Sliced Partial Optimal Transport

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supplementary material

Nicolas Bonneel and David Coeurjolly

Univ. Lyon, CNRS

1 Notations

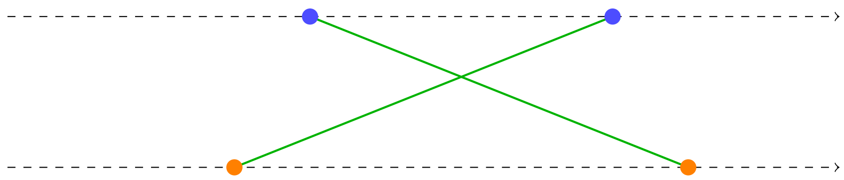
Given a set of points $X = \{x_i \in \mathbb{R}\}_{i=1..m}$ and $Y = \{y_j \in \mathbb{R}\}_{i=1..n}$ on the real line, $m < n$, the goal is to find an injective assignment $a : \mathbb{N} \to \mathbb{N}$ minimizing $\sum (x_i - y_{a(i)})^2$.

For the sake of simplicity, we assume that points of X (resp. Y) are distinct. All these results hold without this assumption by defining a total order on points with multiplicity (*i.e.* when $x_i = x_j$).

2 Warm-up

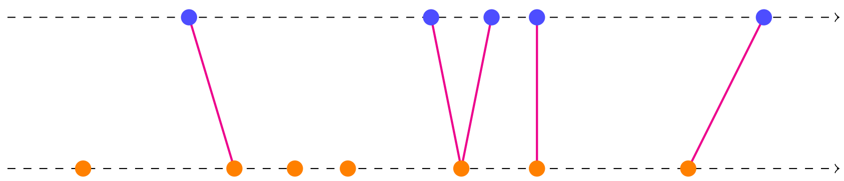
2.1 Miscellaneous

For convex cost, no such crossings can occur in the assign map (resvoling the crossing will always lead to lower cost).



2.2 Injective nearest neighbor assignment

We consider the nearest neighbor assignment $t : \mathbb{N} \to \mathbb{N}$ between X and Y.



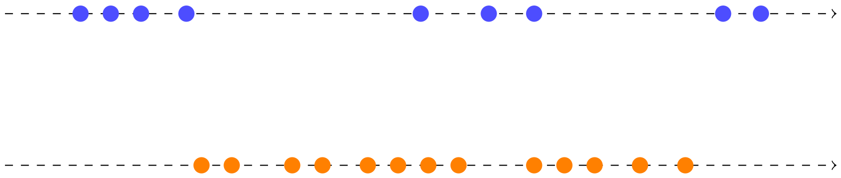
The nearest neighbor assignment can be obtained by a simple 2-sweep algorithm in $O(n + m)$.

Proposition 1. *If the t assignment $X \to Y$ is injective, then the optimal assignment a is given by t.*

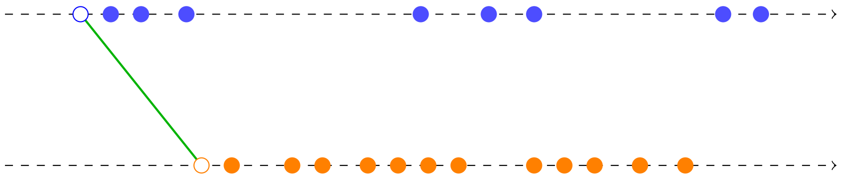
Proof. The nearest neighbor match already minimizes $\sum (x_i - y_{t(i)})^2$. Hence, if t is injective, it trivially corresponds to the optimal assignment a. □

3 Reducing the ranges of X and Y

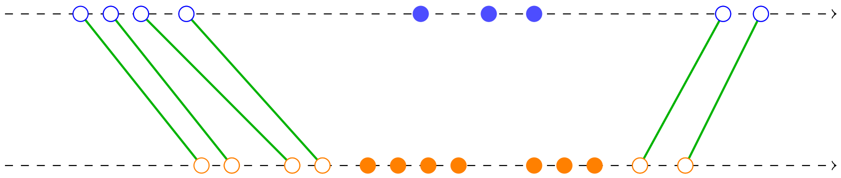
Let us consider the example:



