conclude similarly.

- $$\begin{bmatrix} A & B & A+B & A & B & A+B \\ 0 & C & C & 0 & C & C \\ 0 & 0 & A & A+B & A & A+B \\ 0 & 0 & 0 & C & 0 & C \\ 0 & 0 & 0 & 0 & A & B \\ 0 & 0 & 0 & 0 & 0 & C \end{bmatrix}$$
  When the size of the considered submatrices $5m < w \leq 6m$, we rely on the following transformation



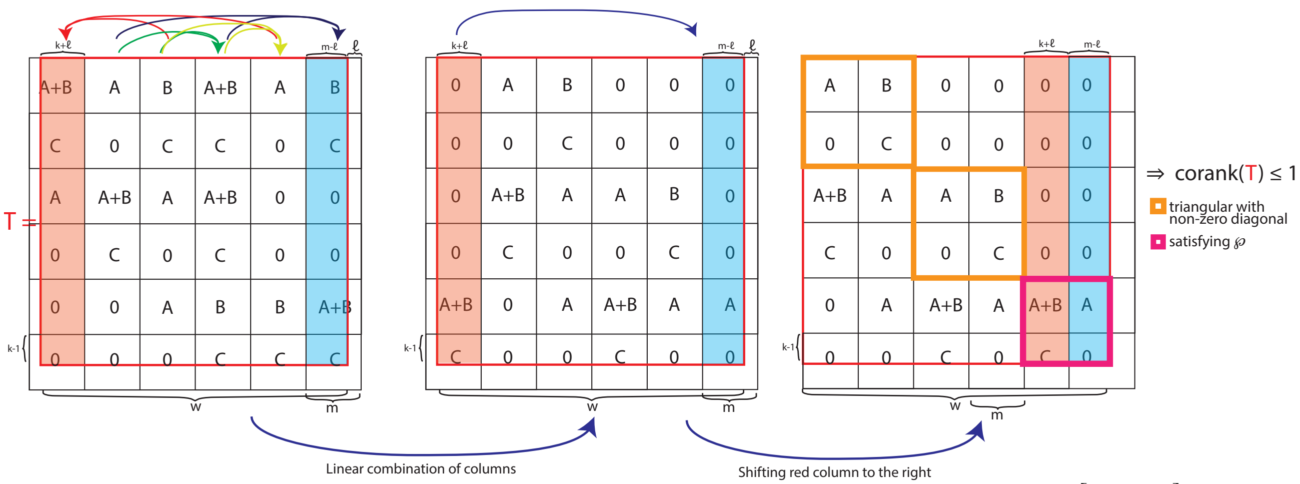
  This results in a block triangular matrix with one block-triangular block that has two diagonal blocks of the form $\begin{bmatrix} B & A \\ C & 0 \end{bmatrix}$ highlighted in bold orange that have non-zero determinants, and another block on the diagonal consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.
- $$\begin{bmatrix} B & A+B & A & B & A+B & A \\ C & C & 0 & C & C & 0 \\ 0 & A & A+B & A & A+B & 0 \\ 0 & 0 & C & 0 & C & 0 \\ 0 & 0 & 0 & A & B & B \\ 0 & 0 & 0 & 0 & C & 0 \end{bmatrix}$$
  When the size of the considered submatrices $5m < w \leq 6m$, we rely on the following transformation



This results in a block triangular matrix with one block-triangular block that has two diagonal blocks of the form $\begin{bmatrix} B & A \\ C & 0 \end{bmatrix}$ (up to permutation) highlighted in bold orange that have non-zero determinants, and another block on the diagonal consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} B & A+B \\ C & C \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

- $\begin{bmatrix} A+B & A & B & A+B & A & B \\ C & 0 & C & C & 0 & C \\ A & A+B & A & A+B & 0 & 0 \\ 0 & C & 0 & C & 0 & 0 \\ 0 & 0 & A & B & B & A+B \\ 0 & 0 & 0 & C & 0 & C \end{bmatrix}$

When the size of the considered submatrices $5m < w \leq 6m$, we rely on the following transformation



