SLICED PARTIAL OPTIMAL TRANSPORT

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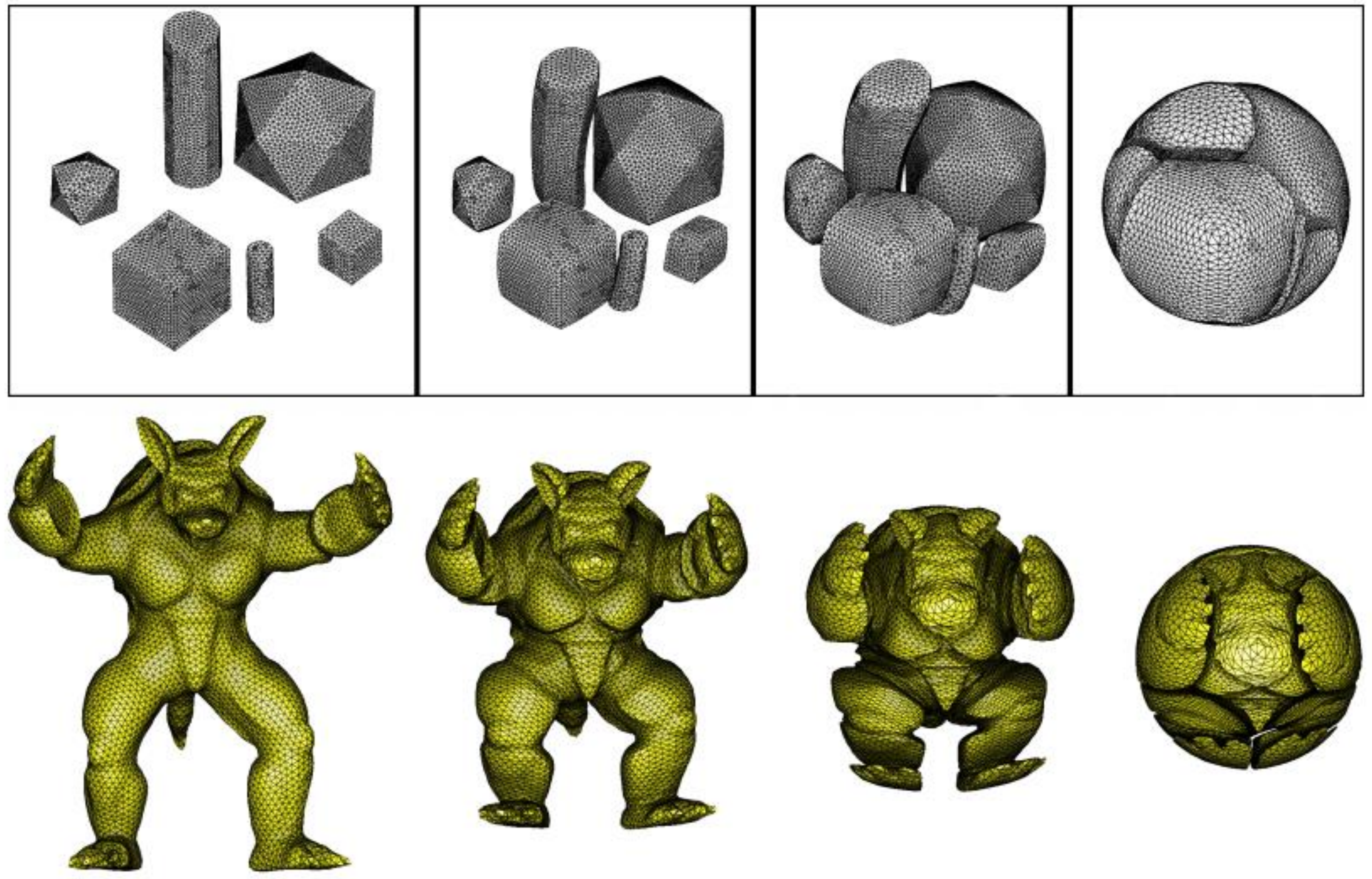
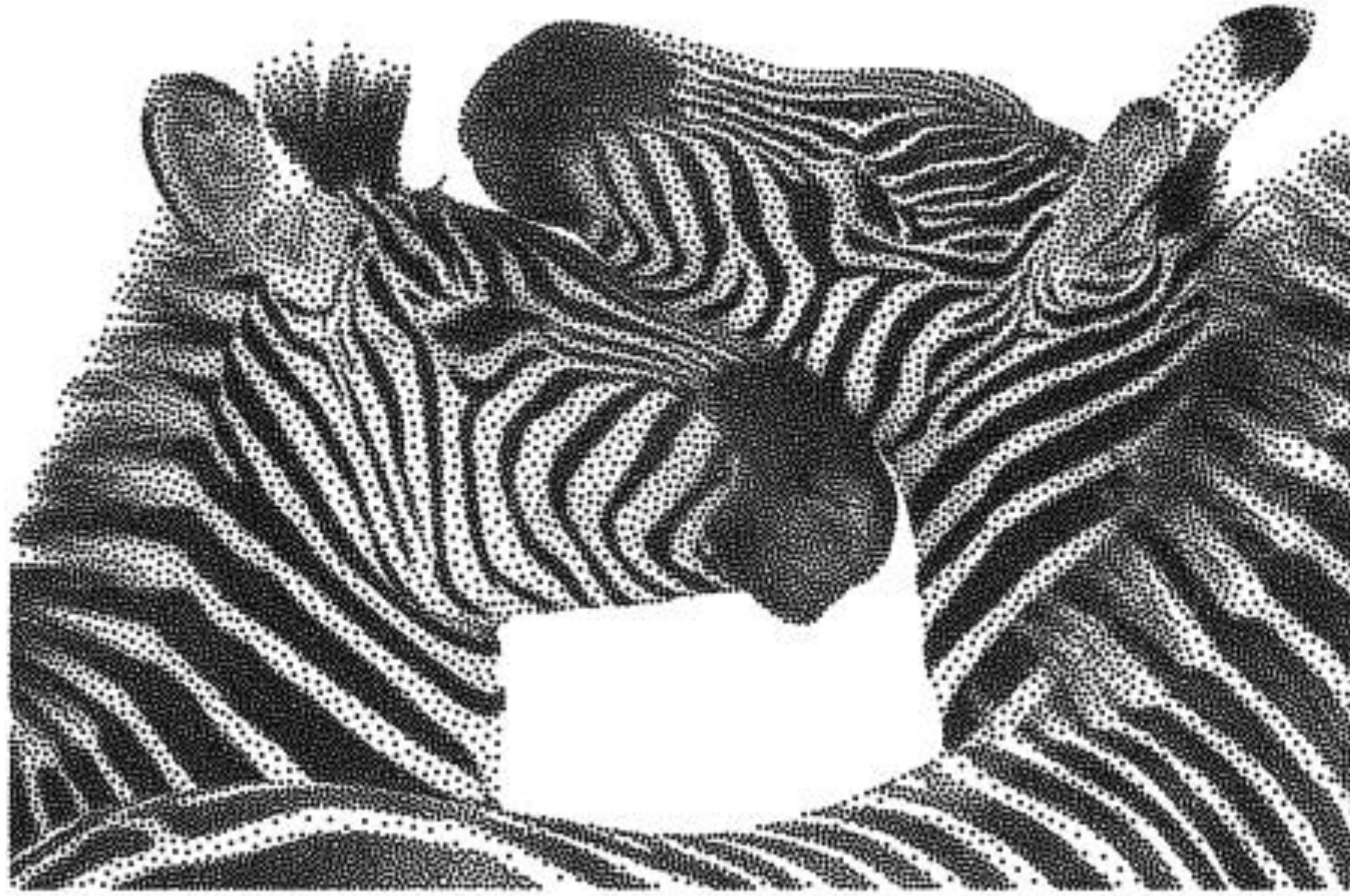
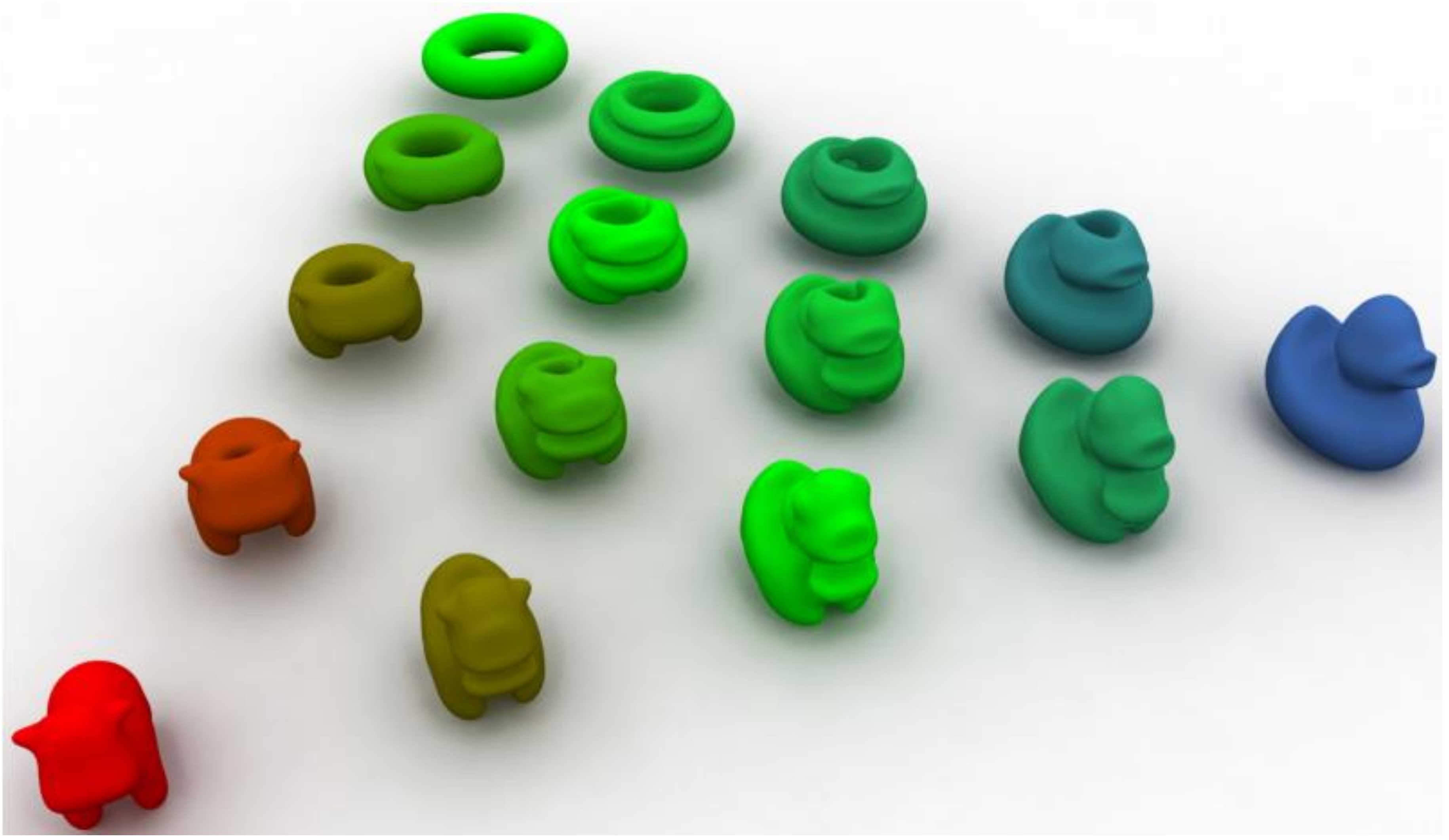
*CNRS, Univ. Lyon.



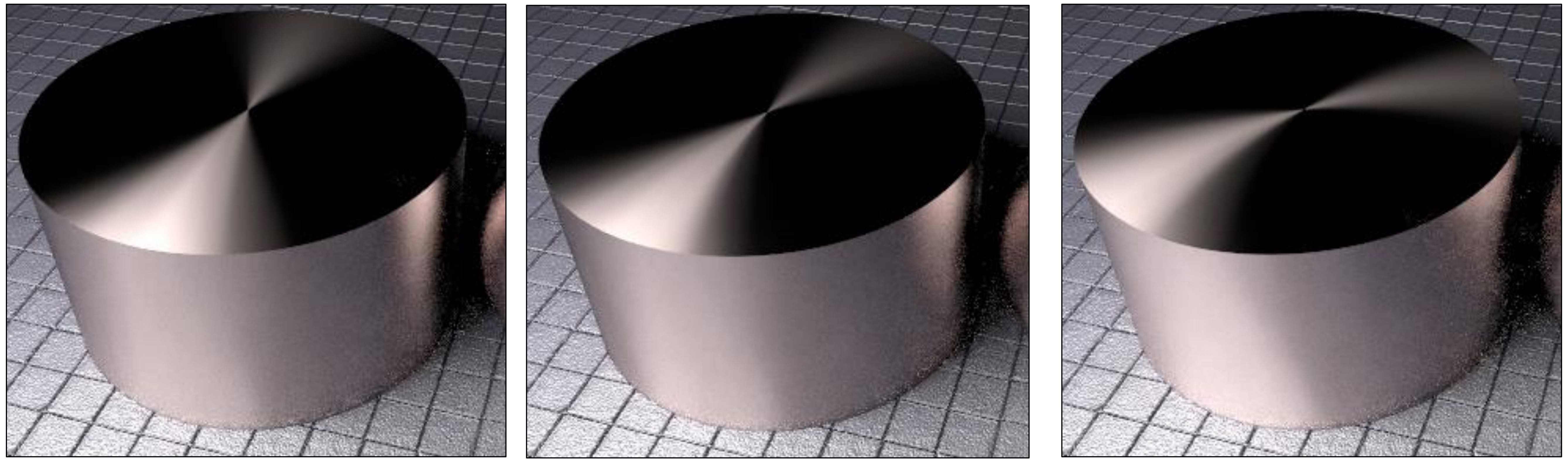
[Levy 2015]



[de Goes et al. 2012]



[Solomon et al. 2015]



[Bonneel et al. 2011]

Matching points with optimal transport

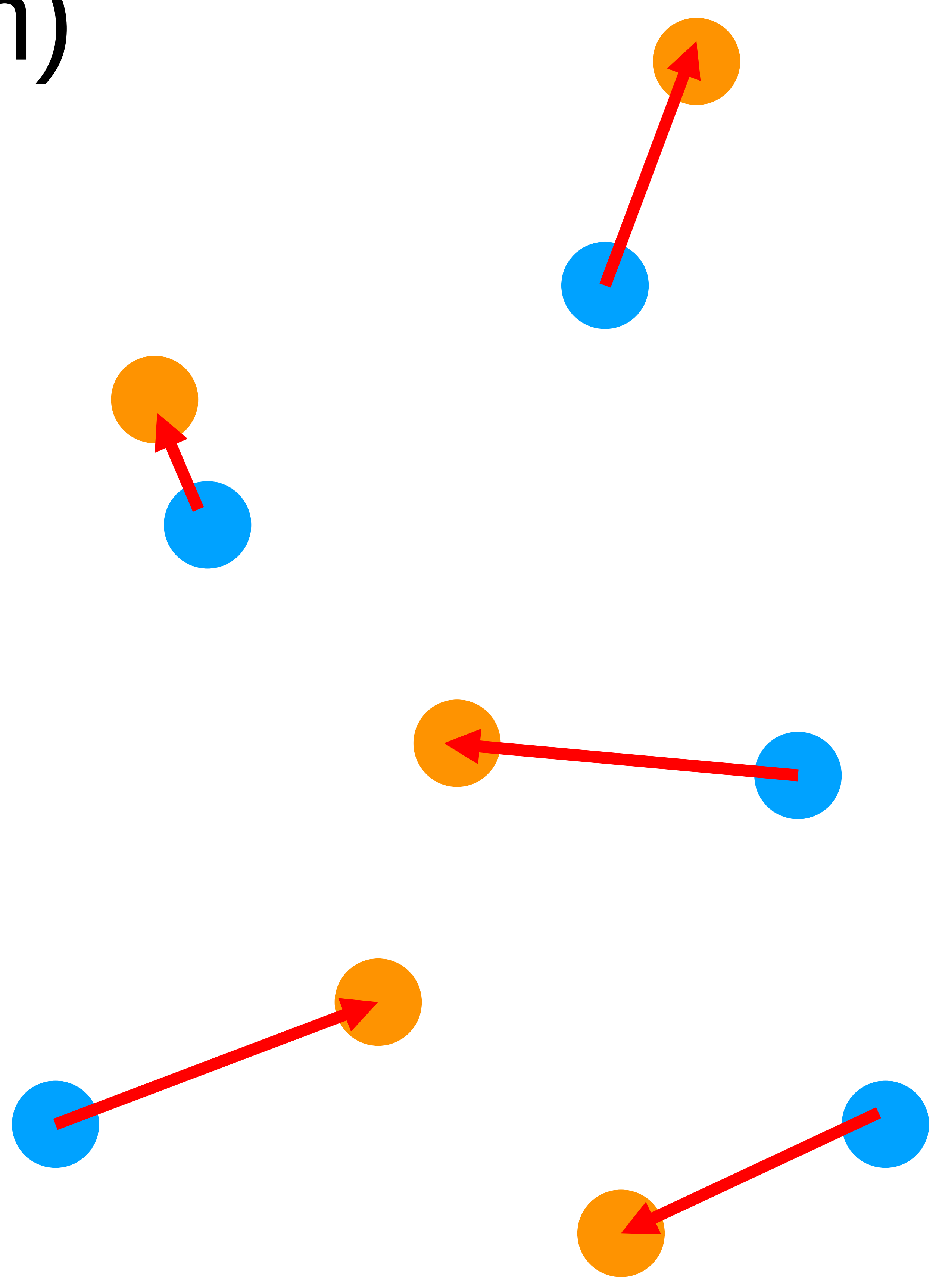
- Monge (Linear Assignment Problem)

$$\min_{T \text{ bijective}} \sum_i c(x_i, y_{T(i)})$$

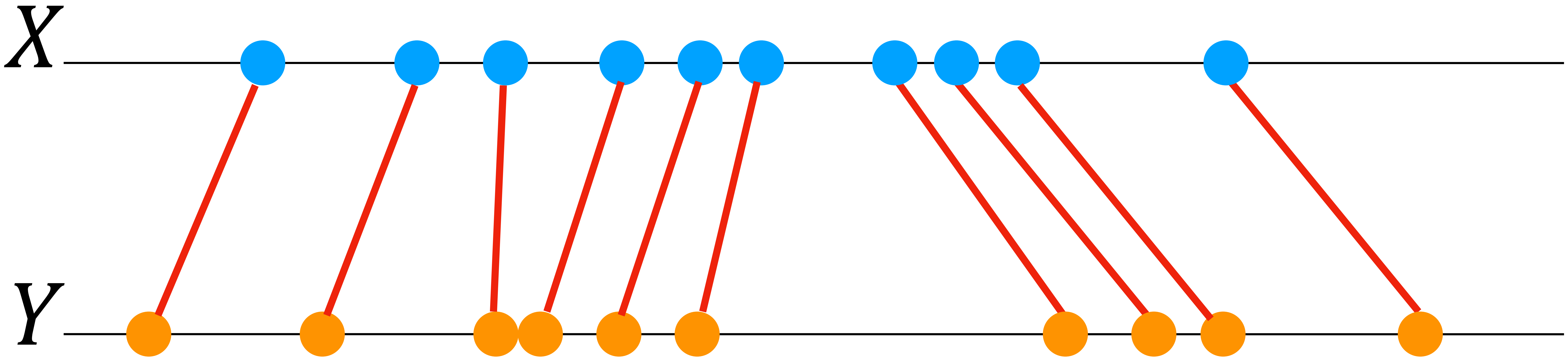
- Kantorovich

$$W(f,g) = \min \sum_{i,j} c_{i,j} \pi_{i,j}$$

s.t.
$$\sum_j \pi_{i,j} = 1$$
$$\sum_i \pi_{i,j} = 1$$
$$\pi_{i,j} \geq 0$$

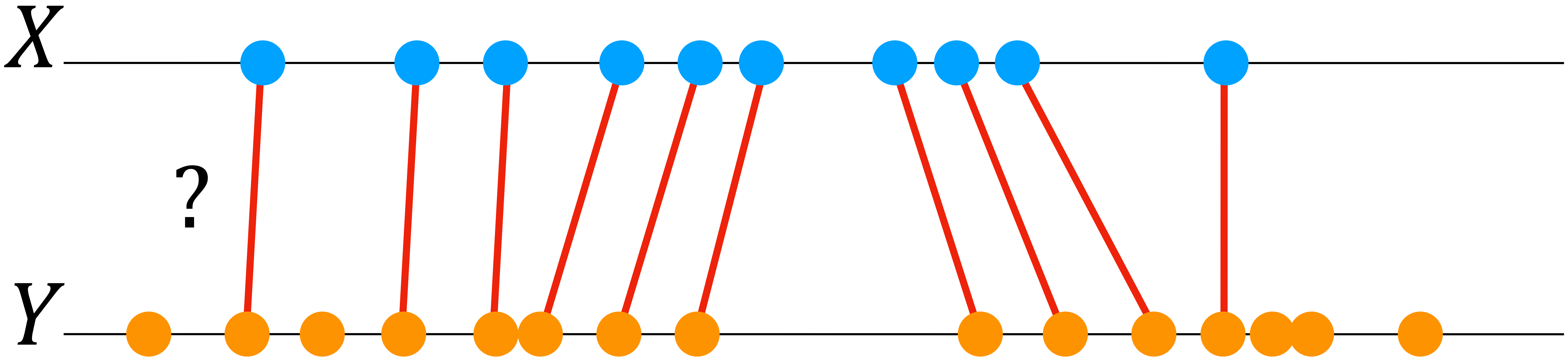


1-d Linear Assignment Problem is trivial*



*assuming the cost c is a convex function of |x-y|

Partial optimal assignment ?