Since $x_1 < y_1$, the assignment $x_1 \to y_1$ (thus setting $a(1) = 1$) is optimal since any other assignment of x_1 than y_1 would have higher cost. We can then proceed with $X = X \setminus \{x_1\}$ and $Y = Y \setminus \{y_1\}$:



We can repeat this process on both sides of X and Y ($x_1 < y_1$ on the left and $x_m > y_n$ on the right). We finally end up with a smaller optimal assignment problem:



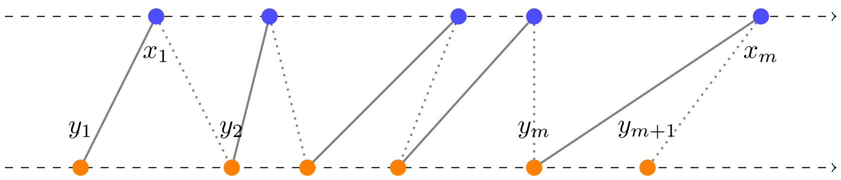
The problem is not solved yet but reducing the ranges allows us to considerably reduce the size the problem using trivial optimal assignments.

Proposition 2. *If $x_1 < y_1$, the optimal assignment of X to Y is given by setting $a(1) = 1$ and solving the optimal assignment of $X \setminus \{x_1\}$ to $Y \setminus \{y_1\}$. Similarly, if $x_m > y_n$, we can set $a(m) = n$ and solve the problem of $X \setminus \{x_m\}$ to $Y \setminus \{y_n\}$.*

Proof. Let us prove the first assert, the proof for the second one is similar. By contradiction, let us assume that $x_1 \to y_k$ with $k > 1$ (*i.e.* $a(1) = k$). Since no crossing can occur in the optimal assignment, the point y_1 is not assigned to any point in X. Since $x_1 < y_1 < y_k$, $(x_1 - y_1)^2 < (x_1 - y_k)^2$ which contradicts the fact that the assignment as minimal cost. So we necessarily have $x_1 \to y_1$ in the optimal transport. □

The algorithm iterates from the left to right until $x_i > y_i$, construct the assignment $\{x_j \to y_j\}$ of all $1 \leq j < i$, and performs the same from the right. This preprocessing step can be done in $O(m)$.

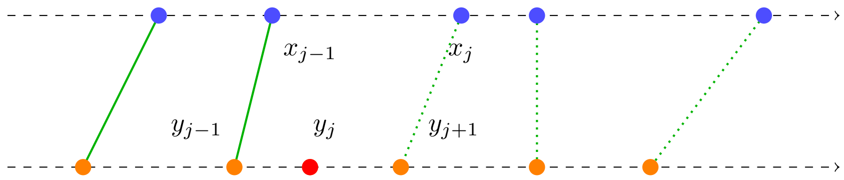
4 Case $n = m + 1$



We consider two assignments: The first one is from left to right, $A : (x_1, \dots, x_m) \to (y_1, \dots, y_m)$, and the second one is from right to left, $B : (x_1, \dots, x_m) \to (y_2, \dots, y_{m+1})$ (resp. in bold and dashed lines). At a point x_i, we define the two costs as

$$C_A(j) = \sum_{i=1}^{j} (x_i - y_i)^2 \quad \text{and} \quad C_B(j) = \sum_{i=m}^{i} (x_i - y_{i+1})^2 .$$

We are looking for the point y_j in Y that is not assigned to any point in X by the optimal assignment a. E.g.



Let y_j be such that

$$j = \underset{1 \leq i \leq m}{\operatorname{argmin}} (C_A(i) + C_B(i)) \,.$$

Proposition 3. *The optimal assignment a is obtained by assignments from A on the left of y_j and from B on its right.*

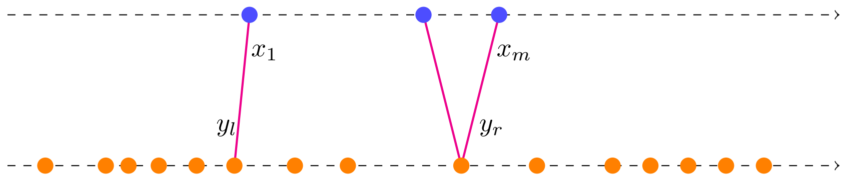
Proof. First, by definition of y_j, $(C_A(j) + C_B(j))$ is exactly the cost of the optimal assignment we are looking for. By contradiction, let suppose that we have a point x_k with $k \geq j$ associated with some point $y_{k'}$ with $k' < j$. As the optimal assignment being a one-to-one and onto mapping from X to $Y \setminus \{y_j\}$, there is a point x_l with $l < k$ that is associated with some $y_{l'}$ with $l' > j$. By convexity of the cost function (Sec. 2.1), we have a contradiction since assignments $x_k \to y_{k'}$ and $x_l \to y_{l'}$ are crossing. □

The algorithm is now very simple and can be done in $O(m)$: For all j, compute the $C_A(j)$ and $C_B(j)$, get the index j minimizing the sum of $C_A(j)$ and $C_B(j)$, construct a.