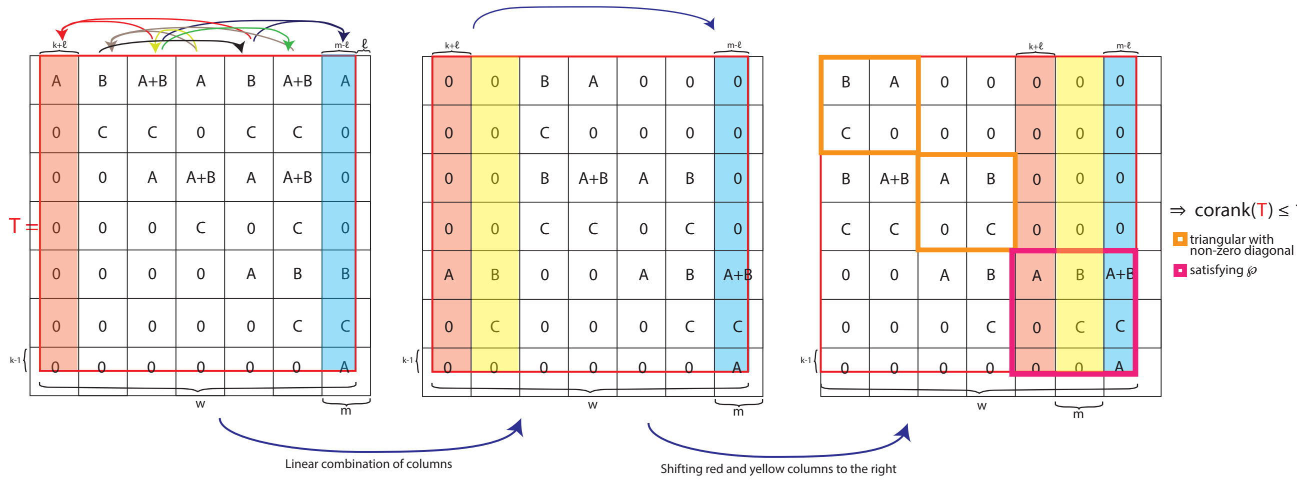
This results in a block triangular matrix with one block-triangular block that has two diagonal blocks of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ highlighted in bold orange that have non-zero determinants, and another block on the diagonal consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} A+B & A \\ C & 0 \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

2.7 $5m \leq w \leq 6m$ and $6m \leq w \leq 7m$: testing $\mathcal{P}$ on all 7x7 blocks (left to right):

Similarly, all our transformations only involve simple (block) column-wise operations. When the size of the considered submatrices $5m < w \leq 6m$, T only involves the first six blocks of rows. We apply the same column transformations as for the case $6m \leq w \leq 7m$, and conclude similarly.

- $$\begin{bmatrix} A & B & A+B & A & B & A+B & A \\ 0 & C & C & 0 & C & C & 0 \\ 0 & 0 & A & A+B & A & A+B & 0 \\ 0 & 0 & 0 & C & 0 & C & 0 \\ 0 & 0 & 0 & 0 & A & B & B \\ 0 & 0 & 0 & 0 & 0 & C & C \\ 0 & 0 & 0 & 0 & 0 & 0 & A \end{bmatrix}$$

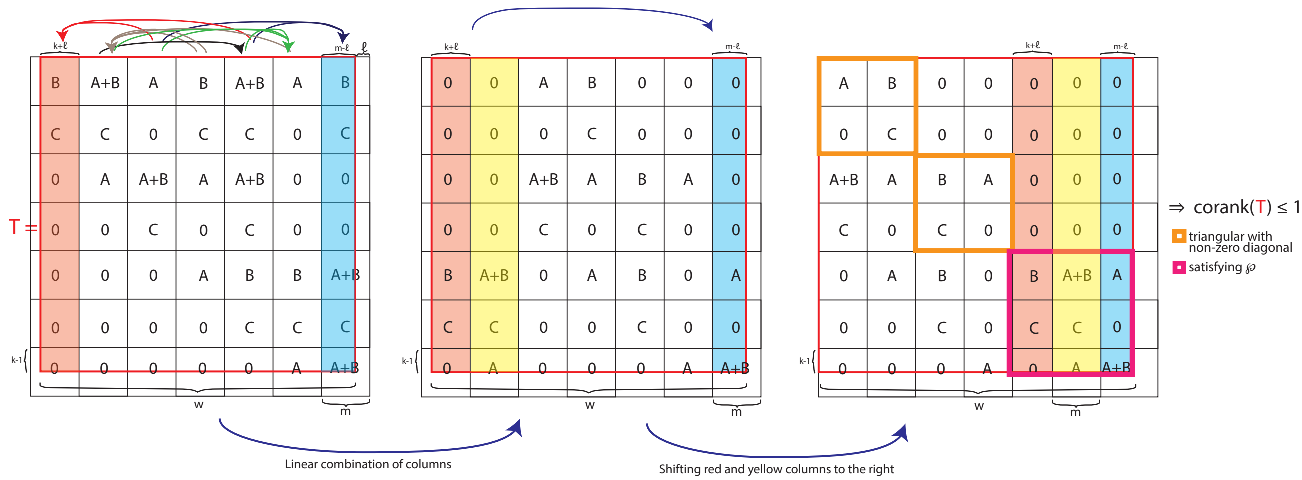
  When the size of the considered submatrices $6m < w \leq 7m$, we rely on the following transformation

