Local certification of forbidden subgraphs

Nicolas Bousquet, Linda Cook, Laurent Feuilloley, Théo Pierron, Sébastien Zeitoun

September 5, 2024

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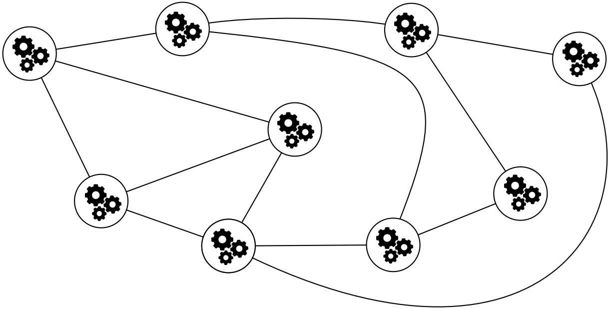
Context: distributed computing

Model: graph, $\begin{cases} \text{vertices = computation units} \\ \text{edges = communication channels} \end{cases}$



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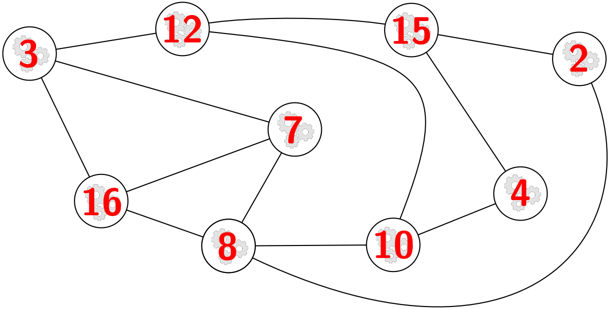

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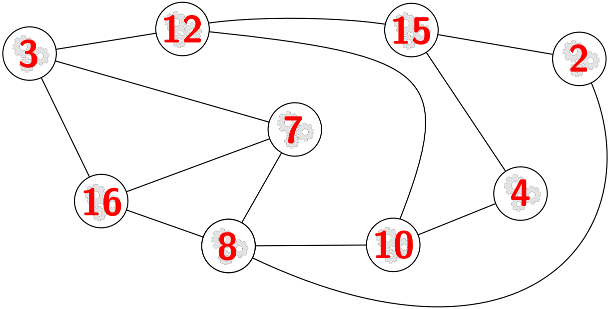

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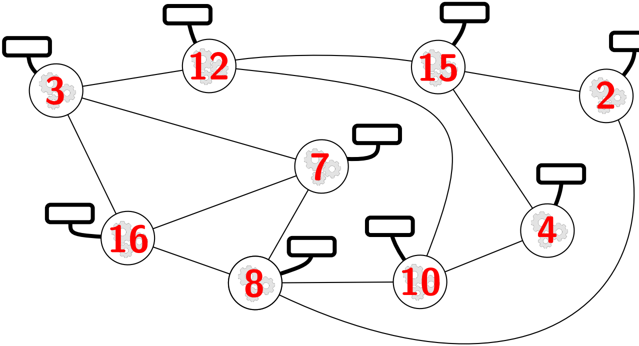

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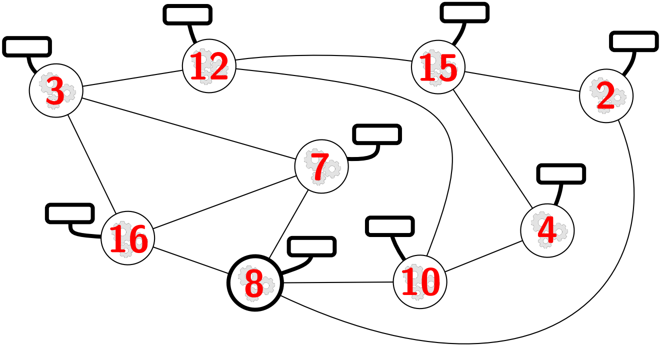