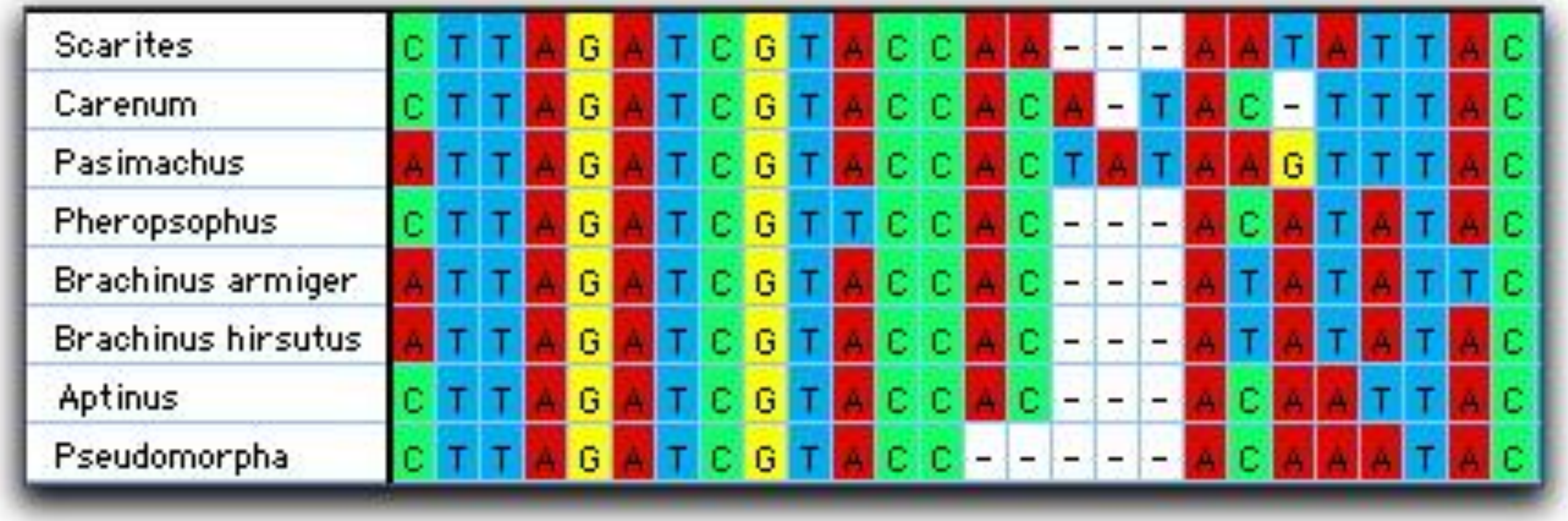


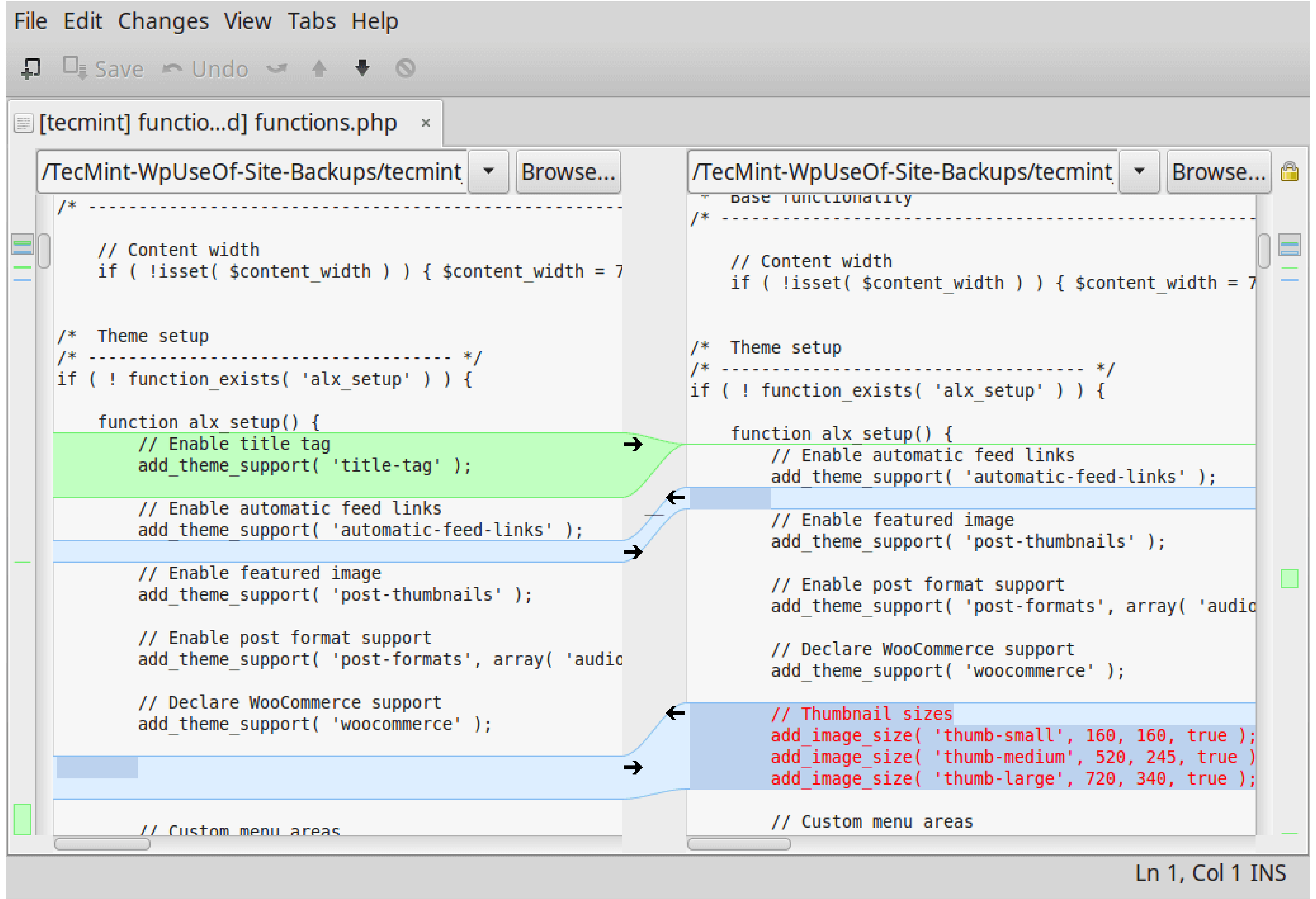
$$W(f,g) = \min \sum_{i,j} c_{i,j}\pi_{i,j} \quad \textbf{s.t.} \quad \sum_j \pi_{i,j} = 1, \quad \sum_i \pi_{i,j} \leq 1, \quad \pi_{i,j} \geq 0$$

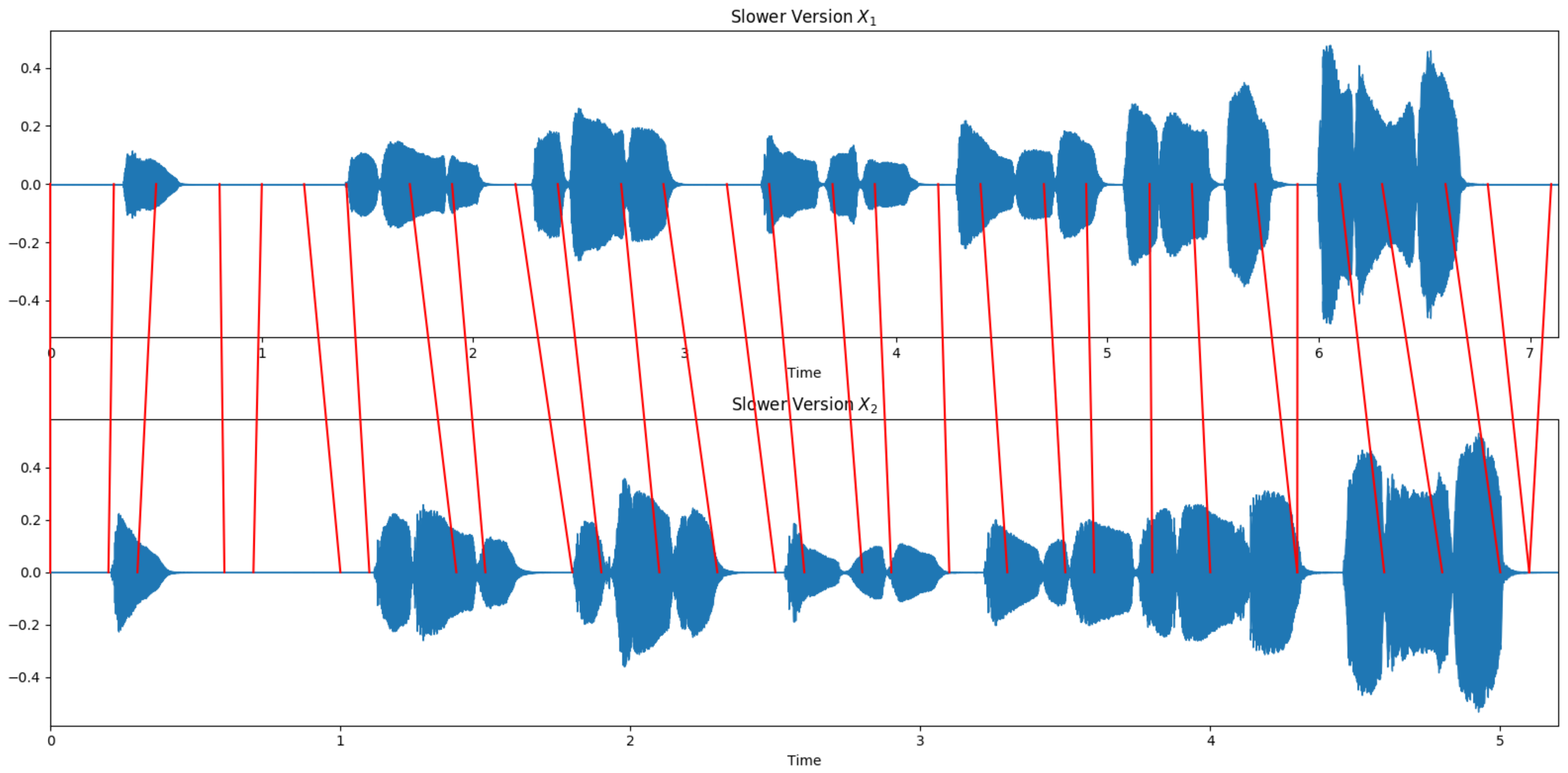
$$\min_{T \text{ injective}} \sum_i c(x_i, y_{T(i)})$$

Similar problems

- DNA sequence alignment
- Text alignment
- Music synchronization
- …

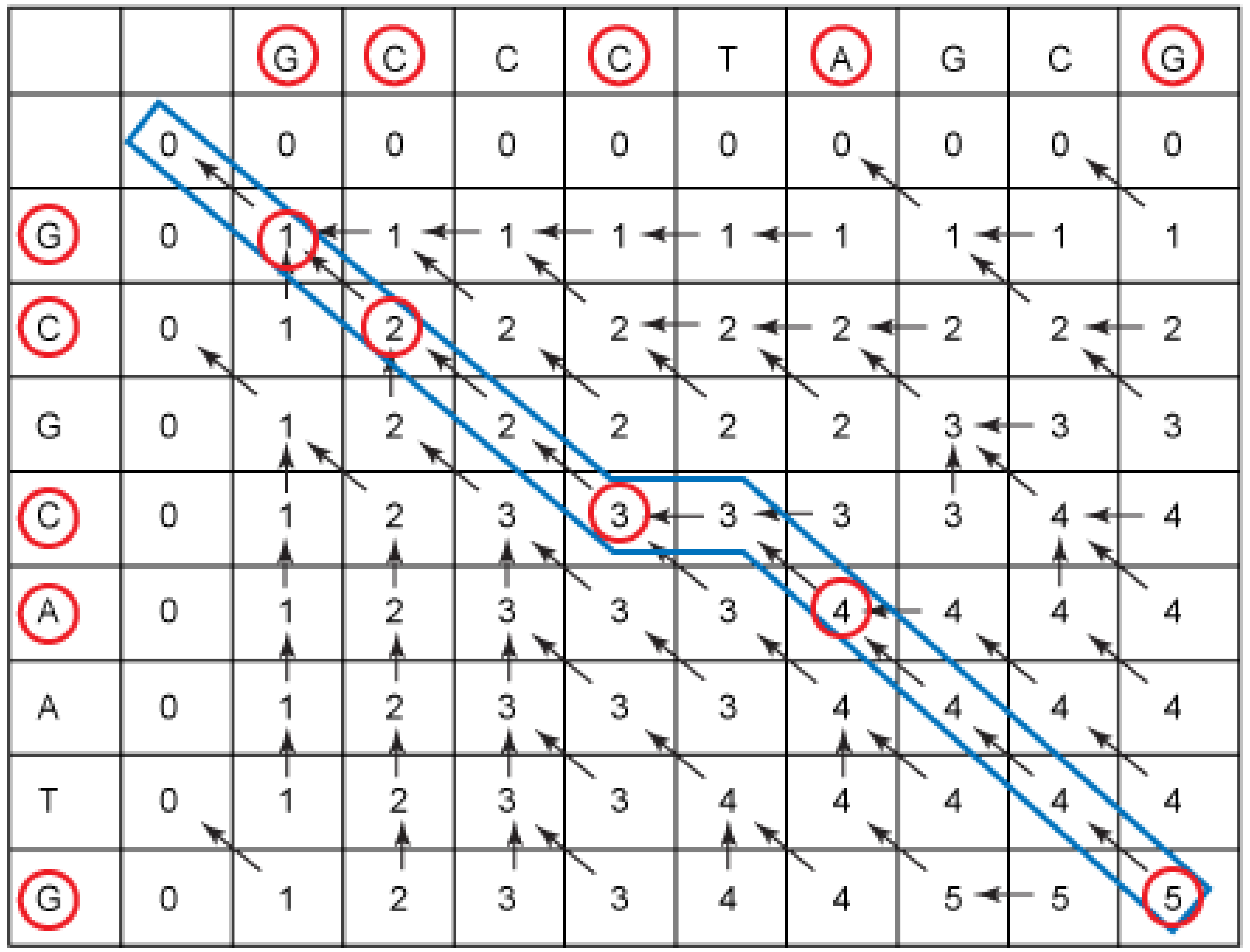






Existing solutions

- Dynamic Time Warping
  - Solves a dynamic programming problem
  - Smith–Waterman algorithm, Needleman–Wunsch algorithm $O(N^2)$ space and time
  - Hirschberg's algorithm $O(N^2)$ time, $O(N)$ space
- All end up doing variants of
  - $A_{i,j} = \min(A_{i-1,j-1} + cost, A_{i-1,j} + cost', A_{i,j-1} + cost'')$



Quadratic time complexity algorithm (linear space)

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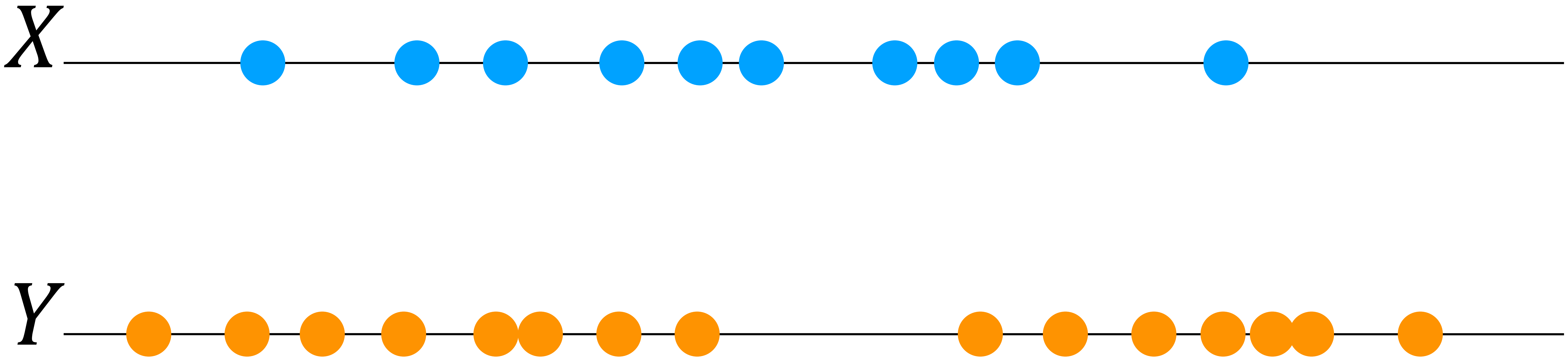

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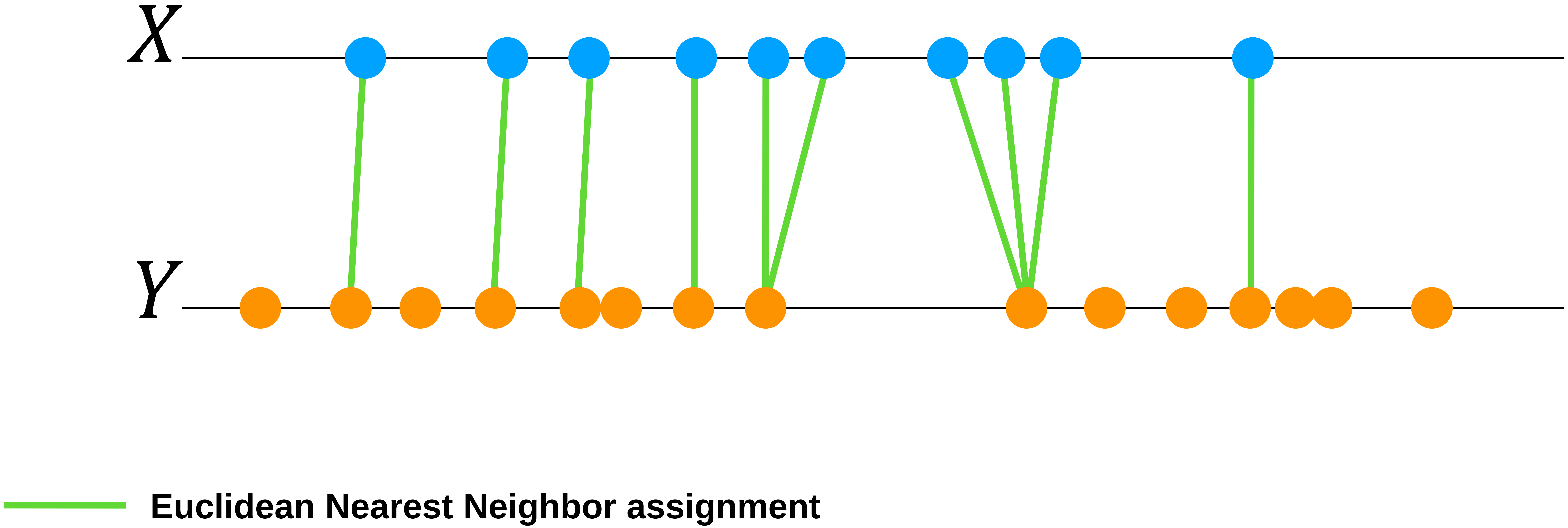

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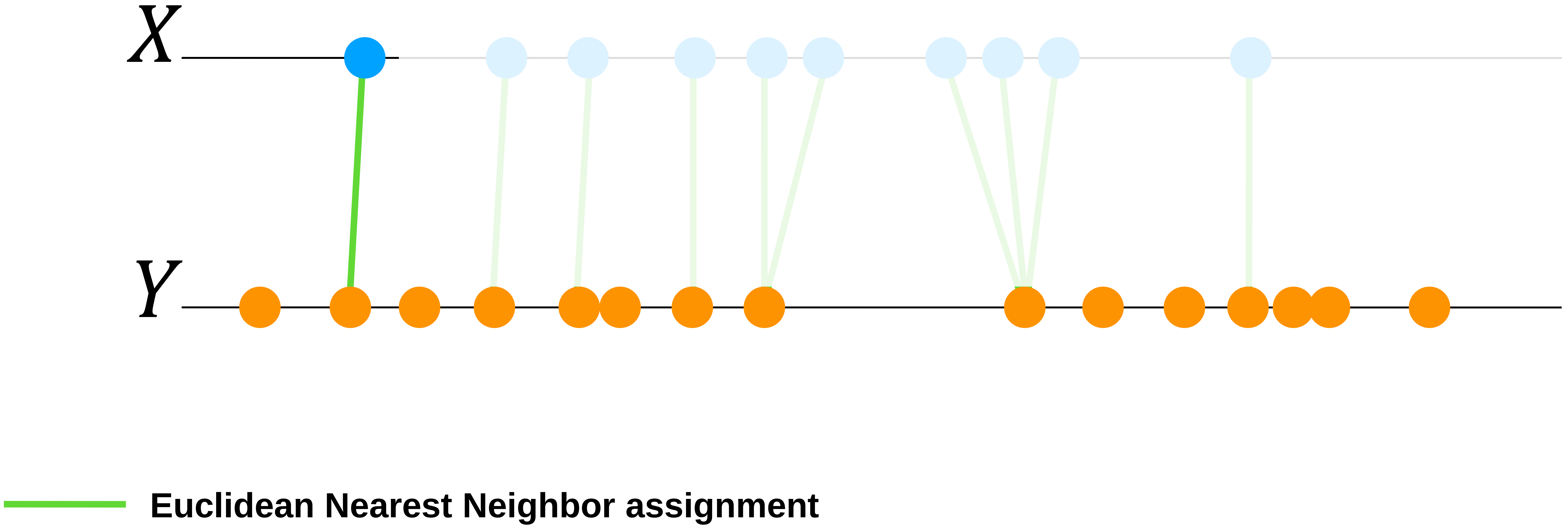