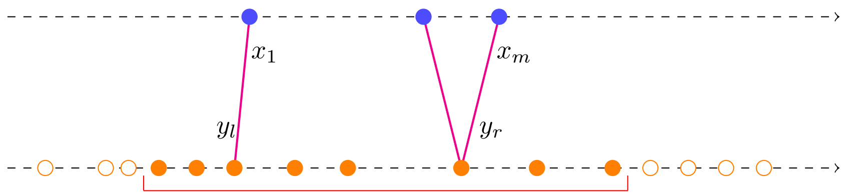
Note: this algorithm is similar to a substep of the Hirschberg's algorithm [**?**] to solve dynamic problems in quadratic time but linear space.

5 Reduction Y using m

We describe here a simple preprocessing that can reduce the set Y in $O(\log n)$. Let us consider the following situation (the nearest neighbor assignment t is given in magenta):



We denote by y_l (resp. y_r) the nearest neighbor of x_1 (resp. of x_m). Then Y can be shrunk to obtain the following Y':



Lemma 1. *In the optimal assignment, x_m cannot be associated to a point y_k with $k < l$. Similarly, x_1 cannot be associated to a point y_k with $k > r$.*

Proof. Let us suppose that we have $x_m \to y_k$ with $k < l$ in the optimal assignment. If $k < m$ we have a first contradiction (as no crossings may occur in the optimal map, the assignment must be injective between points of X and $\{y_1 \dots y_k\}$, which is only possible if $k \geq m$). If $k \geq m$, we have a second contradiction on the cost of the assignment: Since $t(1) = l$, $(x_1 - y_l)^2 \leq (x_m - y_l)^2$ (and by definition, $x_1 < x_m$), we have $x_m \geq y_l$ and $(x_m - y_k)^2 = (x_m - y_l)^2 + (y_l - y_k)^2 > (x_m - y_l)^2$. Hence, the assignment $x_m \to y_l$ would reduce the cost, which is a contradiction. The proof is similar to x_1 that cannot be associated to a point y_k with $k > r$. □

Proposition 4. *The optimal assignment problem of X to Y can be reduced to an assignment problem of X to $\{y_{\max(0,l-m+1)} \cdots y_{\min(n,r+m-1)}\}$.*

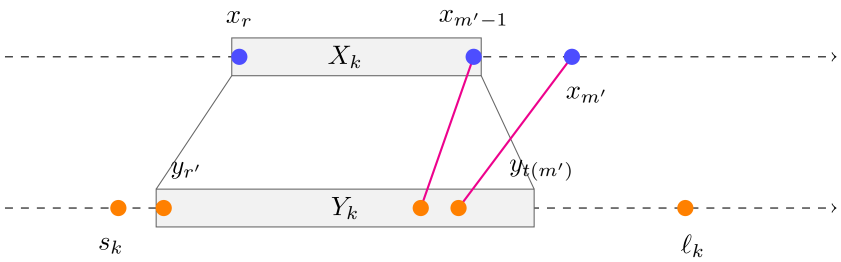
Proof. Using Lemma 1, x_r is associated in the optimal assignment to some y_k with $k \geq l$. Let us consider x_{r-1}, it is necessarily associated $y_{k'}$ with $k' \geq l-1$. Indeed, by contradiction, let assume that $x_r \to y_l$ and that $x_{r-1} \to y_{k'}$ with $k' < l-1$, since assignments cannot cross, points $\{y_{k'+1} \dots y_{l-1}\}$ are not optimally assigned to any point in X. But as $t(r-1) > l$ (nearest neighbor assignments cannot cross too), we have $(x_{r-1} - y_{k'})^2 > (x_{r-1} - y_{l-1})^2$. So $x_{r-1} \to y_{k'}$ is not optimal which contradicts the hypothesis. Similar arguments hold for points x_{r-2} to x_l. Finally, we have that x_l is necessarily associated to a point $y_{k'}$ with $k' \geq \max(0, l-m+1)$. Again, x_l is necessarily associated to a point $y_{k'}$ with $k' \leq \min(n, l+m-1)$, which ends the proof. □