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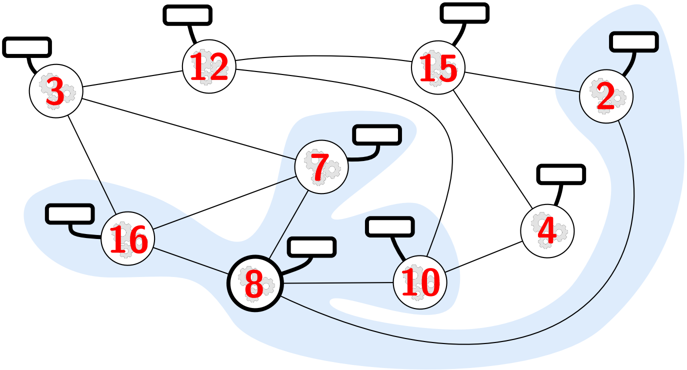

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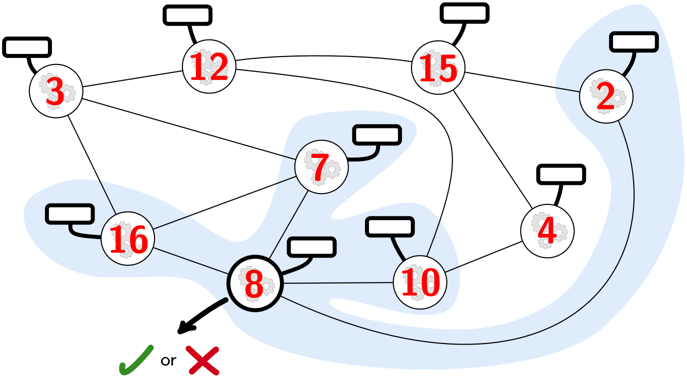

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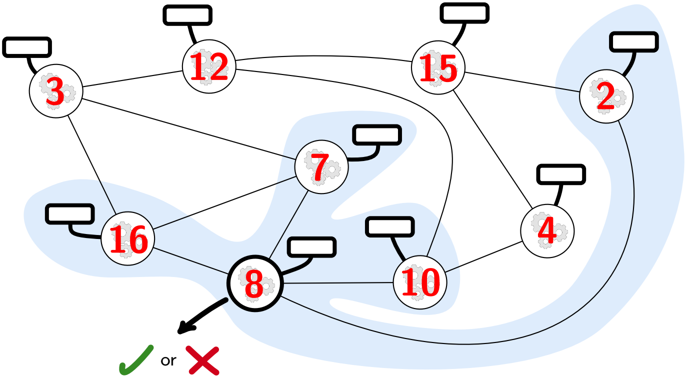

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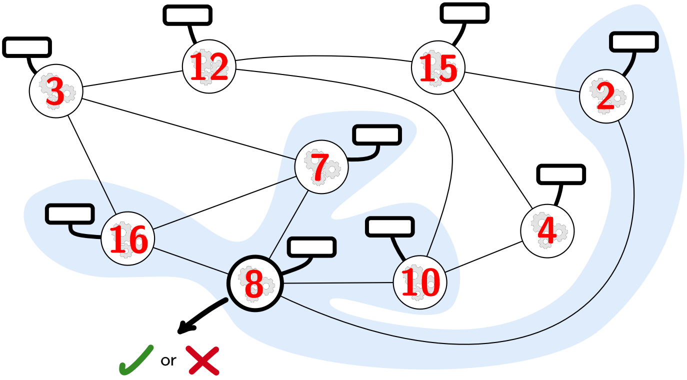

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G satisfies $\mathcal{P}$ $\Longleftrightarrow$ there exists an assignment of the certificates such that G is accepted

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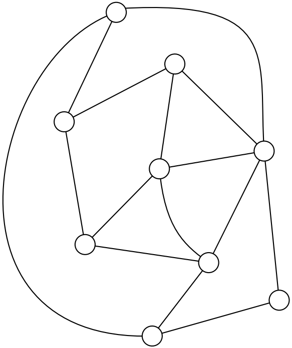