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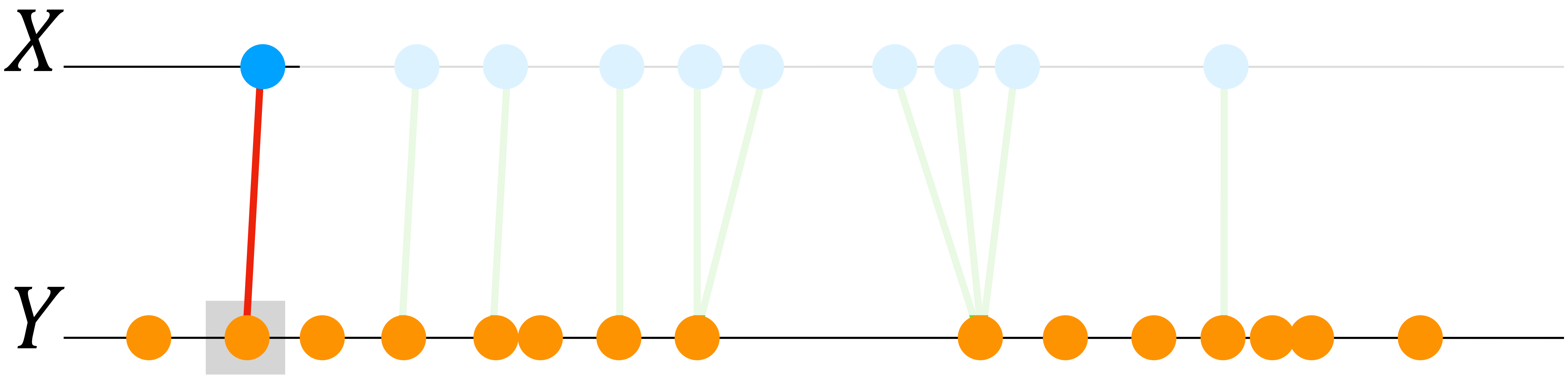
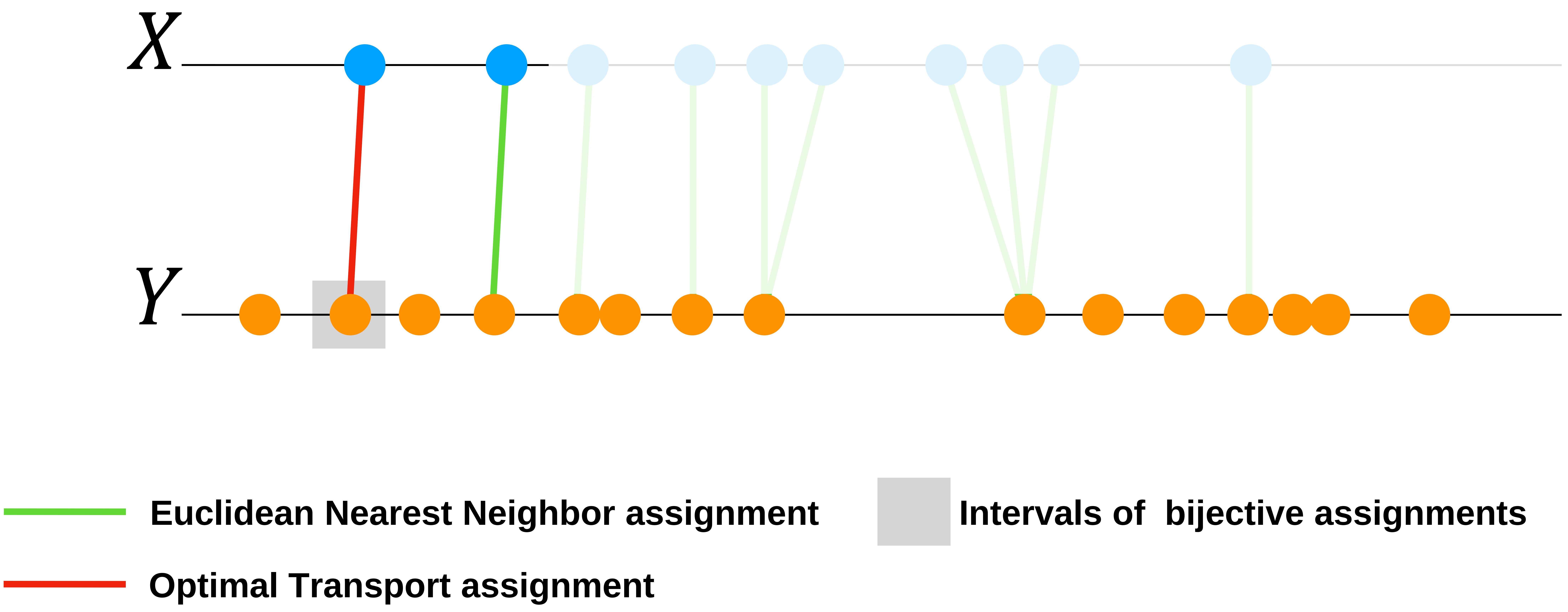


Euclidean Nearest Neighbor assignment

Intervals of bijective assignments

Optimal Transport assignment

Quadratic time complexity algorithm (linear space)



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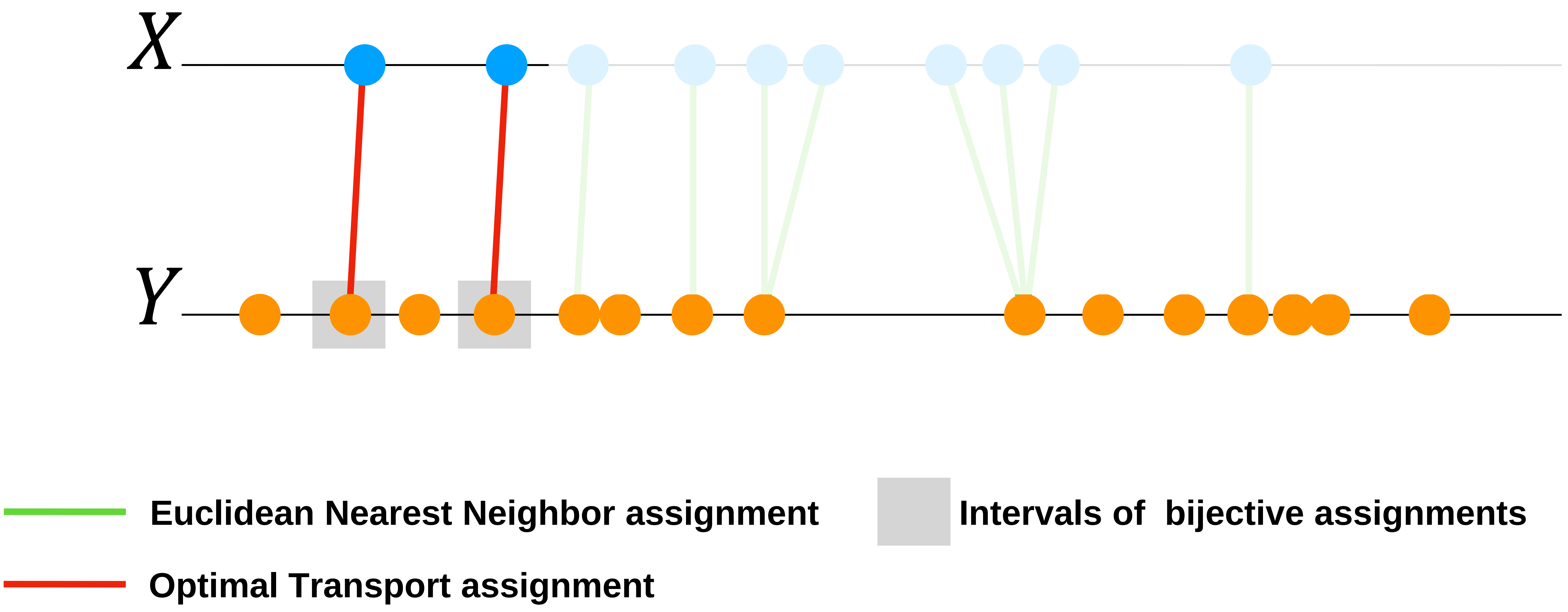

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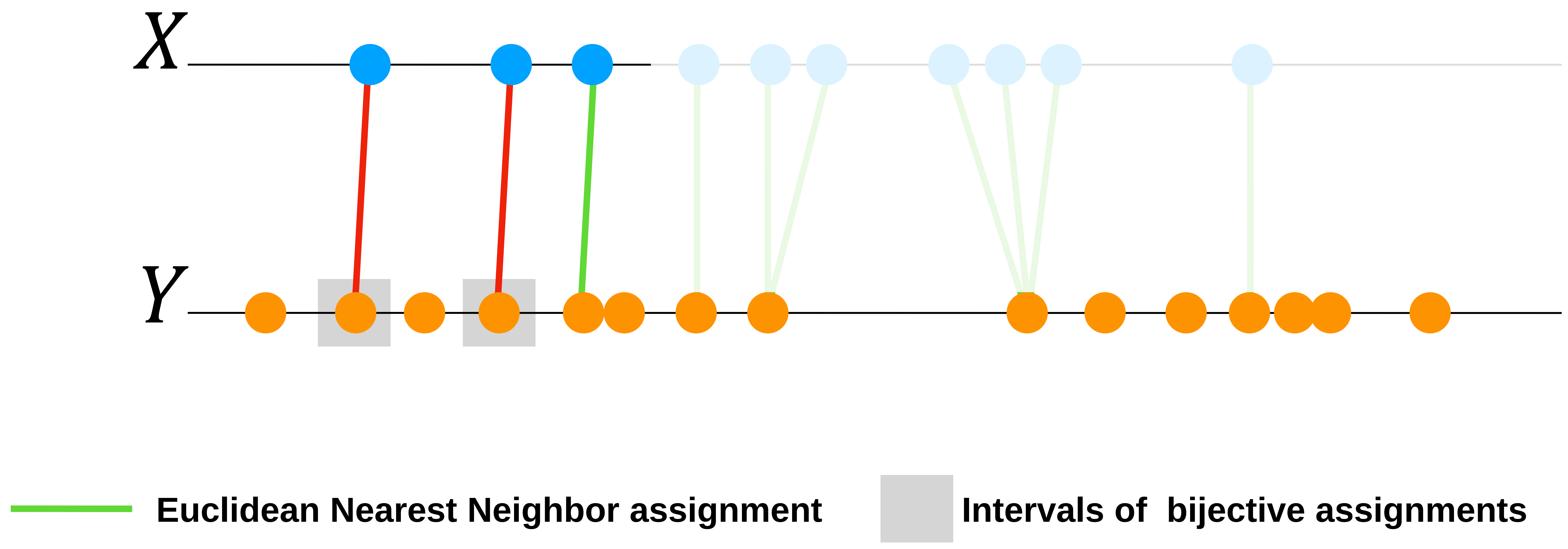

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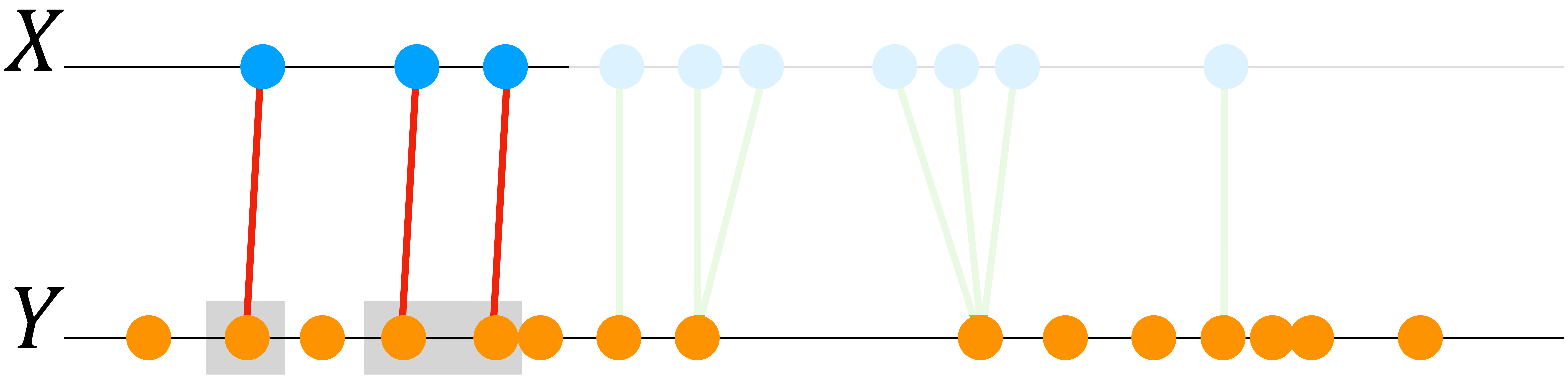

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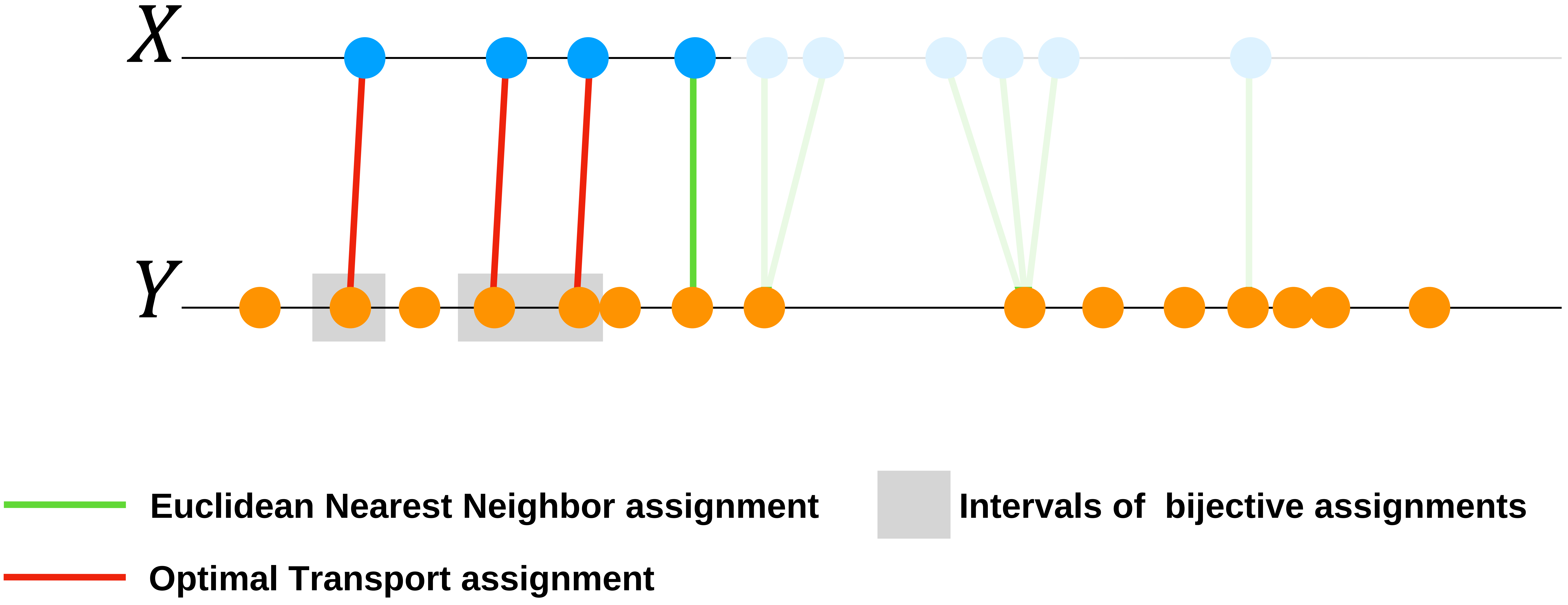

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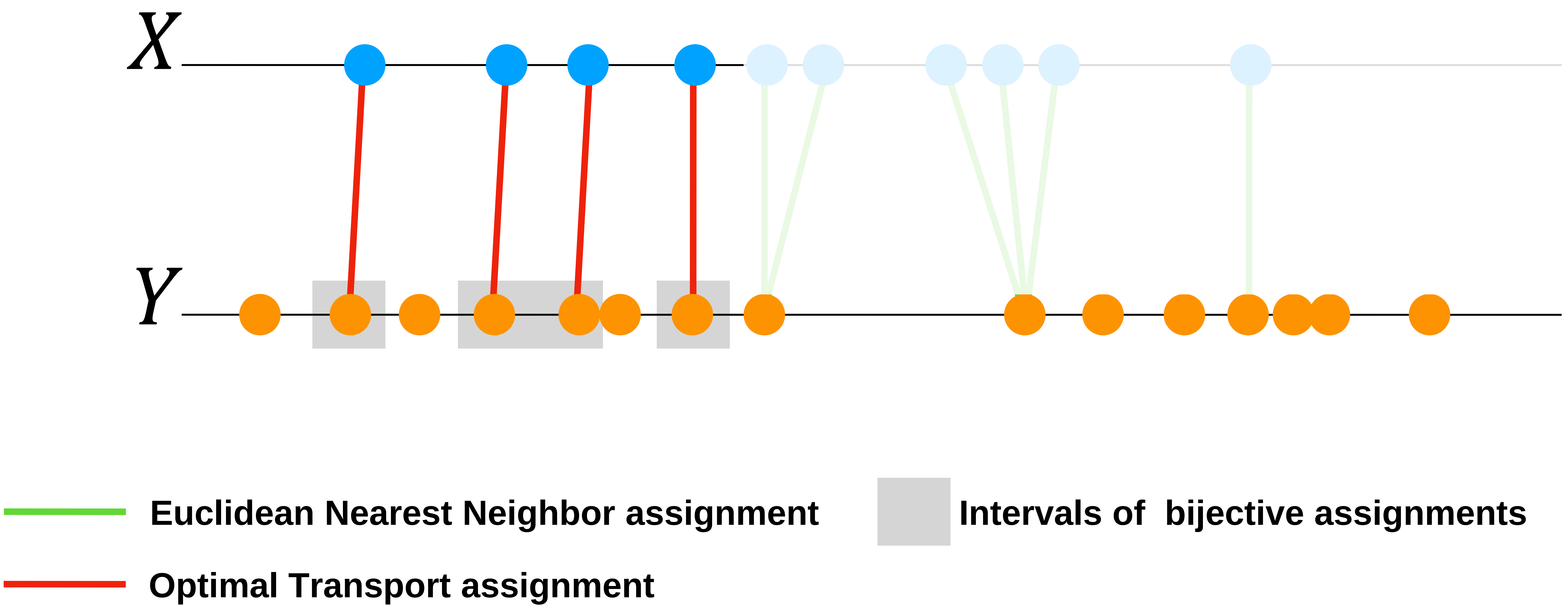

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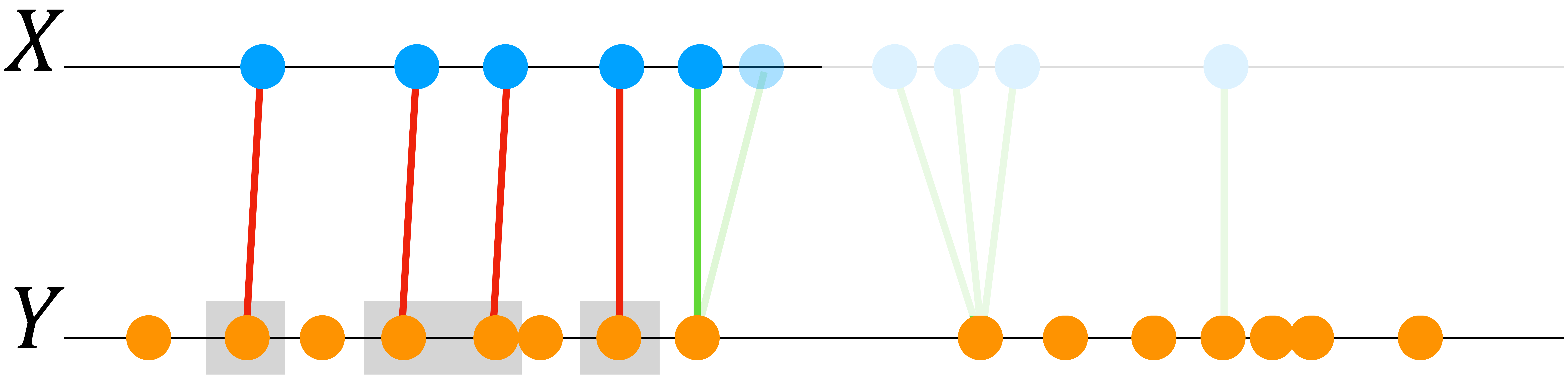

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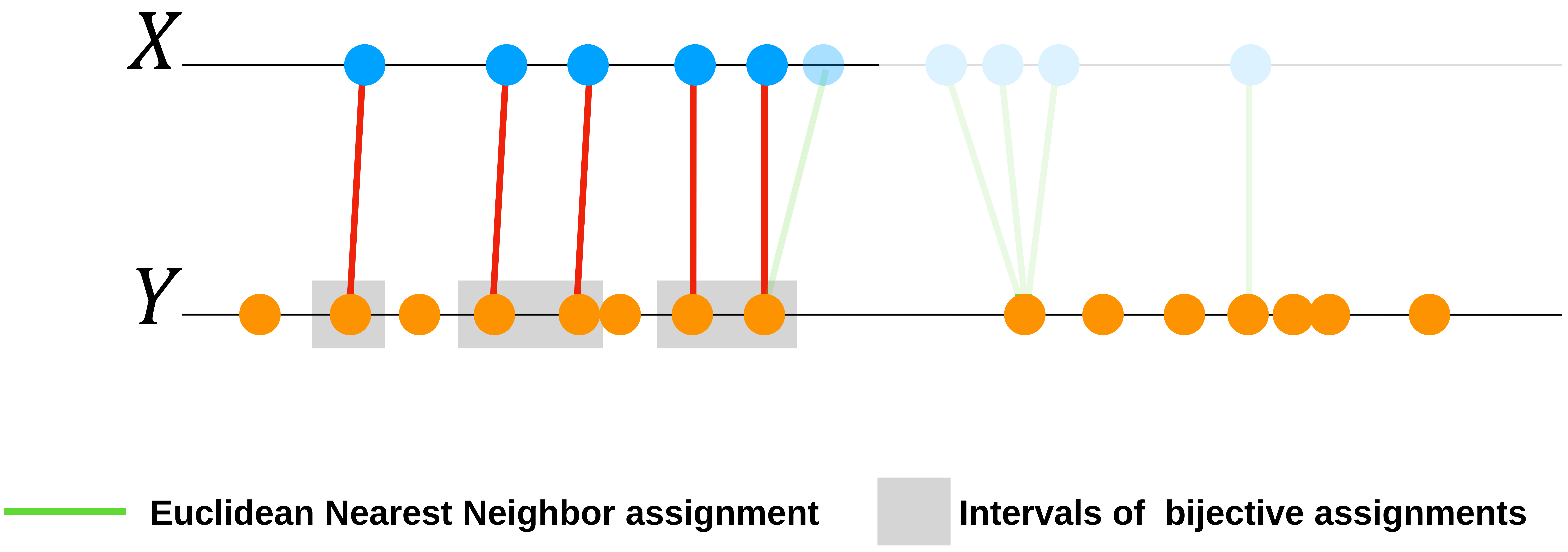