  This results in a block triangular matrix with one block-triangular block that has two diagonal blocks of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ (up to permutations) highlighted in bold orange that have non-zero determinants, and another block on the diagonal consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} A & B & A+B \\ 0 & C & C \\ 0 & 0 & A \end{bmatrix}$ highlighted in bold magenta for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

- $$\begin{bmatrix} B & A+B & A & B & A+B & A & B \\ C & C & 0 & C & C & 0 & C \\ 0 & A & A+B & A & A+B & 0 & 0 \\ 0 & 0 & C & 0 & C & 0 & 0 \\ 0 & 0 & 0 & A & B & B & A+B \\ 0 & 0 & 0 & 0 & C & C & C \\ 0 & 0 & 0 & 0 & 0 & A & A+B \end{bmatrix}$$

  When the size of the considered submatrices $6m < w \leq 7m$, we rely on the following transformation



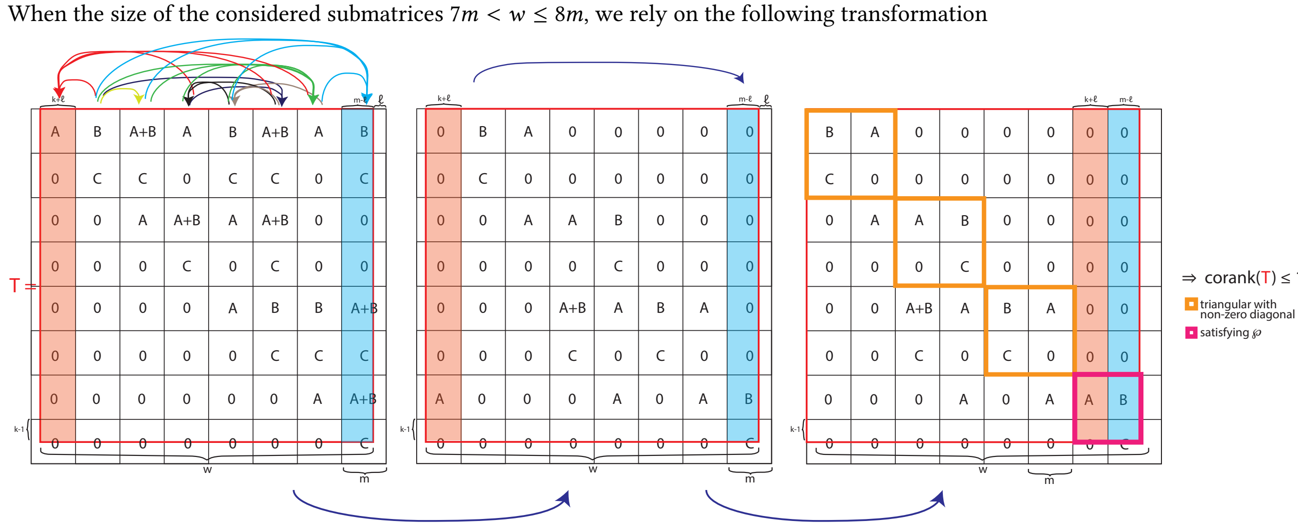
This results in a block triangular matrix with one block-triangular block that has two diagonal blocks of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ (up to permutations) highlighted in bold orange that have non-zero determinants, and another block on the diagonal consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} B & A+B & A \\ C & C & 0 \\ 0 & A & A+B \end{bmatrix}$ highlighted in bold magenta for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

2.8 $6m \leq w \leq 7m$ and $7m \leq w \leq 8m$: testing $\mathcal{P}$ on the entire matrix:

Similarly, all our transformations only involve simple (block) column-wise operations. When the size of the considered submatrices $6m < w \leq 7m$, T only involves the first seven blocks of rows. We apply the same column transformations as for the case $7m \leq w \leq 8m$, and conclude similarly.

- $$\begin{bmatrix} A & B & A+B & A & B & A+B & A & B \\ 0 & C & C & 0 & C & C & 0 & C \\ 0 & 0 & A & A+B & A & A+B & 0 & 0 \\ 0 & 0 & 0 & C & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & A & B & B & A+B \\ 0 & 0 & 0 & 0 & 0 & C & C & C \\ 0 & 0 & 0 & 0 & 0 & 0 & A & A+B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C \end{bmatrix}$$

When the size of the considered submatrices $7m < w \leq 8m$, we rely on the following transformation



This results in a block triangular matrix with one block-triangular block that has three diagonal blocks of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ (up to permutations) highlighted in bold orange that have non-zero determinants, and another block on the diagonal consisting of a top-anchored rectangular submatrix of $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ highlighted in bold magenta, for which $\mathcal{P}$ holds by hypothesis, and has thus a corank lower or equal to 1. This concludes the proof for this case.

References

Carl D Meyer, Jr. Generalized inverses and ranks of block matrices. *SIAM Journal on Applied Mathematics*, 25(4):597–602, 1973.