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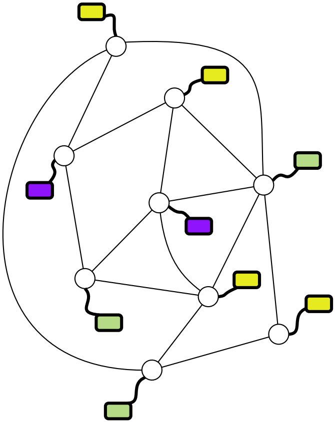

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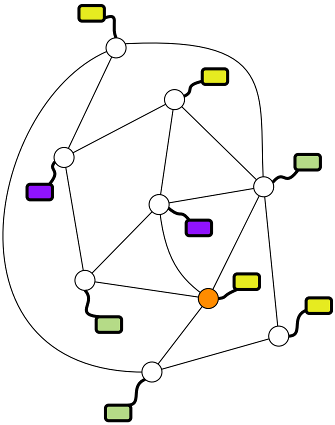

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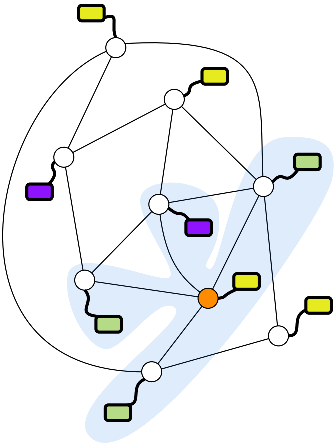

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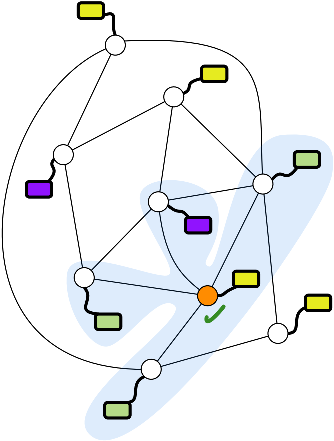

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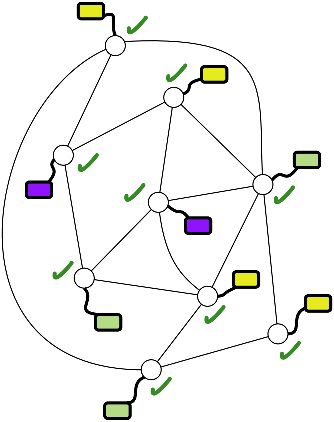

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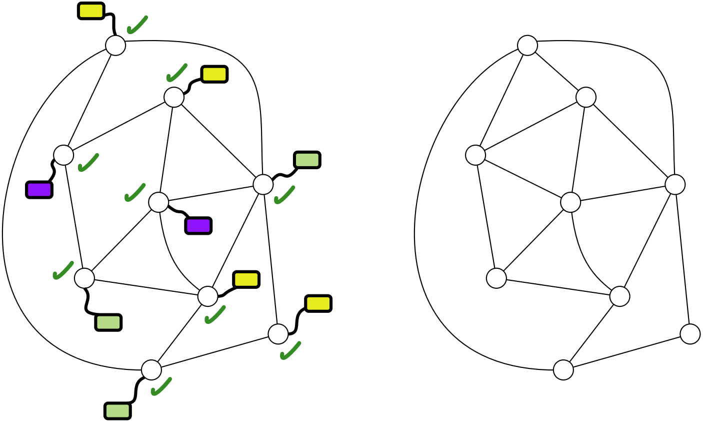