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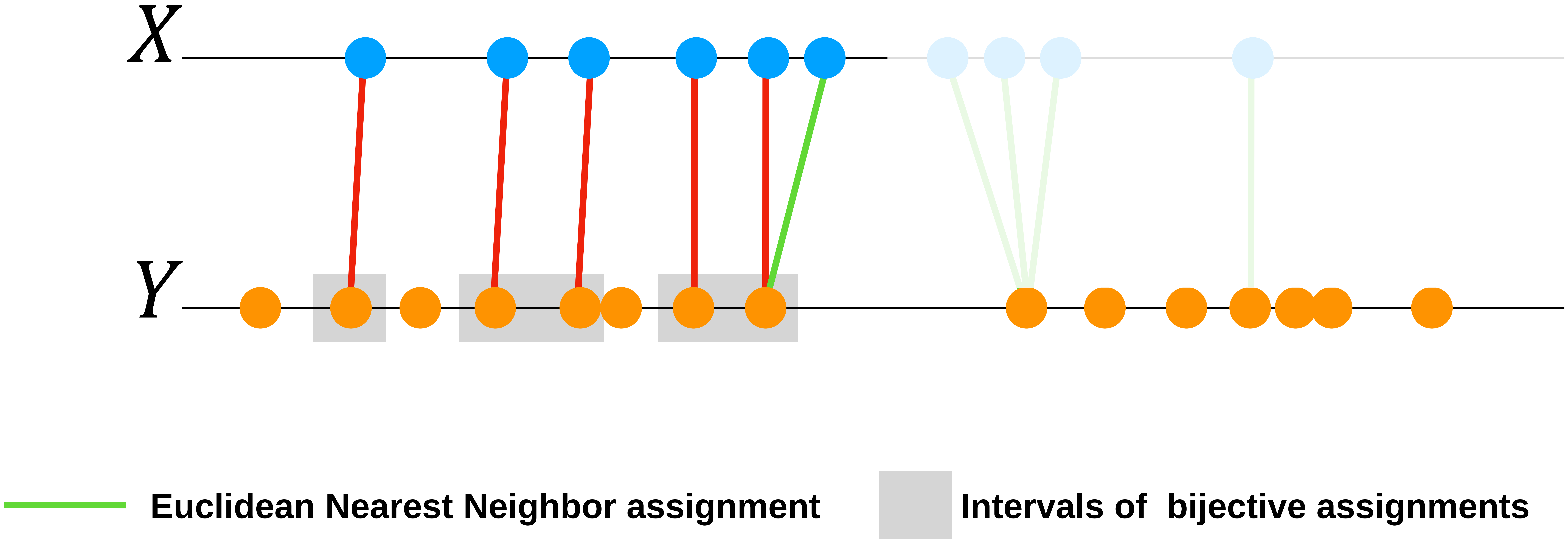

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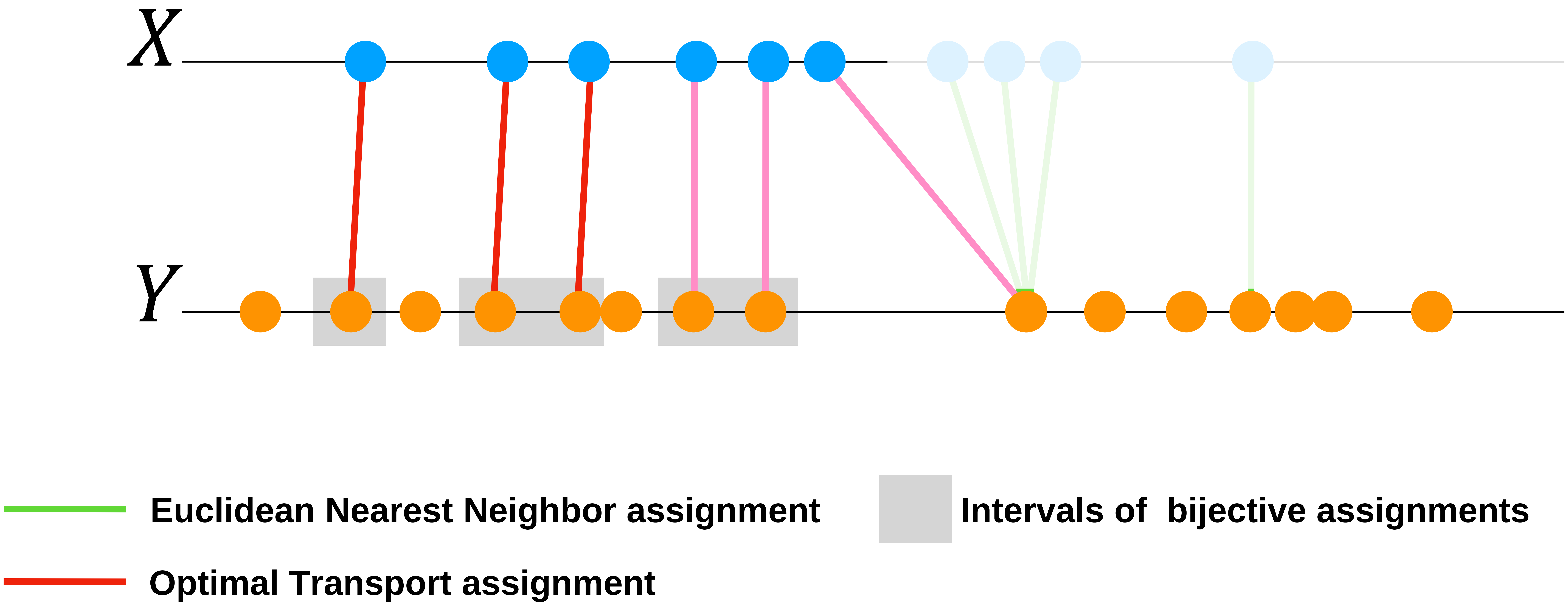

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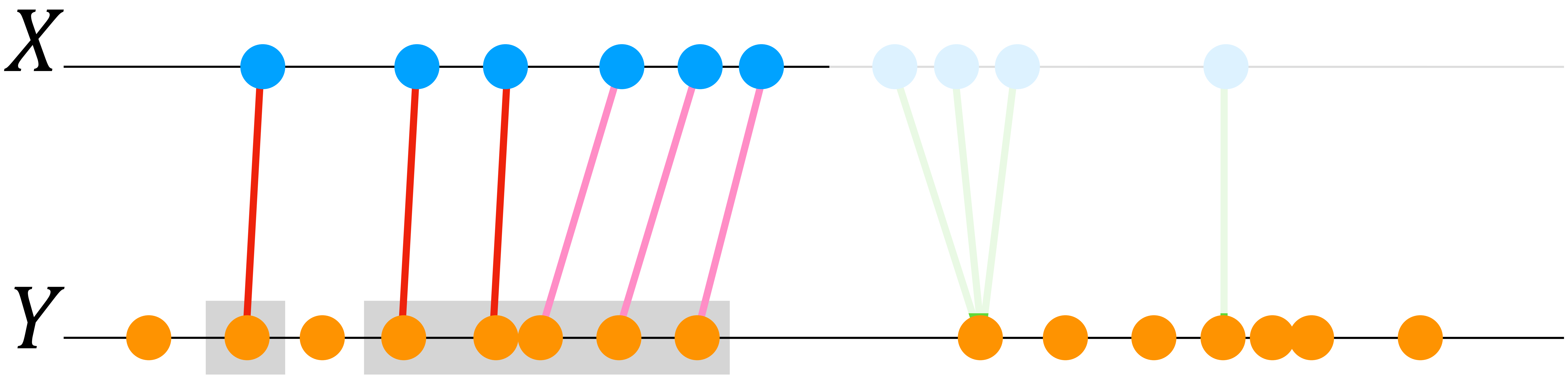

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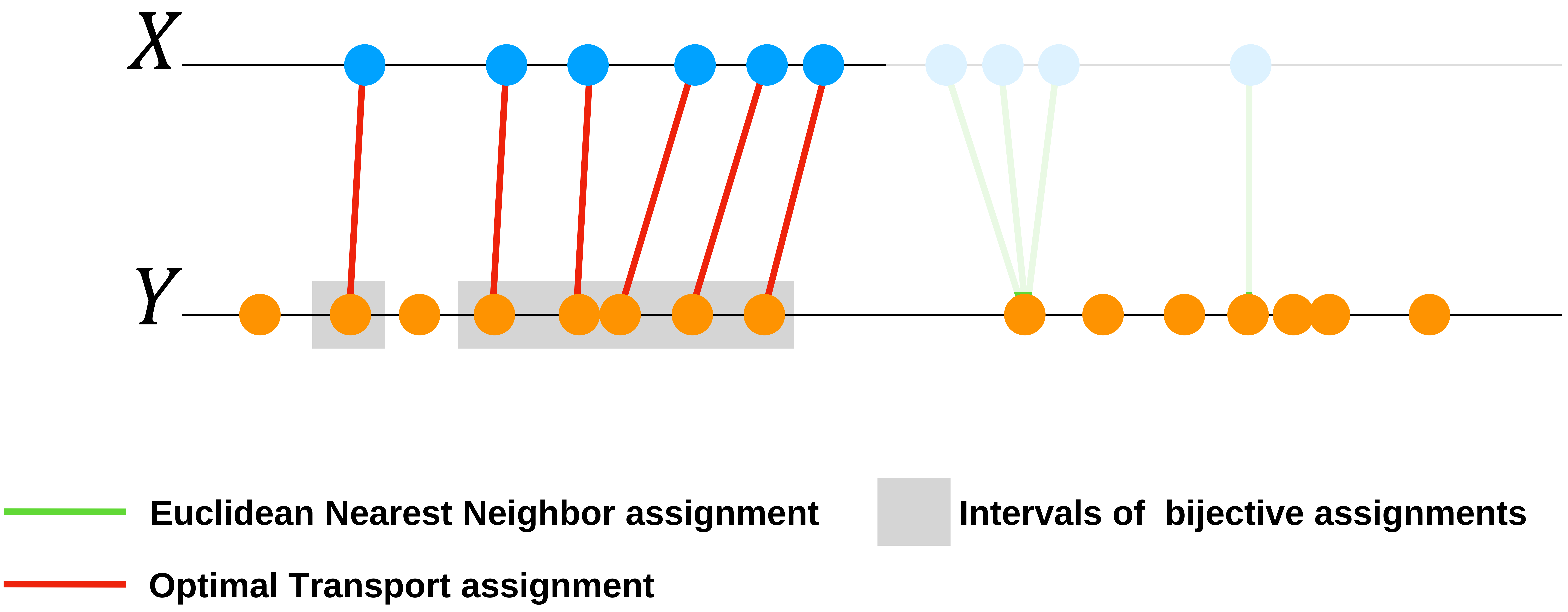

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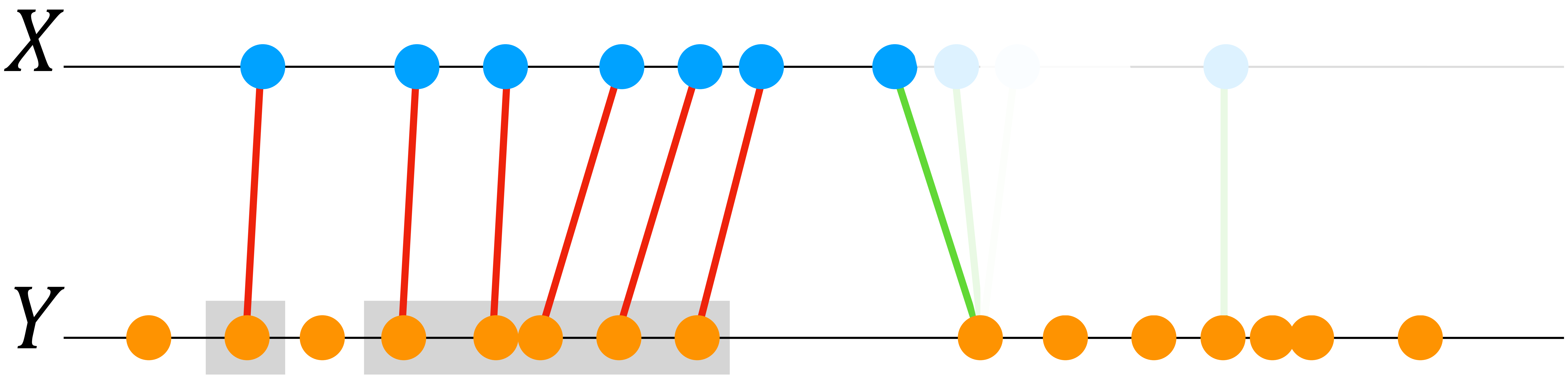

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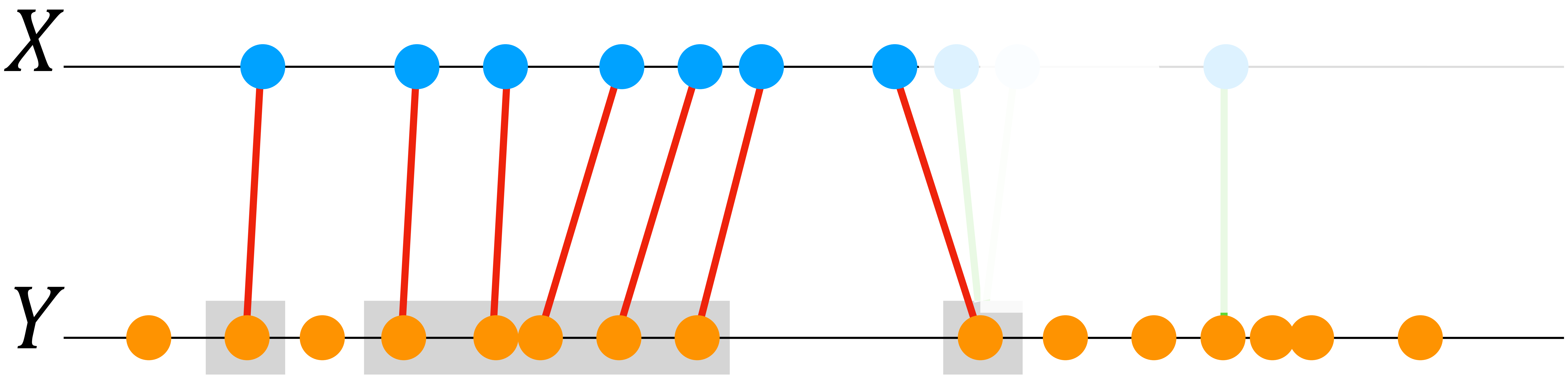

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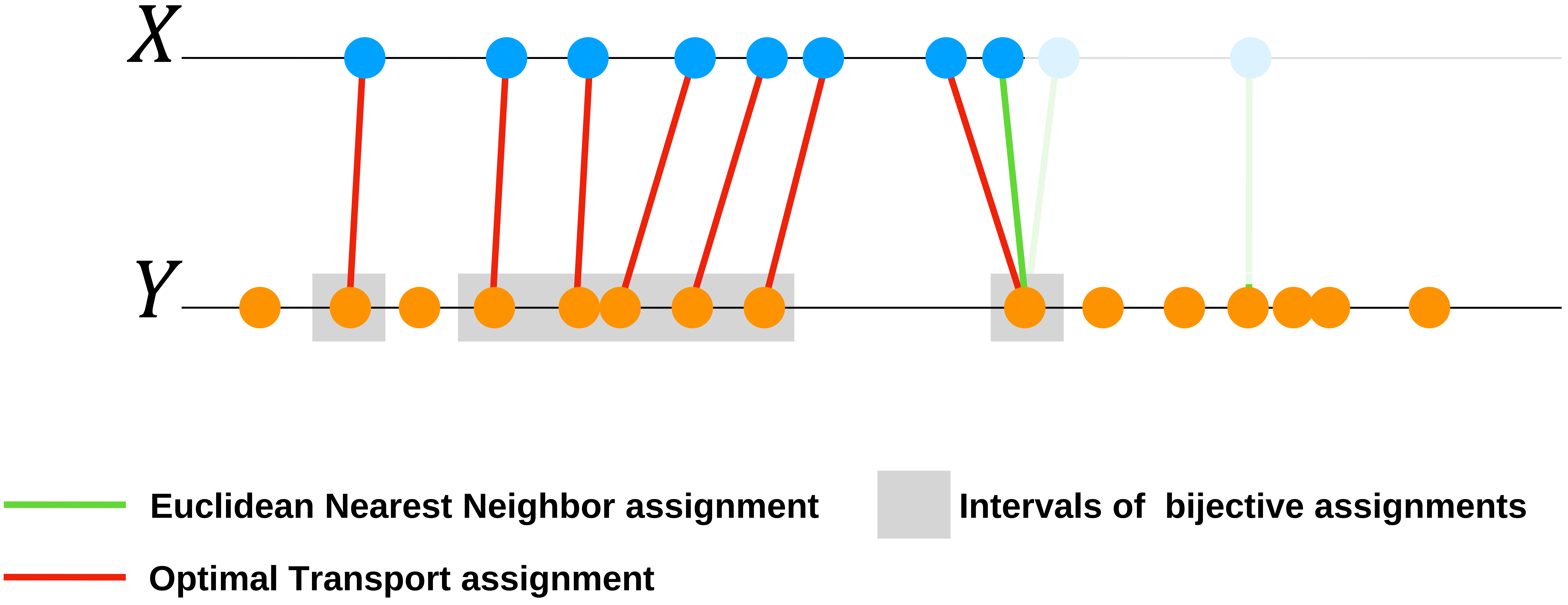

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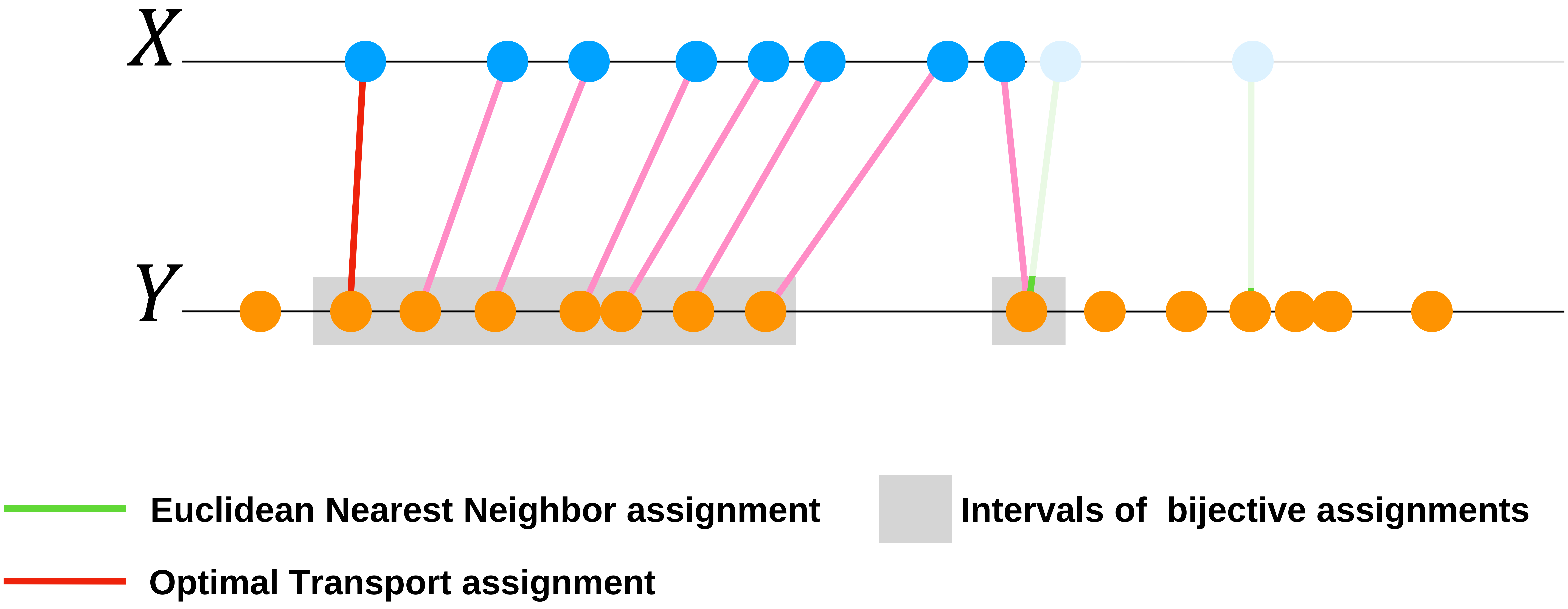