This proposition can be used as a preprocessing step and only requires to have the nearest neighbor assignment $t(1)$ and $t(m)$ of x_1 and x_m, which can be done in $O(\log n)$. Note that or the other preprocessings, we may already have the complete nearest neighbor assignemnt in $O(n+m)$ which makes this preprocessing be in $O(1)$.

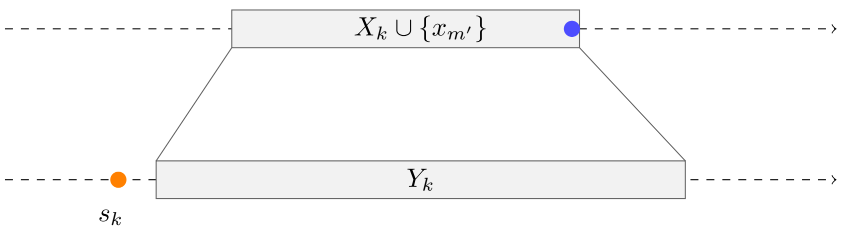
6 Splitting the problem (Algorithm 2)

We prove in this section the correctness of Algorithm 2. First, let us suppose that we always extend $Y_{k'}$ on both sides (lines 16-17). Then, Algorithm 2 corresponds to Algorithm 1 where both options are applied (without the need of computing the option costs). Hence, by correctness of Algorithm 1 at each step of Alg.2, for each sub-problem (X_k, Y_k), the optimal assignment of X_k will only consider points in Y_k. Furthermore, by construction, all $\{X_k\}$ form a disjoint partition of X. Hence $\{A_k\}$ sub-problems are independent and can be solved in parallel.

We now consider the special case of lines 13-14 whose objective is to only extend Y_k on one side, leading to shorter sub-problems. We thus are in the situation where the nearest neighbor of $x_{m'}$, $y_{t(m')}$. Since $y_{t(m')} \in Y_k$, $f_{y_{t(m')}}$ is `false`) but there is no point in X_k whose nearest neighbor is $t(m')$ (for $k < m'$, nearest neighbors cannot cross so we only need to check $t(m'-1)$):



We claim that we just need to extend Y_k with y_{ℓ_k} instead of adding both y_{s_k} and y_{ℓ_k}, to obtain:



Let us suppose by induction that (X_k, Y_k) is a valid optimal assignment sub-problem, meaning that in the optimal assignment of X to Y, all points if X_k are assigned to some points of Y_k. We say that Y_K is *tight* if $|X_k| = |Y_k|$.

- Let suppose that Y_k is tight
  - Let us suppose that optimal assignment is the nearest neighbor one (*i.e.* $a = t$).

    ⇒ By hypothesis, we have $t(m') \neq t(m'-1)$, so $t(m')$ is in fact greater or equal to ℓ_k. Hence $x_{m'}$ can only be optimally assigned to y_{ℓ_k} in the sub-problem $X_k \cup \{x_{m'}\}$ to Y (the only free spot is y_{ℓ_k} and for y_j with $j > \ell_k$ the cost would be higher). So the sub-problem $(X_k \cup \{x_{m'}\}, Y_k \cup \{y_{\ell_k}\})$ is a valid sub-problem (we can just extend Y_k to its right).

- The case Y_k is tight but $a \neq t$ occurs when we have several non-injective assignments in t but a tight Y_k. This situation can only occur when the first point of Y_k is y_1 (the non-injective steps in Y_k would have required an extension on both sides but no more points exist at the left of Y_k).

- If Y_k is not tight, then there exists $y_q \in Y_k$ which is not optimally assigned to any point in X_k. If such y_q is not unique, we consider the rightmost one. There are two options when we add $x_{m'}$:
  - Either the optimal assignment of $x_{m'}$ is y_{ℓ_k} ($a(m') = \ell_k$) ;
  - or the optimal assignment will shift all assignments for $\{y_{q+1}, \dots, y_{\ell_k - 1}\}$ by one to the left (y is now assigned) and we set $a(m') = \ell_k - 1$. Any other assignment of that would have skipped y has necessarily a higher cost and is not optimal.
  - For these two cases, y_{s_k} will never be considered and the optimal assignment of $X_k \cup \{x_{m'}\}$ can be obtained by solving the optimal assignment with Y_k extended only on one at its right.