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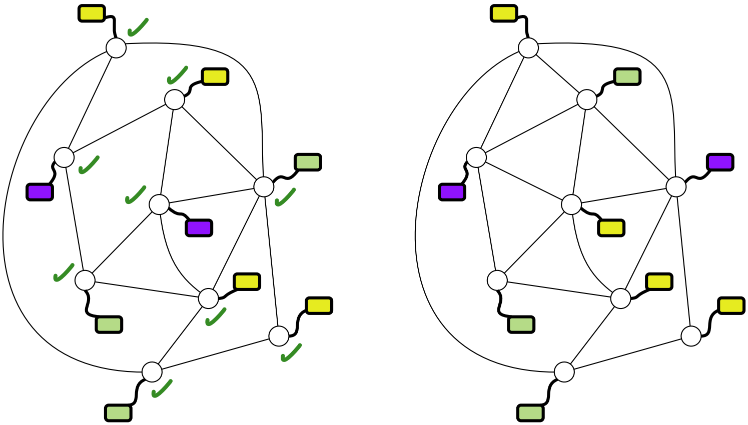

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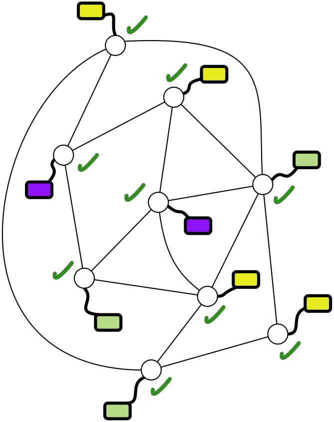

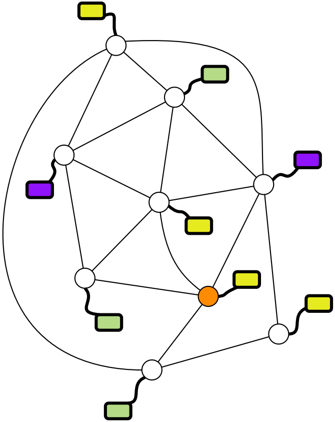

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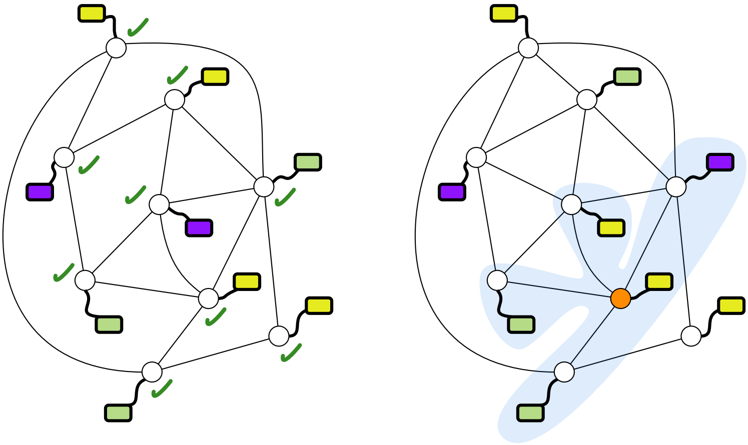

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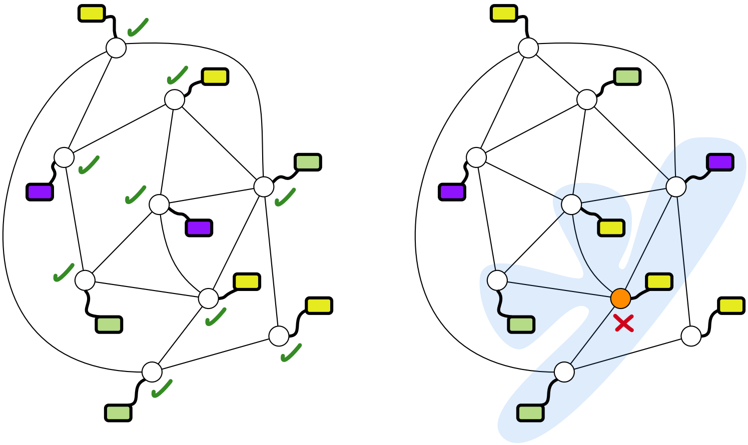