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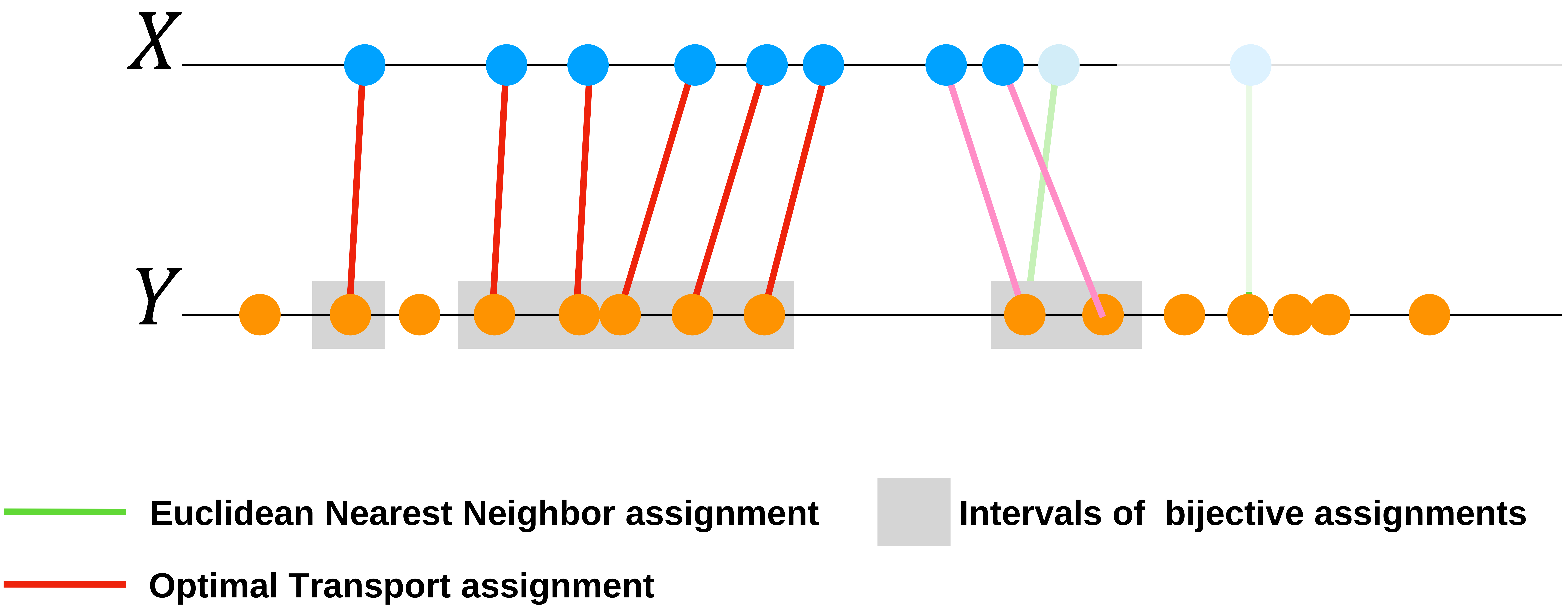

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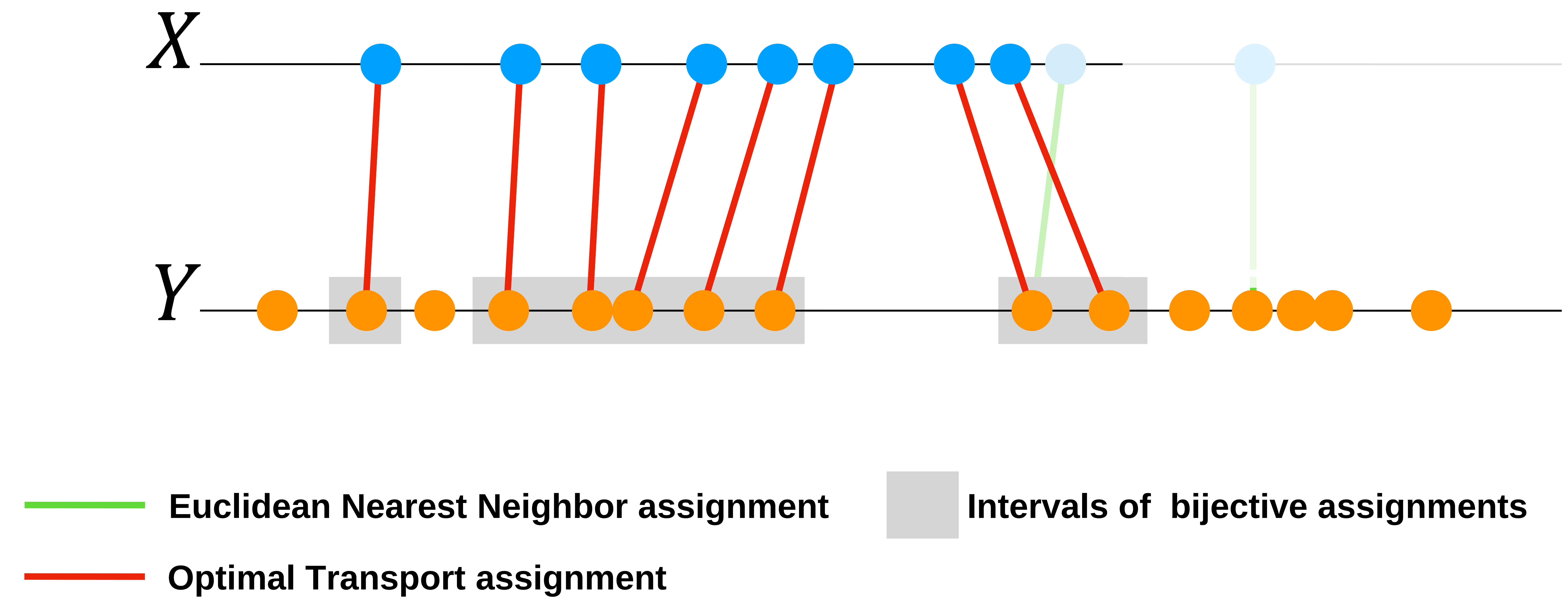

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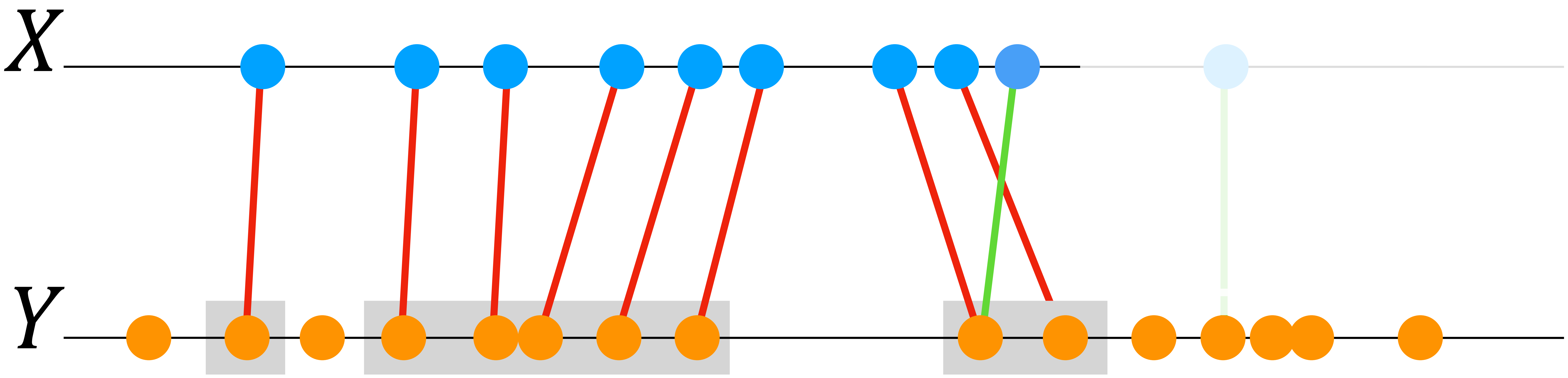

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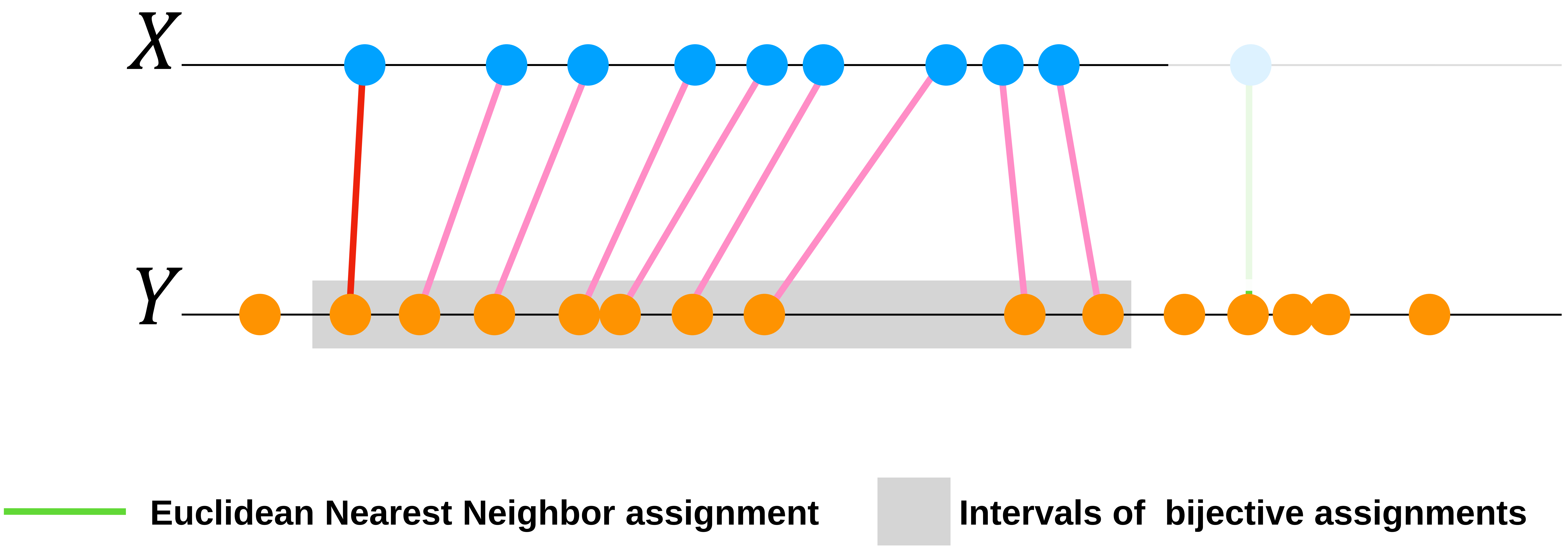

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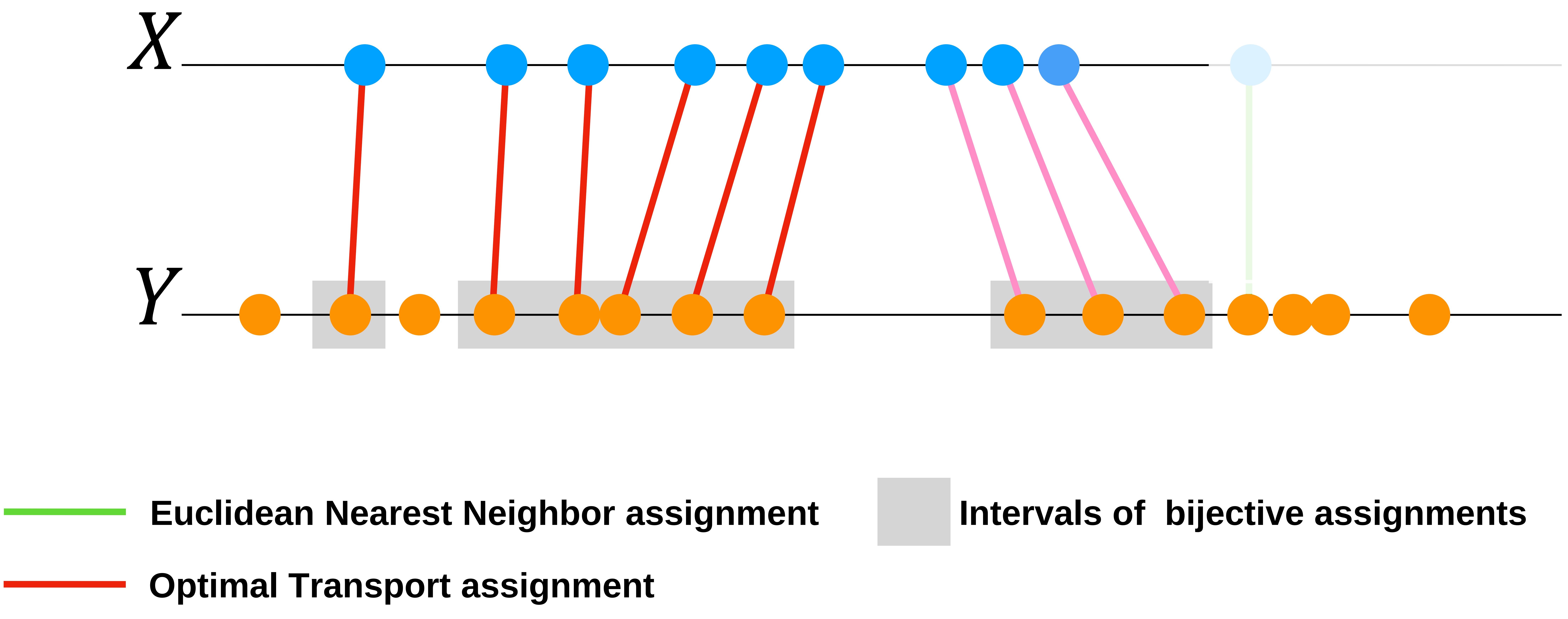

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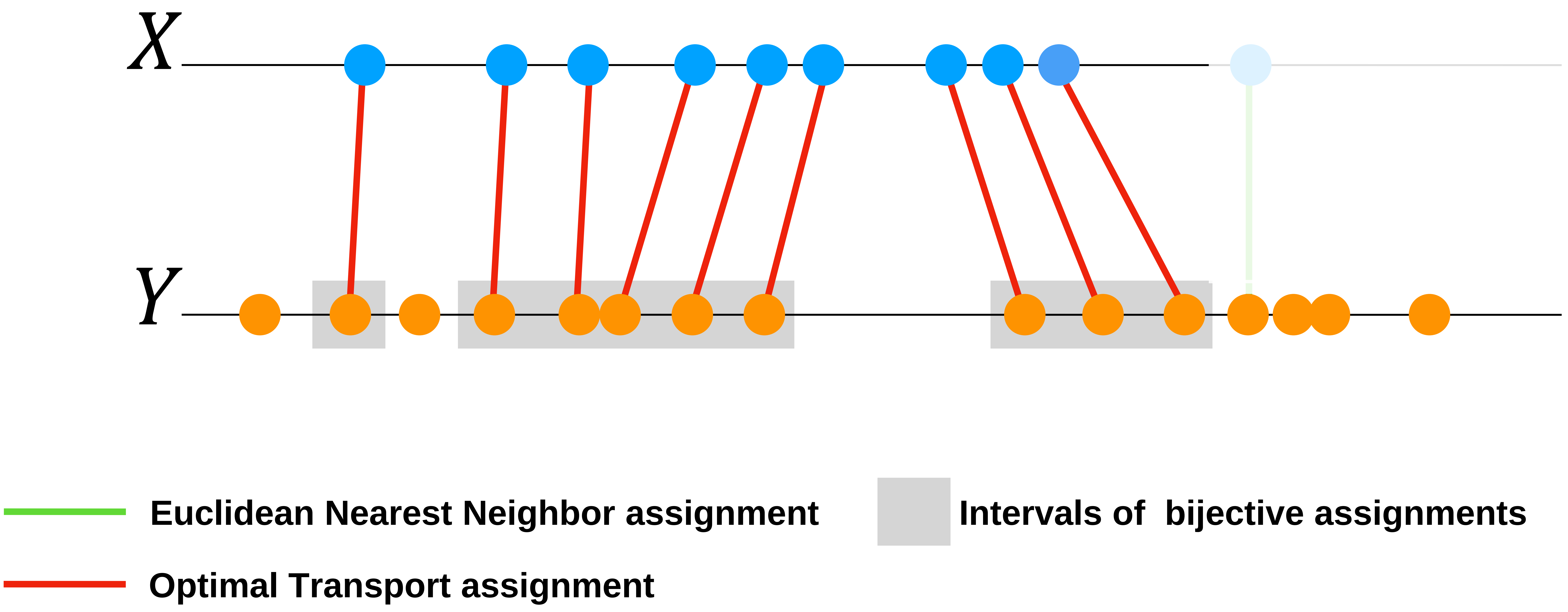

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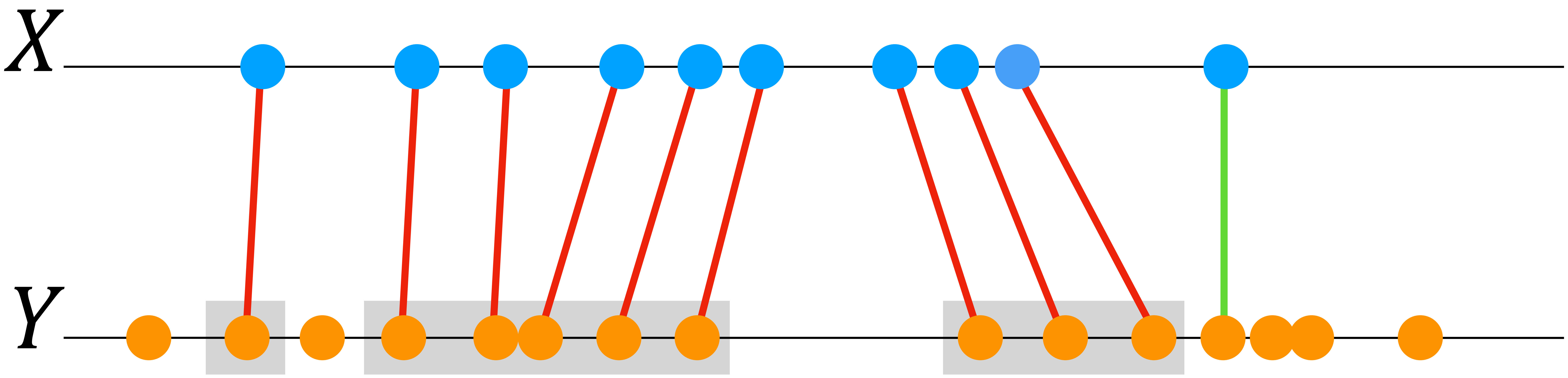

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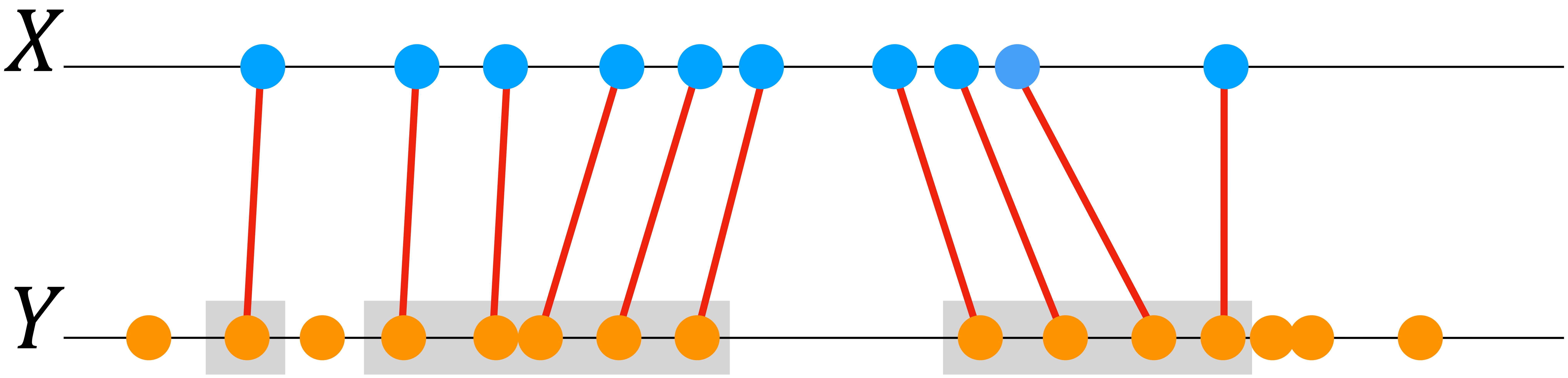