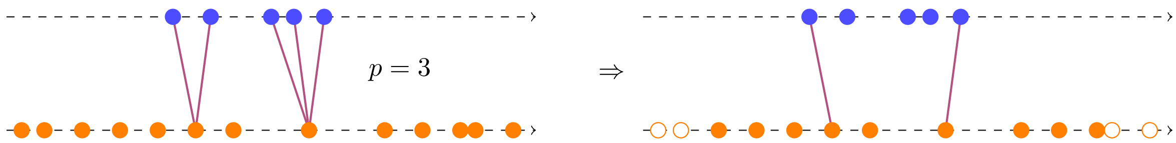
Finally, if $t(m') \neq t(m'-1)$, we just have to extend Y_k on one side to define a valid optimal assignment sub-problem of $X_k \cup \{x_{m'}\}$ to $Y_k \cup \{y_{\ell_k}\}$, which ends the proof of the validity of Alg.2. □

7 Reduction Y using non-injectivity counters

We prove the reduction of Y using the number of times the nearest neighbor assignment t between X and Y is non-injective.

Let $p = \text{card}\{t(i) = t(i+1), \forall i < m\}$. Then,

Proposition 5. *The optimal assignment of X to Y can be reduced to the optimal assignment problem of X to $Y' = \{y_j\}_{j=\max(1,t(1)-p)..\min(t(m)+p,n)}$.*



Proof. The proof uses similar arguments as for Algorithm 2. Let us consider two sets X ($|X| = m$) and Y. If $m = 1$, the proposition is true, a corresponds to the nearest neighbor assignment t. By induction, we suppose that the proposition is true for X and Y, and we add an extra point x_{m+1}.

- If $t(m+1) > min(t(m)+p, n)$, there is no collision in the nearest neighbor assignment of $X \cup \{x_{m+1}\}$ to Y. Thus, p remains the same, x_{m+1} is optimally assigned to $y_{t(m+1)}$ (any other assignment of x_{m+1} would have higher cost). Hence, the optimal assignment of $X \cup \{x_{m+1}\}$ to Y can be reduced to $Y' = \{y_j\}_{j=\max(1,t(1)-p)..\min(t(m+1)+p,n)}$.

- Otherwise, we have two possibilities,
  1. there exists x_i in X such that $t(i) = t(m+1)$ (*i.e.* we have a collision). As t cannot cross, such i is necessarily equal to m (*i.e.* $t(m) = t(m+1)$ hereafter).
     We define $p' := p + 1$. Hence, p' is the new number on non-injectivity in t for the problem $(X \cup \{x_{m+1}\}, Y)$. Furthermore, if we denote Y'' the extension of Y' by on point in Y on both sides. From the correctness proof of Algorithm 2, we know that if the optimal assignment points of X belongs to Y', then Y'' contains the optimal assignments of points $X \cup \{x_{m+1}\}$. By construction, we have $Y'' = \{y_j\}_{j=\max(1,t(1)-p-1)..\min(t(m)+p+1,n)}$ which is equal to $\{y_j\}_{j=\max(1,t(1)-p')..\min(t(m+1)+p',n)}$, which proves the proposition by induction ($t(1)$ remains the same and $t(m+1) = t(m)$).
  2. There is no such x_i. This means that the non-injectivity counter p' when considering x_m equals p. Furthermore, as no crossing occurs in t, we necessarily have $t(m+1) = min(t(m)+1, n)$. By induction, the optimal assignments a of X to Y consider points in Y'. From Algorithm 2, we know that extending Y' by one only on its right, denotes Y'', contains the optimal assignment of $X \cup \{x_{m+1}\}$ (we are in the case of lines 13-14)). As $t(1)$ is unchanged and $t(m+1) = min(t(m)+1, n)$, we have $Y'' = \{y_j\}_{j=\max(1,t(1)-p')..\min(t(m+1)+p',n)}$, and Y'' contains the optimal assignments of $X \cup \{x_m\}$, which completes the proof by induction.

□