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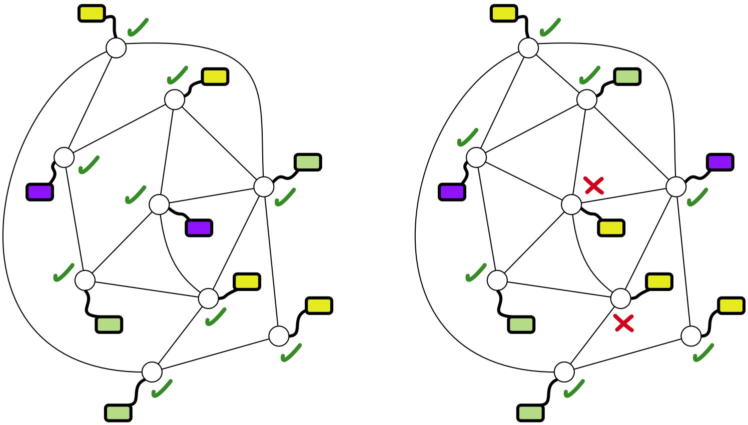

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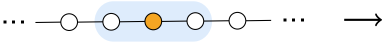

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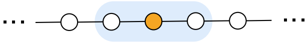

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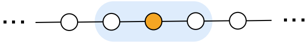

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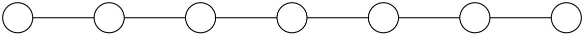

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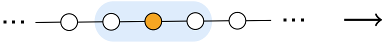

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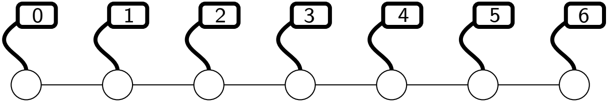

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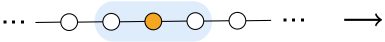

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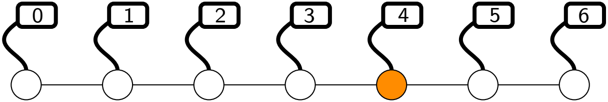

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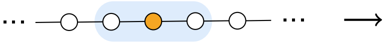

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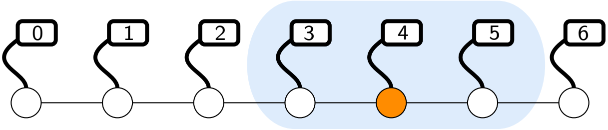

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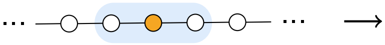

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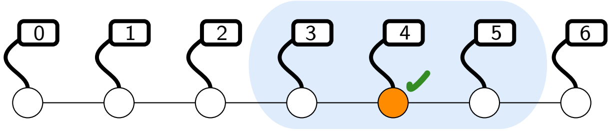

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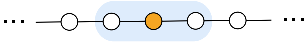

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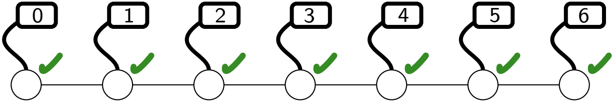

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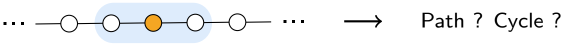

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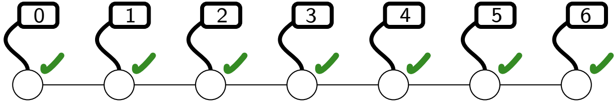

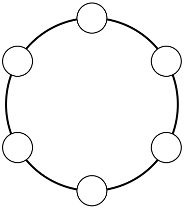

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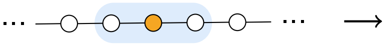

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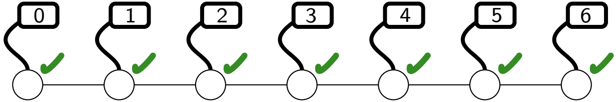

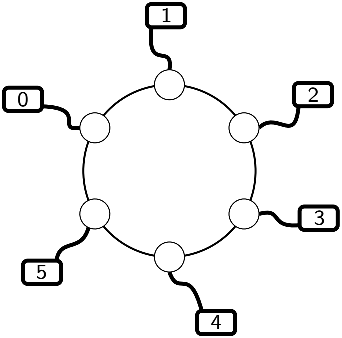

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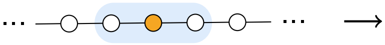