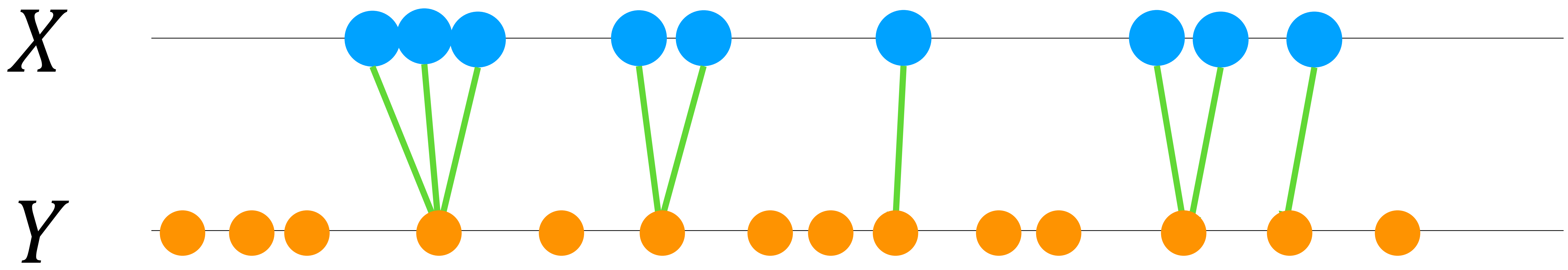
Euclidean Nearest Neighbor assignment

Intervals of bijective assignments

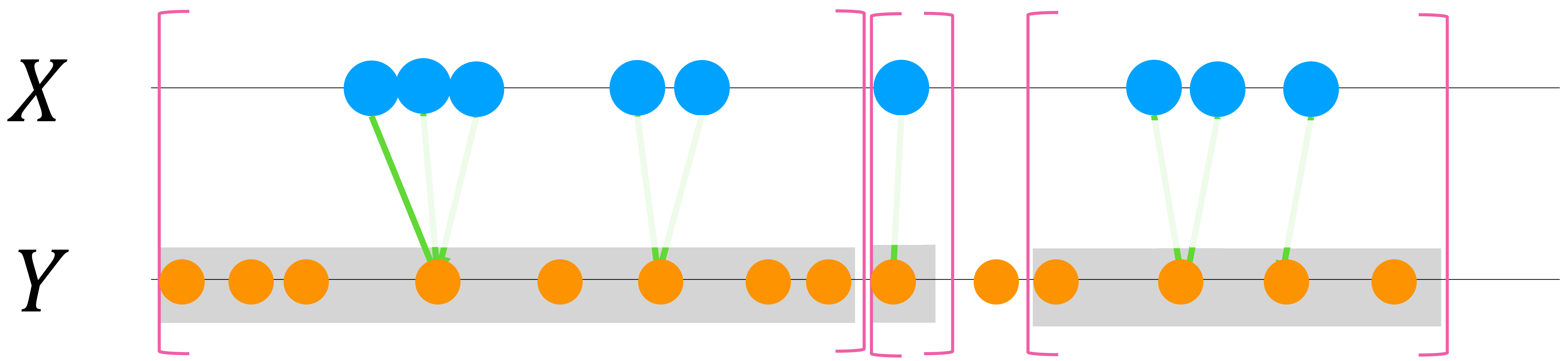
Optimal Transport assignment

Linear time problem decomposition

Problem decomposition



Problem decomposition



Problem decomposition

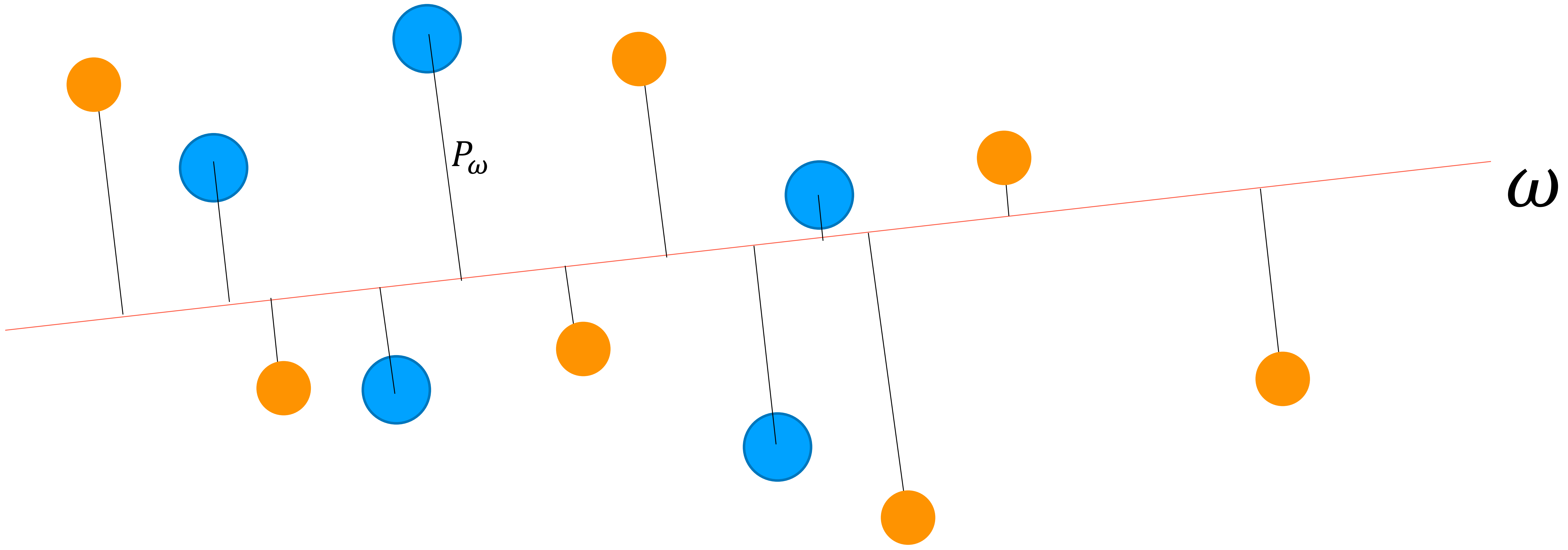
- Computed in quasi-linear time via Union-Find
  - 1 M points in a fraction of a second
- Yields independent subproblems
  - Solvable in parallel
  - That can be further simplified (see paper)

Sliced Partial Optimal Transport (SPOT)

Extension to d dimensions

- Sliced optimal transport

$$E = \int_{\mathbb{S}^{d-1}} W(P_\omega X, P_\omega Y)\mathrm{d}\omega = \int_{\mathbb{S}^{d-1}} \min_T \sum_i \left(P_\omega x_i - P_\omega y_{T(i)}\right)^2 \mathrm{d}\omega$$

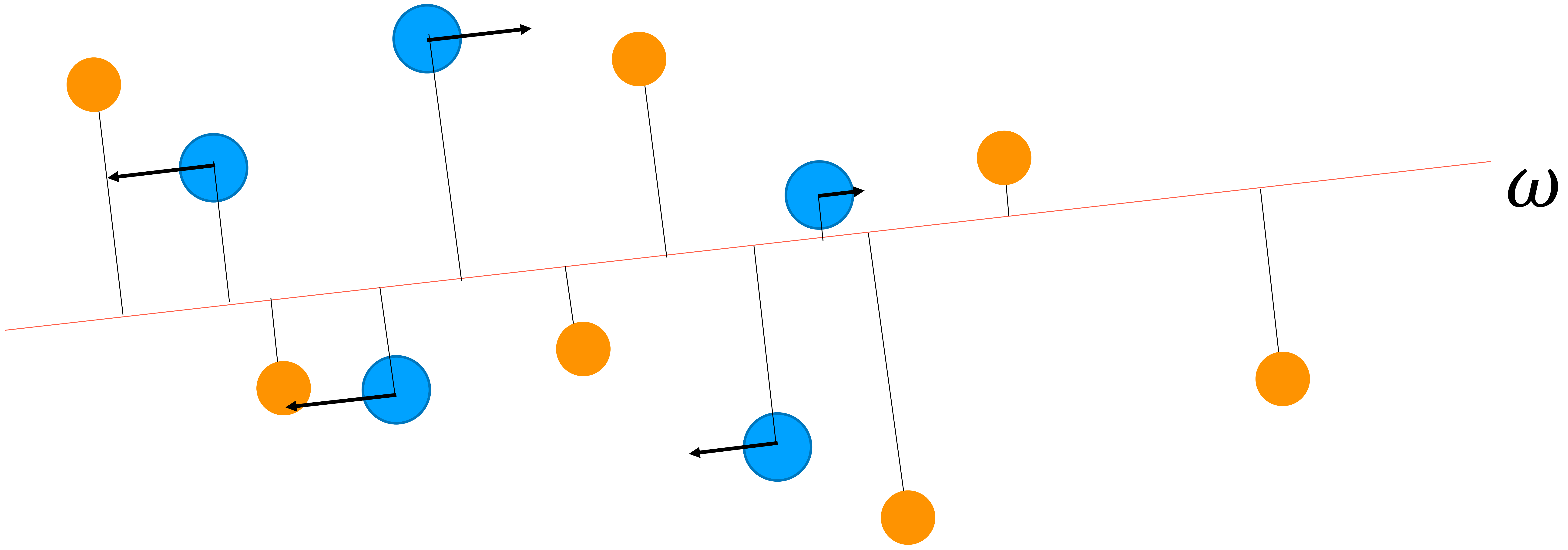


Gradient flow

- Sliced optimal transport

$$X^{n+1} = X^n - \nabla E$$

Stochastic descent: $X^{n+1} = X^n - \nabla \mathrm{W}(P_{\omega^n}X, P_{\omega^n}Y).\omega^n$



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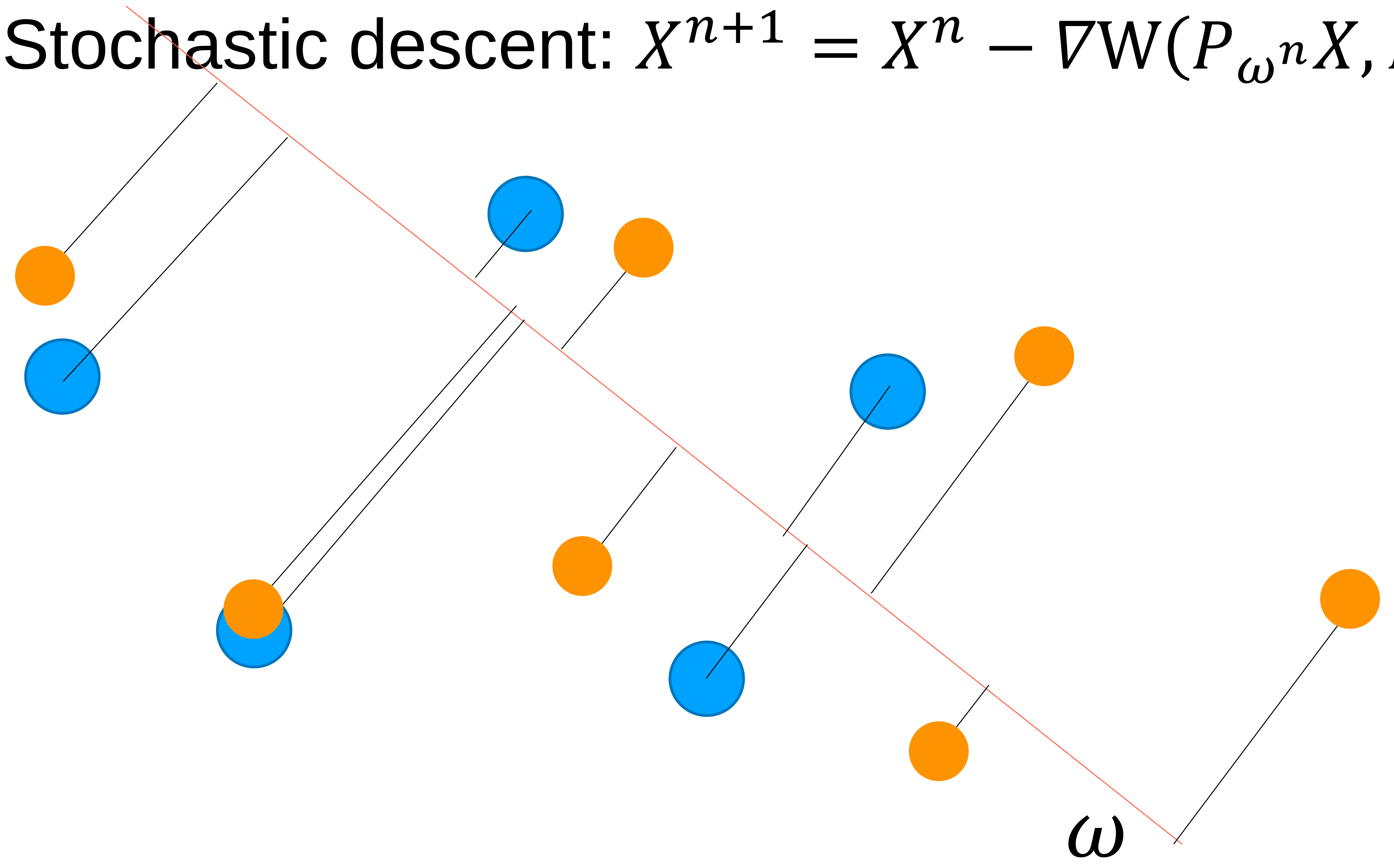
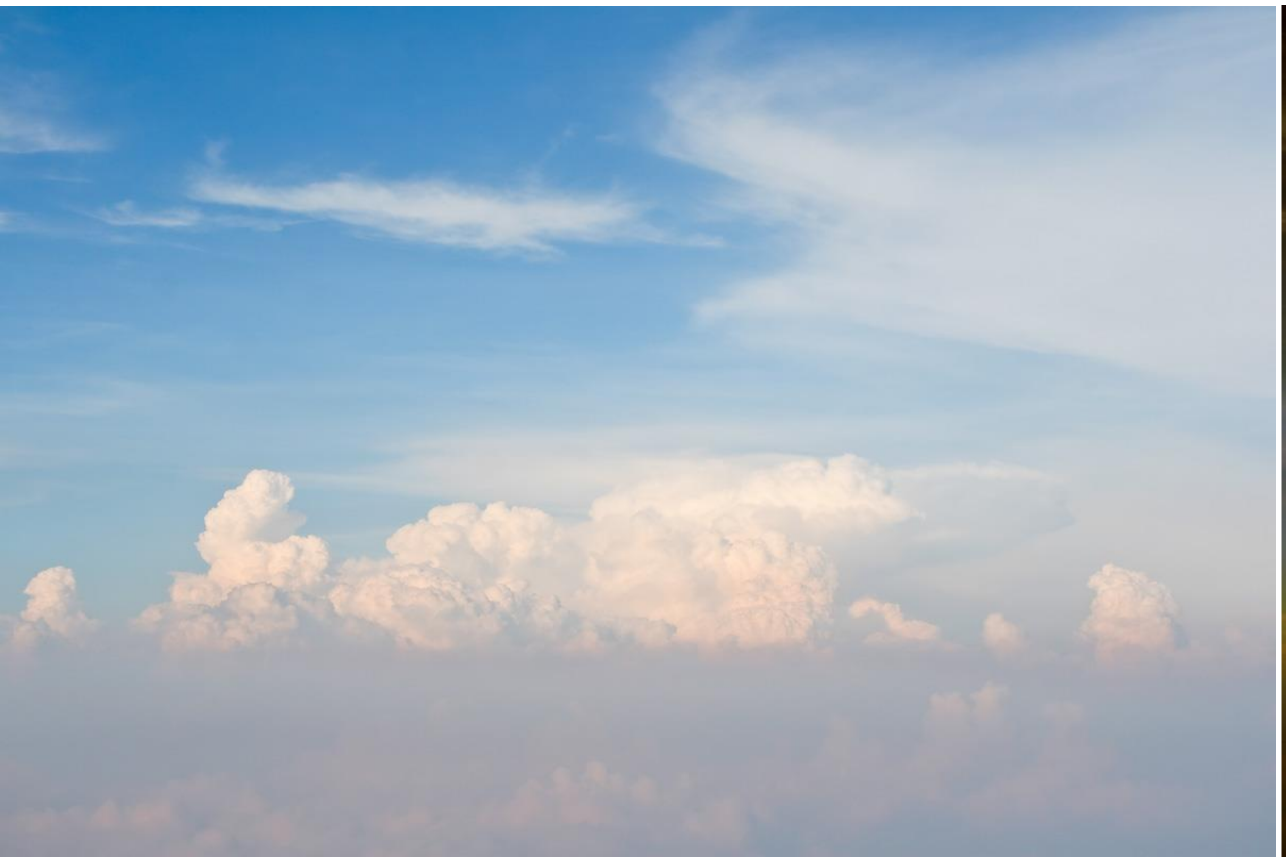
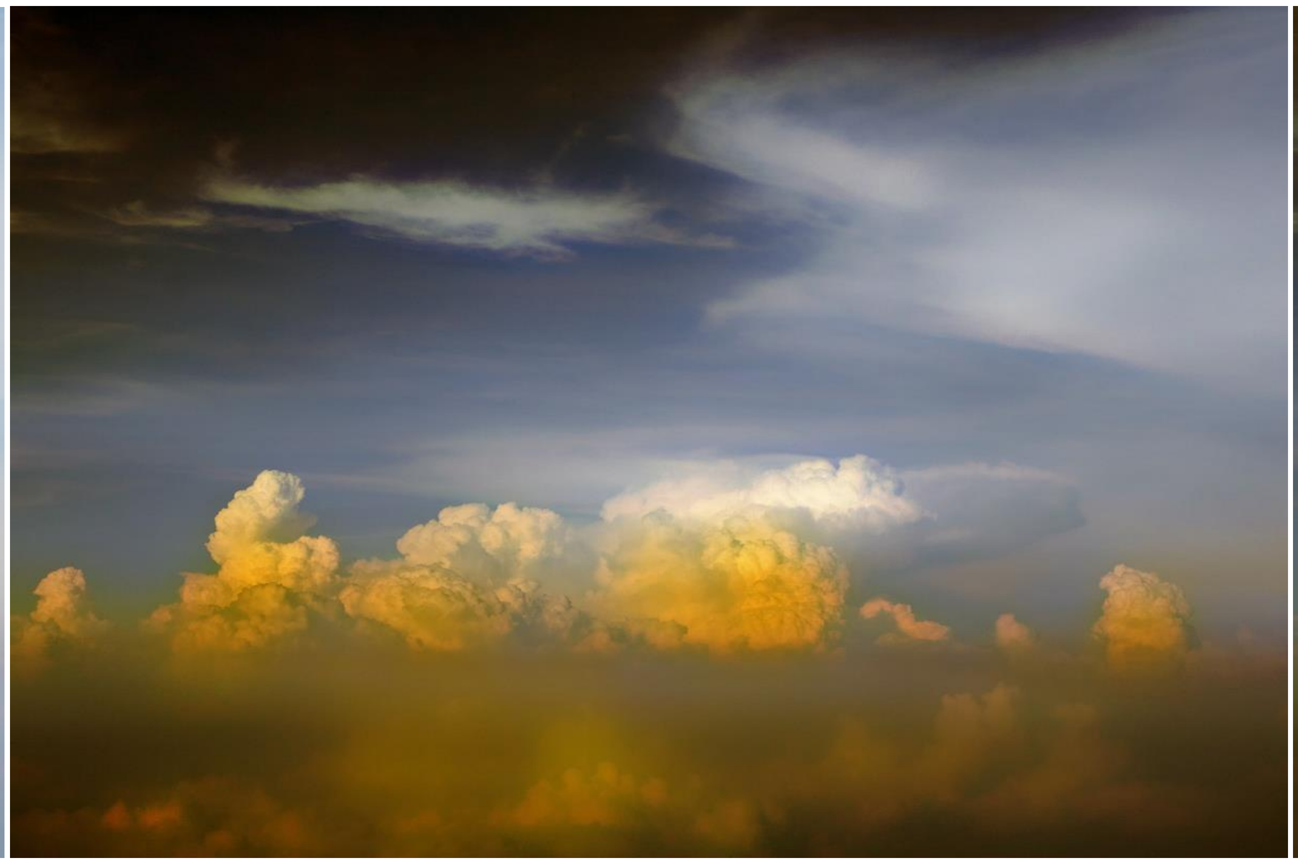


Color transfer application



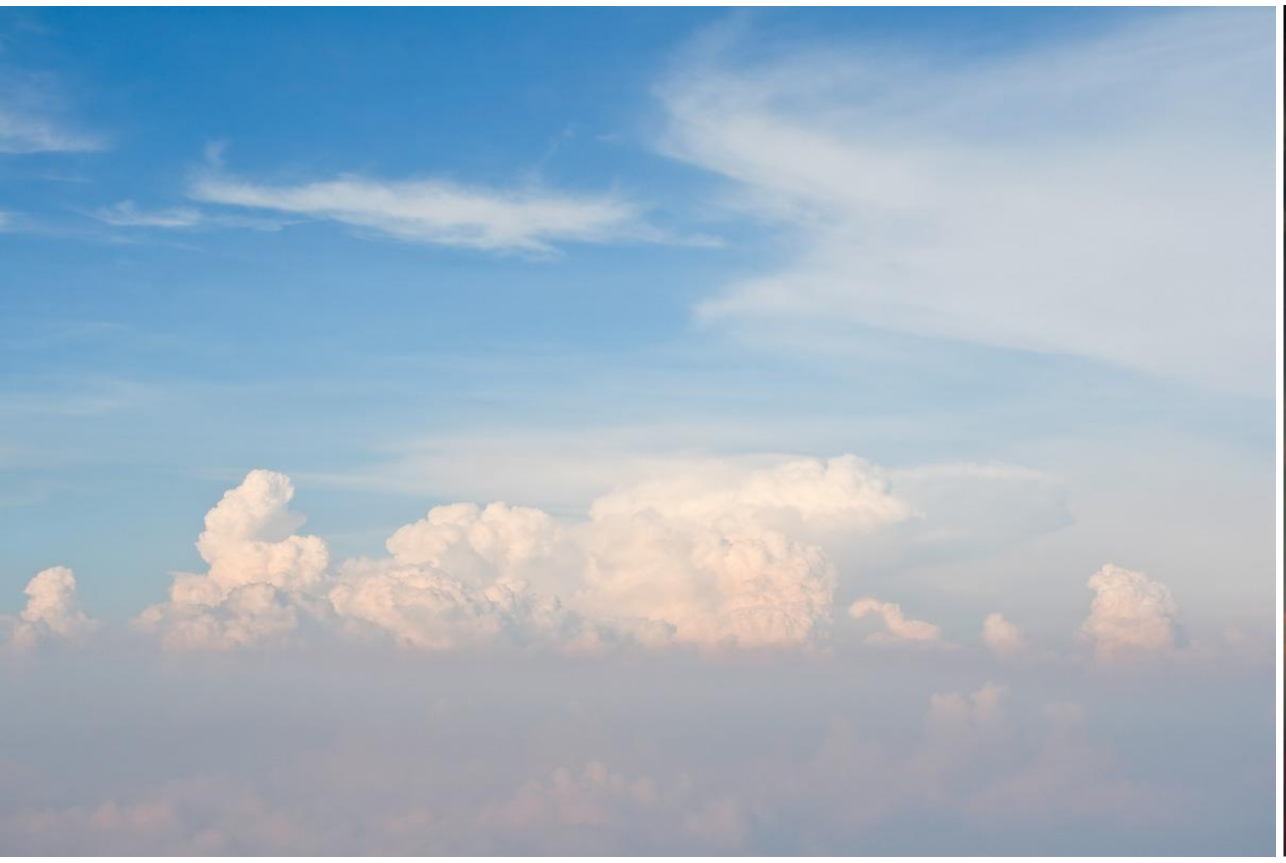


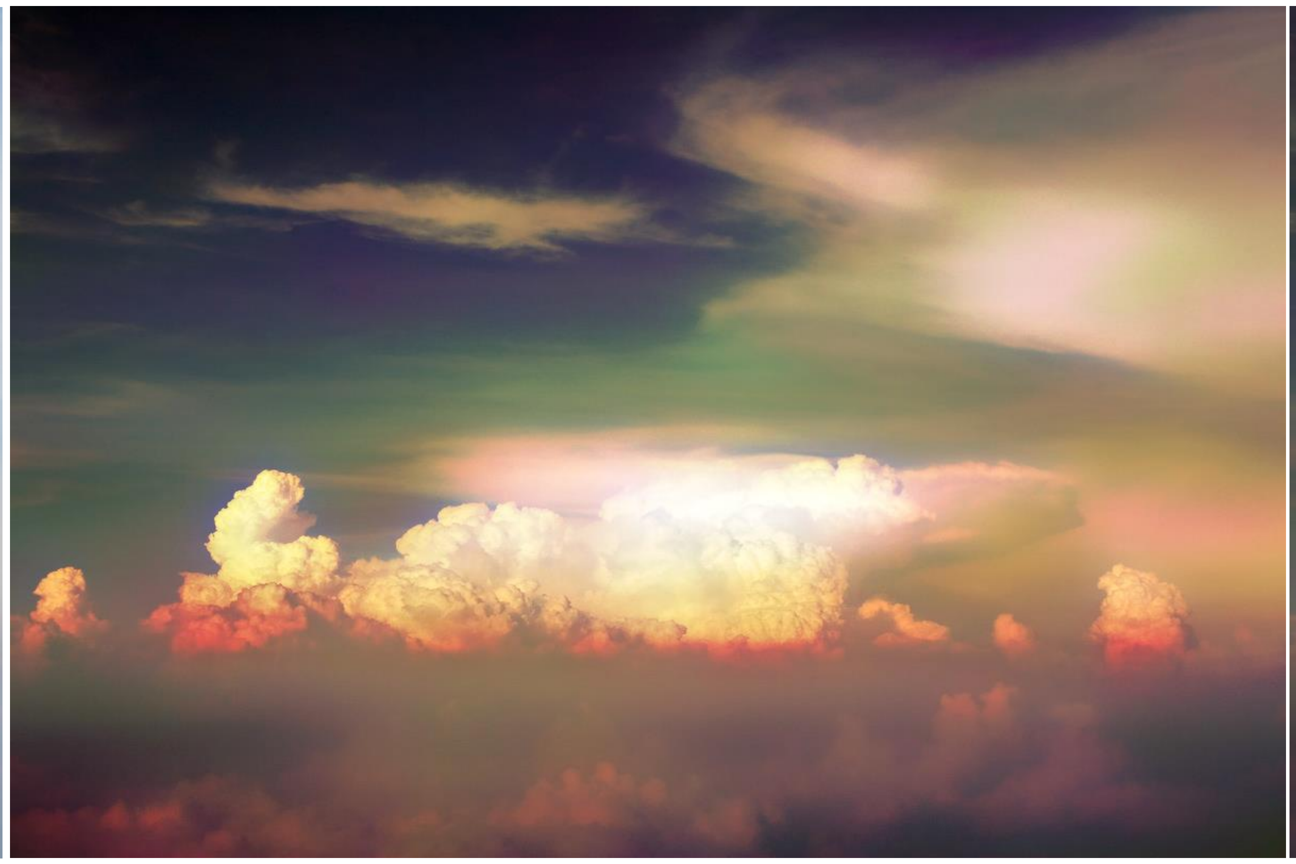
Full Transfer

Target 20% larger

Target 40% larger

Color transfer application





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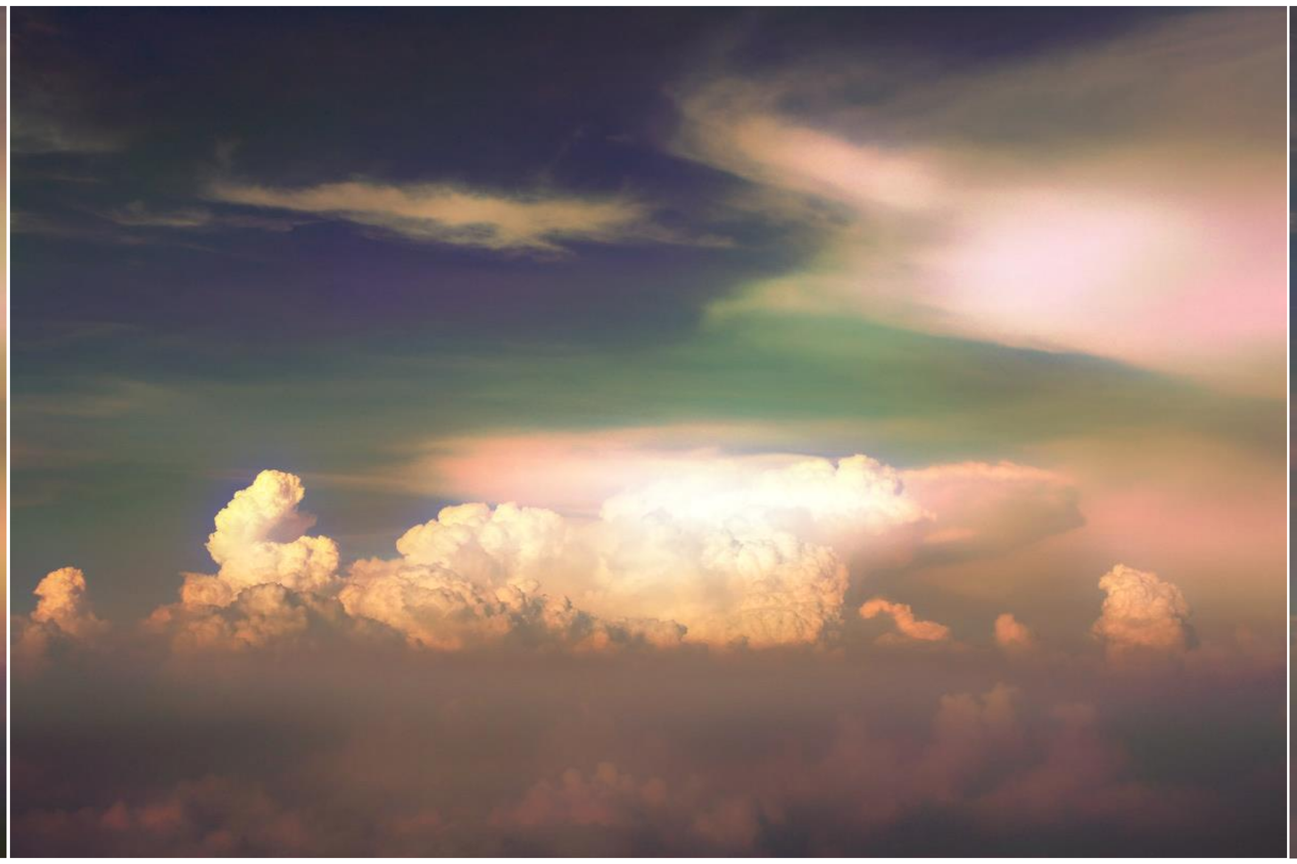
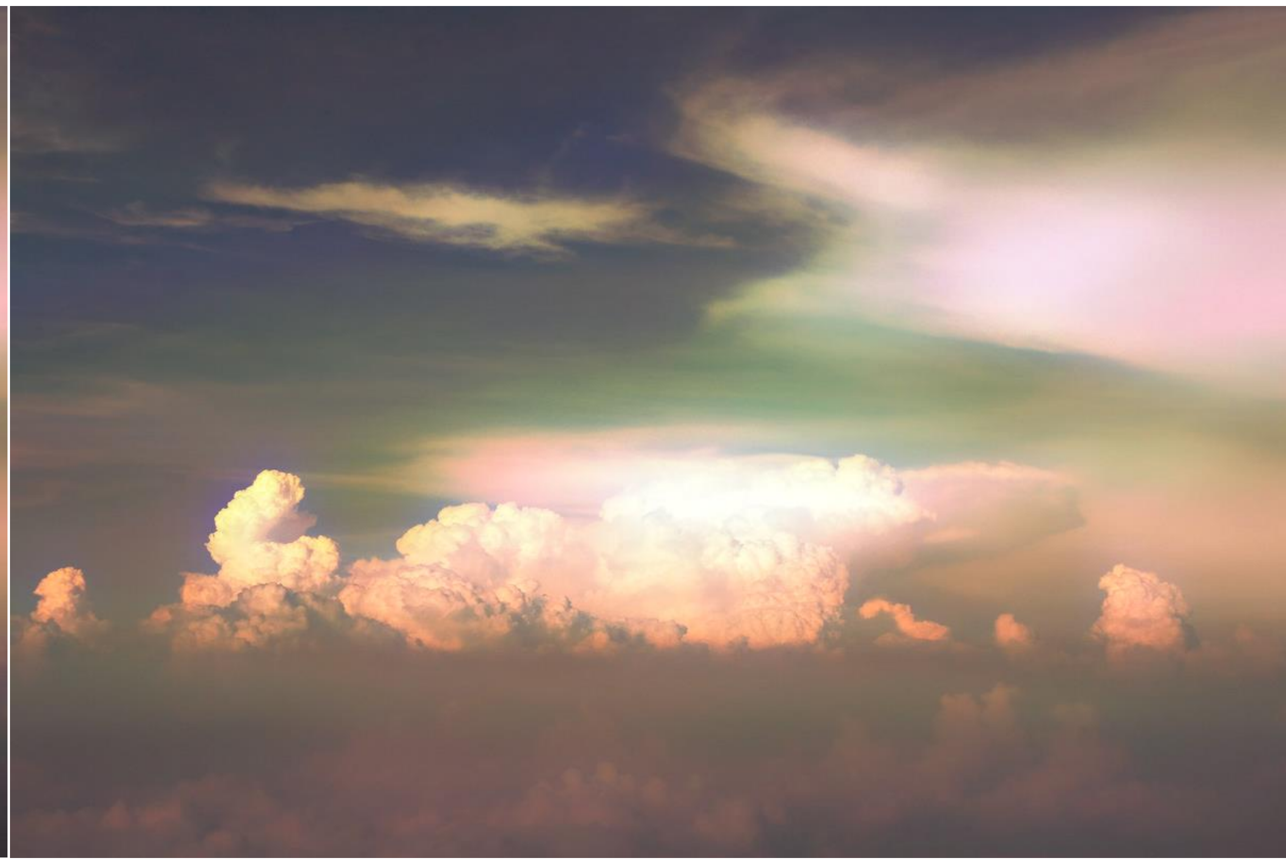


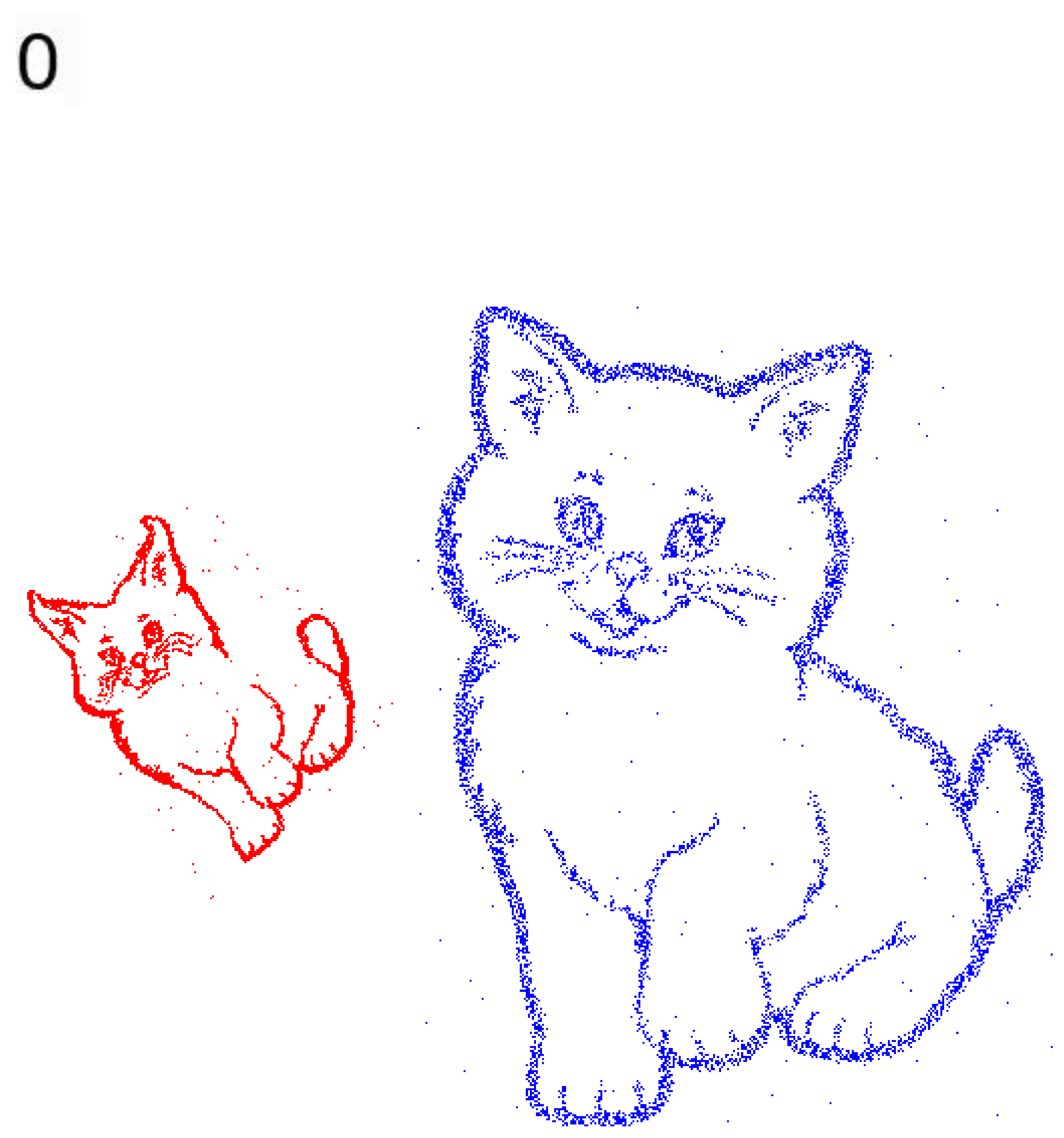
Target 20% larger



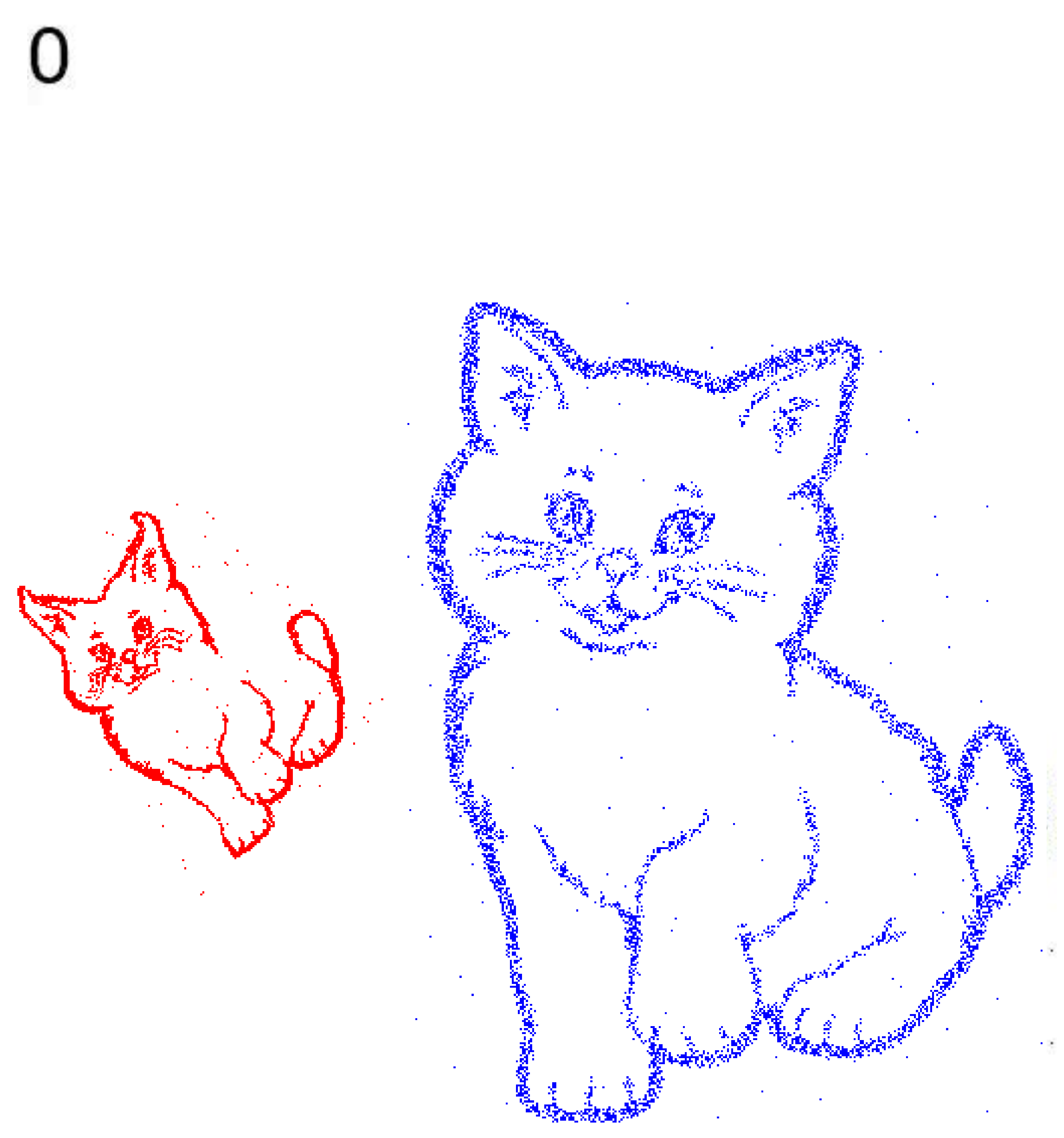
Target 40% larger

Fast Iterative Sliced Transport (FIST)

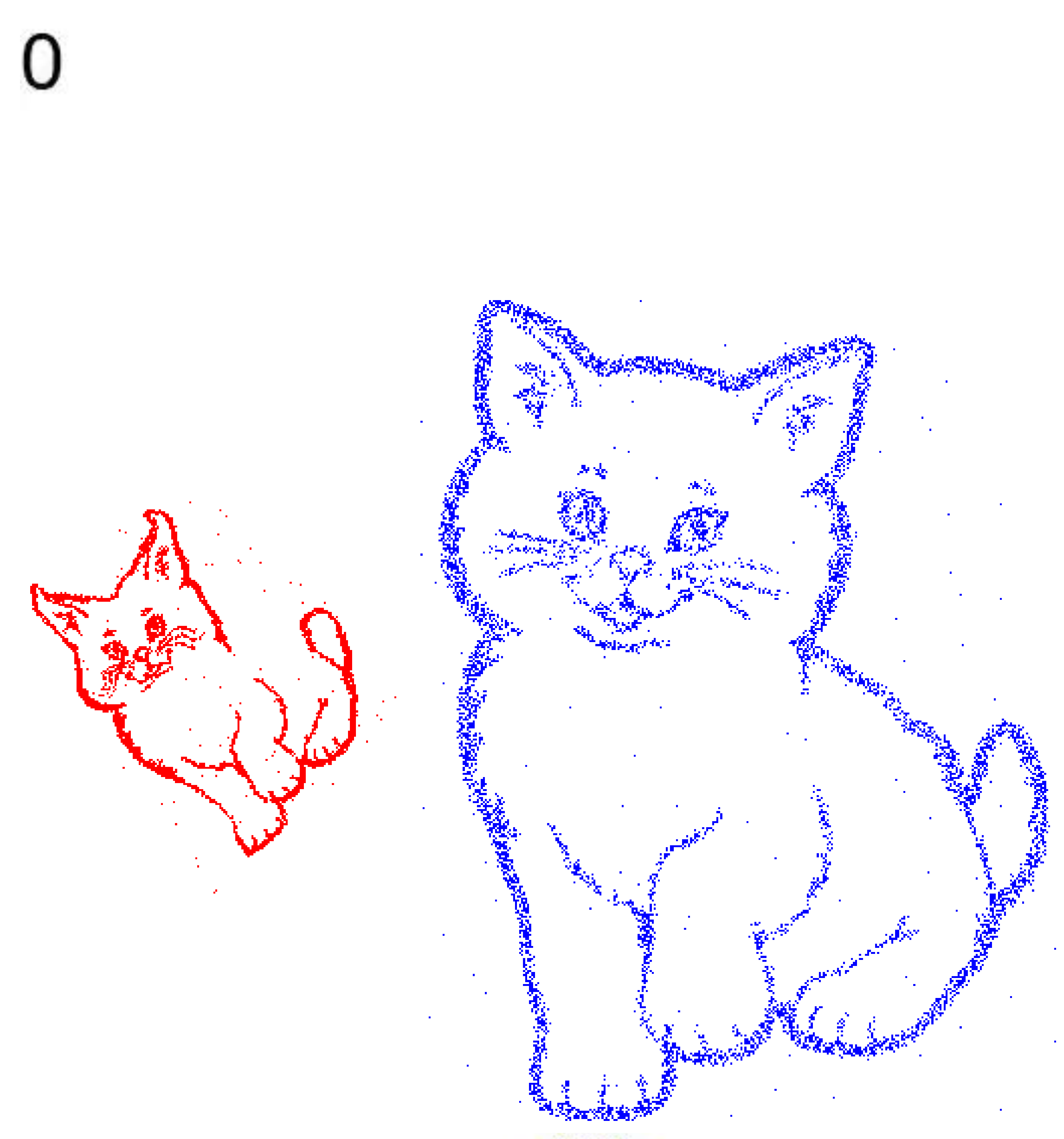
Source: 8k samples
Target: 10k samples



ICP
(0.005 s / iteration)