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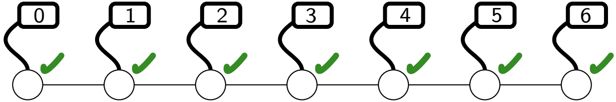

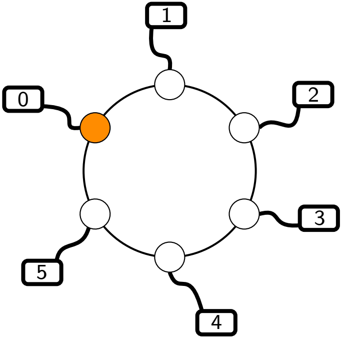

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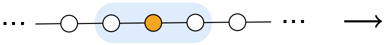

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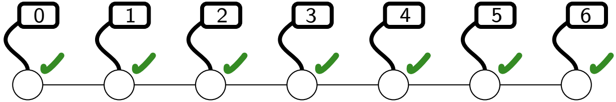

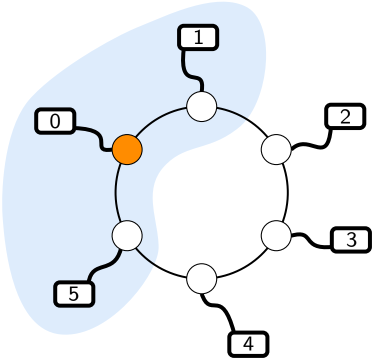

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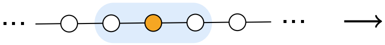

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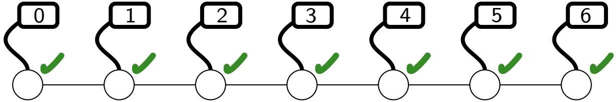

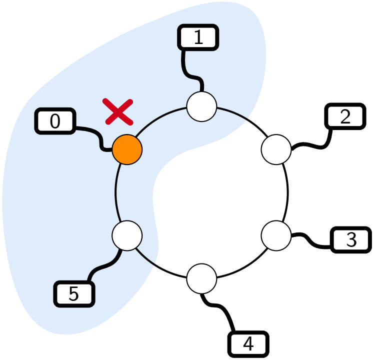

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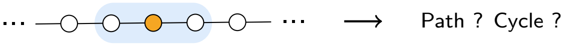

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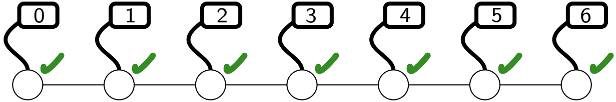

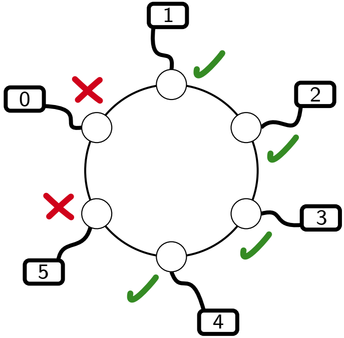

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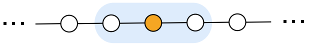

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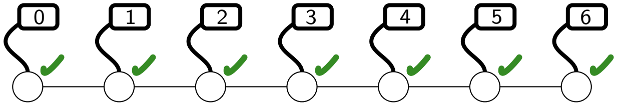