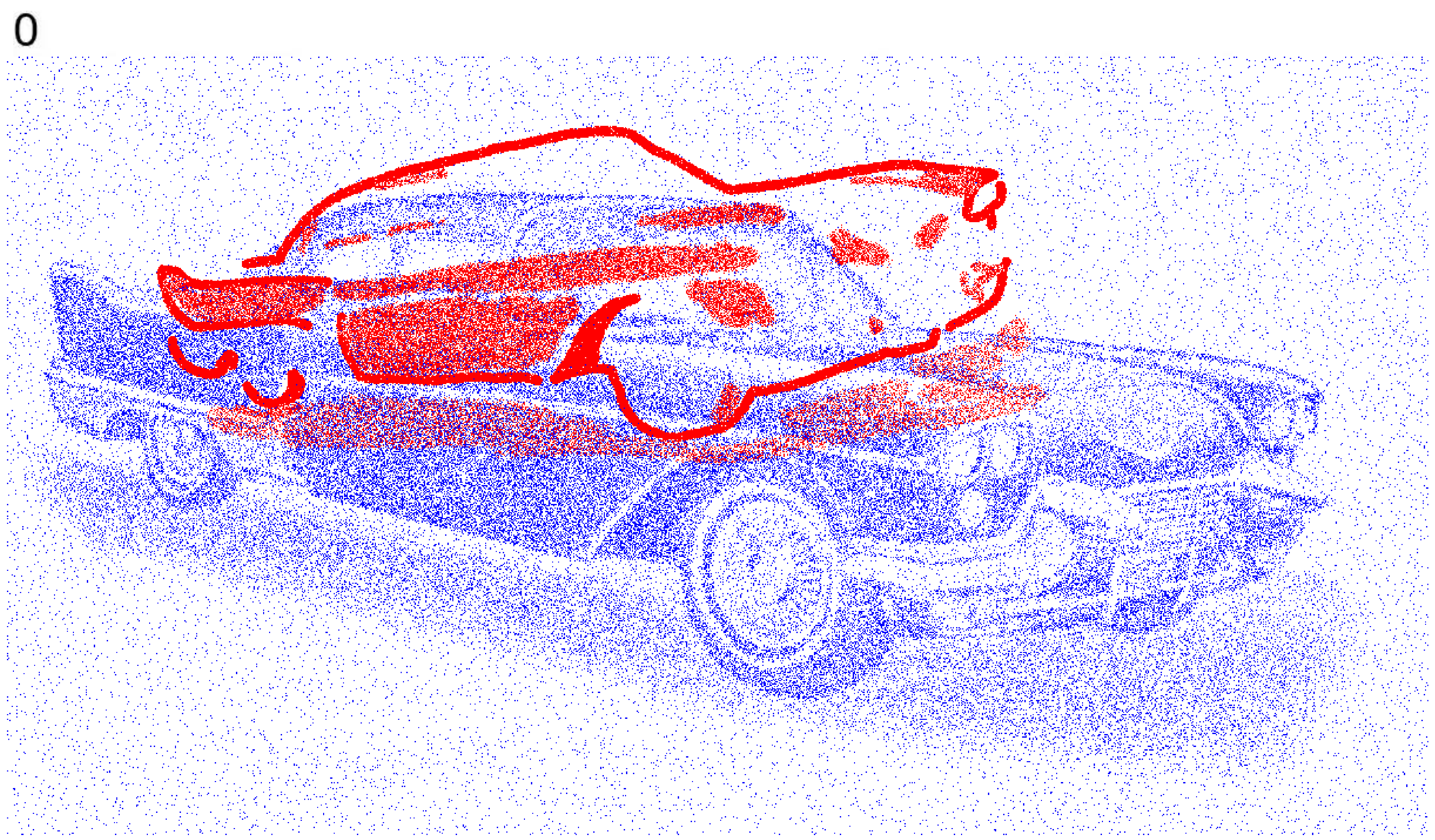
Iterative Transport
with network simplex
(40 s / iteration)



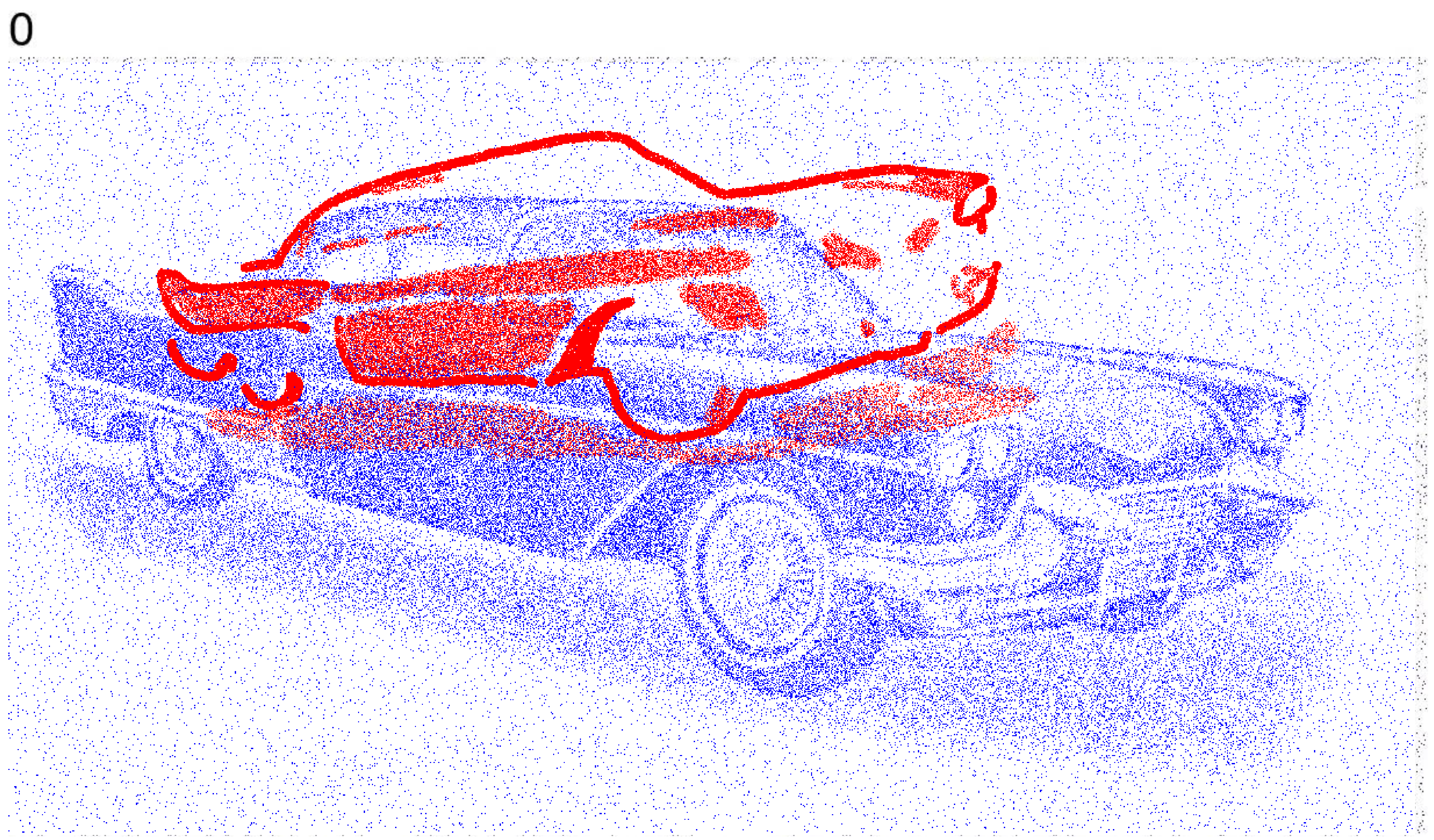
Our FIST algorithm
(0.04 s / iteration)

Source: 90k samples
Target: 100k samples

(input too large for iterative transport with network simplex)



ICP
(0.05 s / iteration)



Our FIST algorithm
(0.66 s / iteration)

Source: 90k samples
Target: 100k samples

(input too large for iterative transport with network simplex)

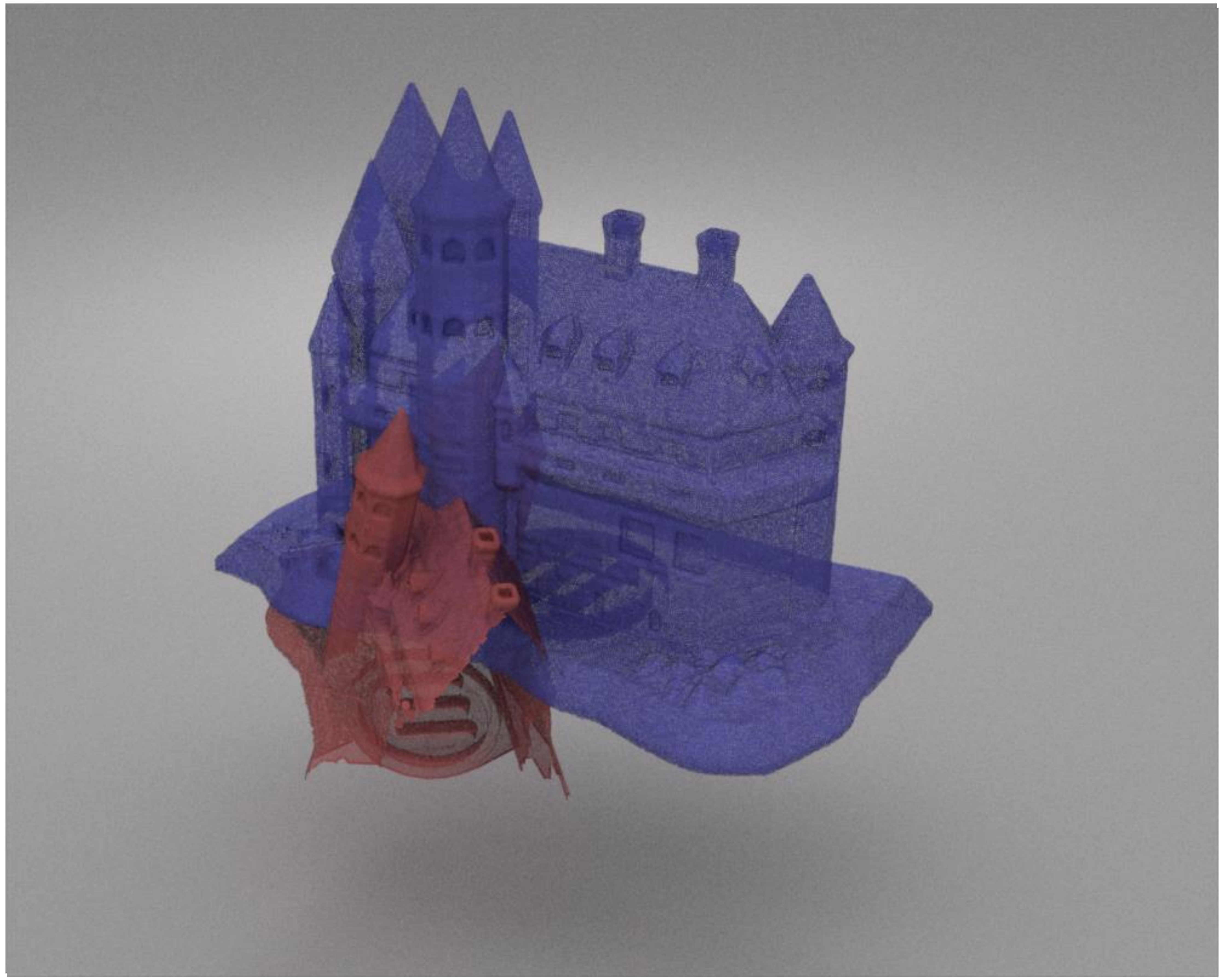
ICP
(0.05 s / iteration)

Our FIST algorithm
(0.69 s / iteration)

Source: 150k samples
Target: 200k samples

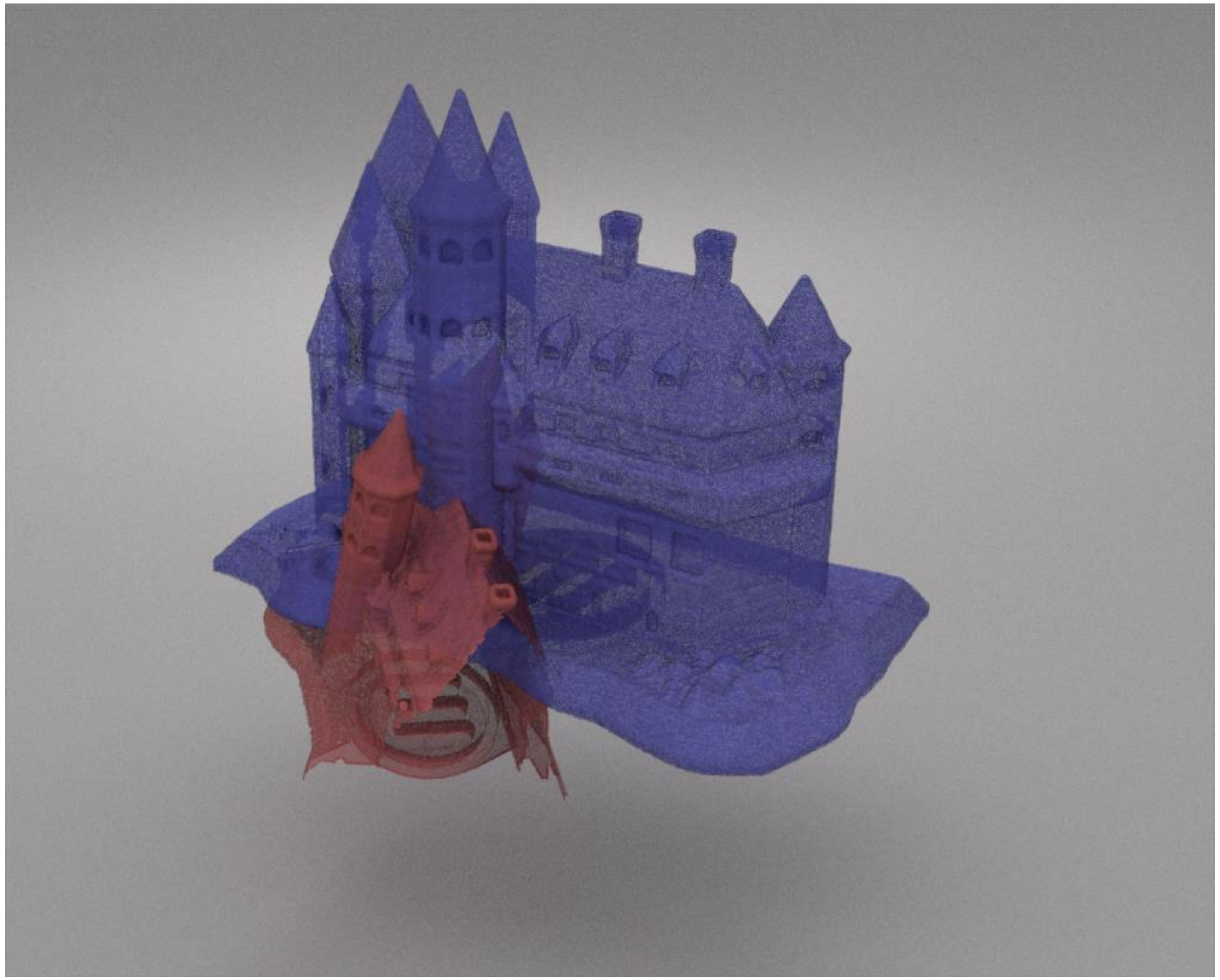
(input too large for iterative transport with network simplex)

0



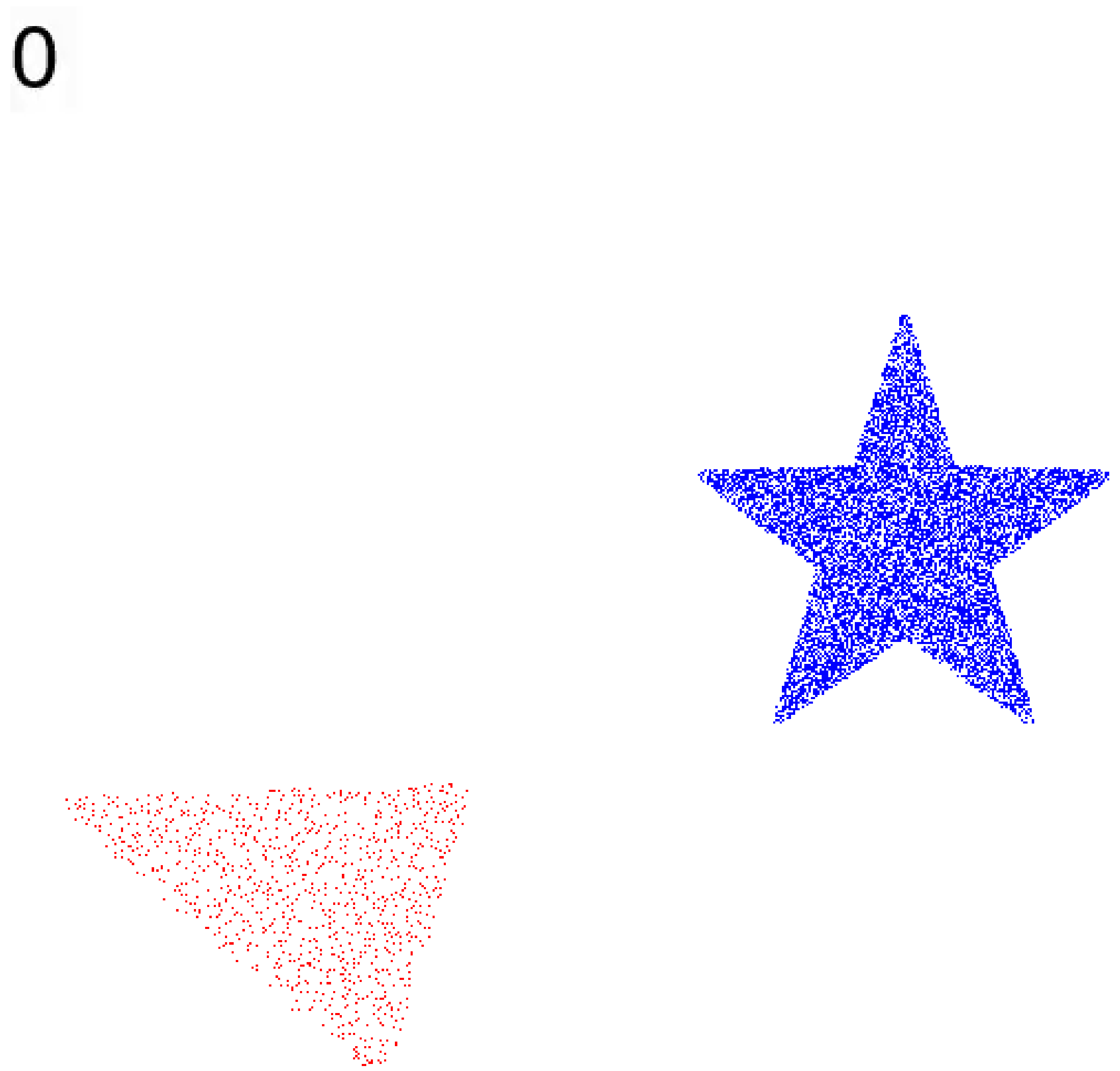
ICP
(0.09 s / iteration)

0

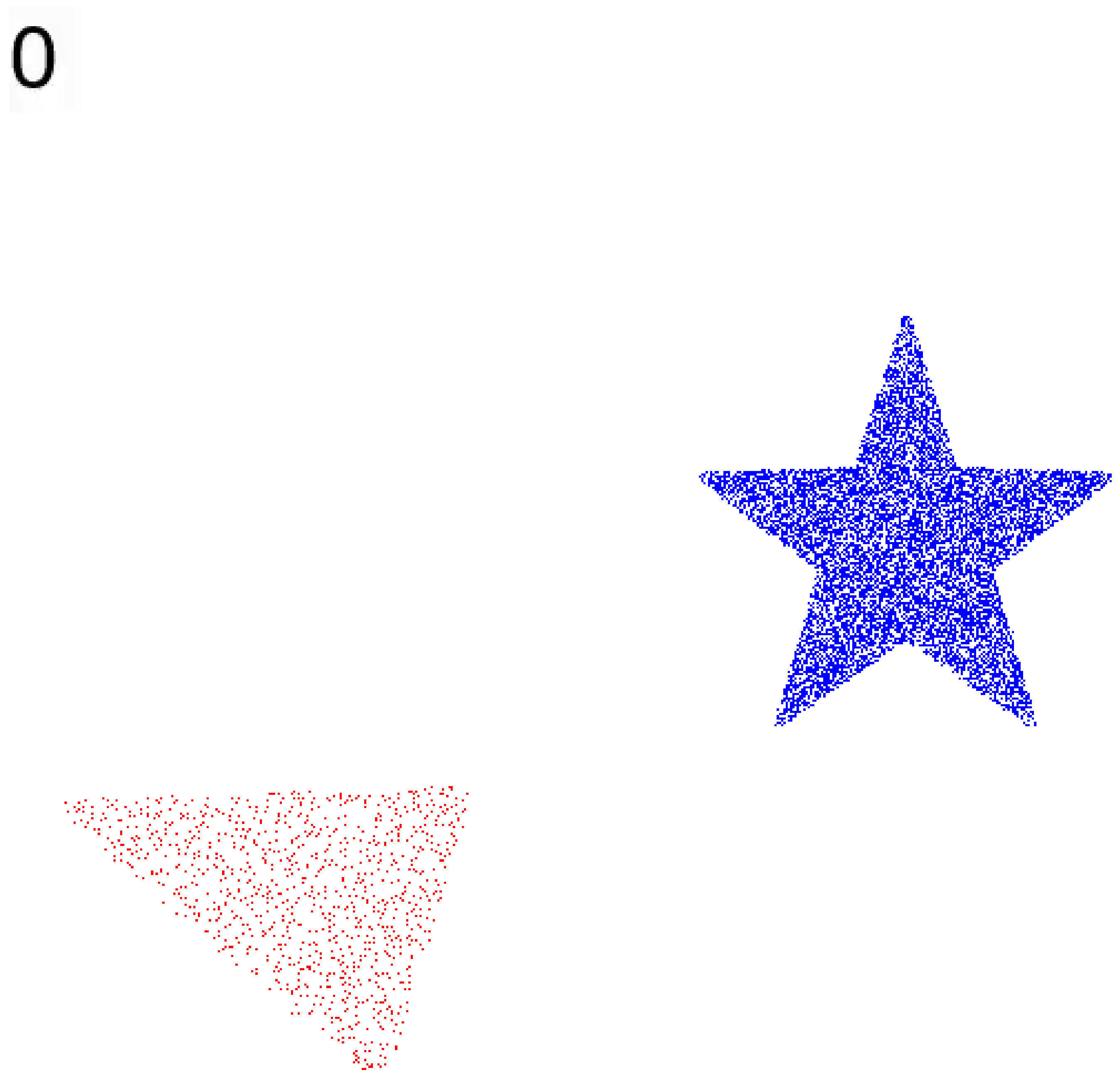


Our FIST algorithm
(2.18 s / iteration)

Failure case: the transport is optimal only on projections



Iterative Transport with Network Simplex



Our FIST algorithm

Conclusions

- Fast partial optimal transport in 1d
  - Quadratic-time algorithm (worst case)
  - Quasi-linear time decomposition
- Sliced Partial Optimal Transport
- Fast Iterative Sliced Transport
- Applications: point cloud registration, color matching
- Code available: https://perso.liris.cnrs.fr/nicolas.bonneel/spot/