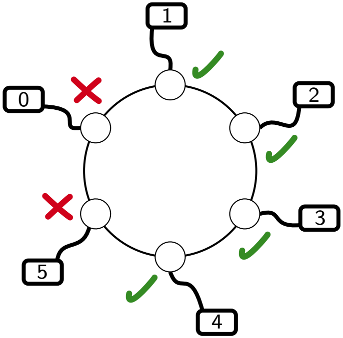

Size of the certificates: $\lceil \log n \rceil$

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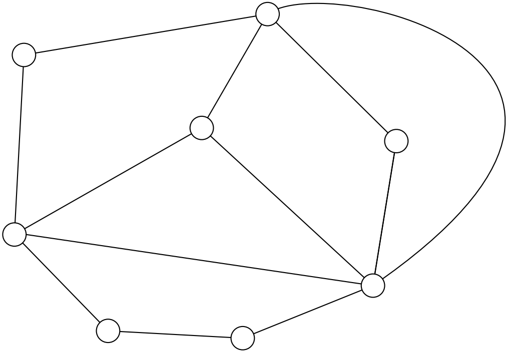

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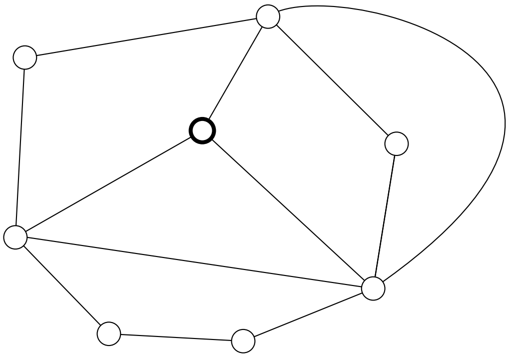

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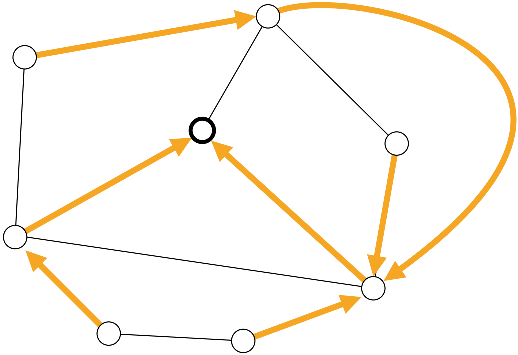

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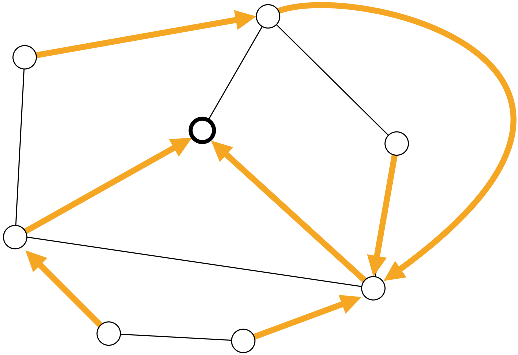

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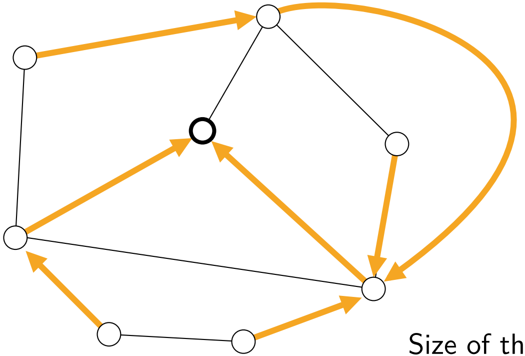

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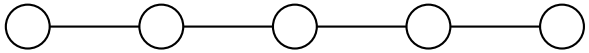
$P_5 =$ 

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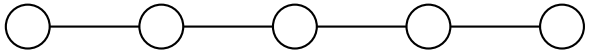
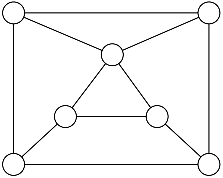
$P_5 =$ 



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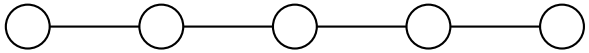
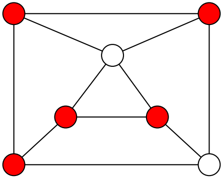
$P_5 =$ 



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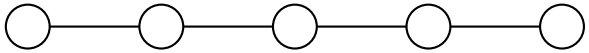
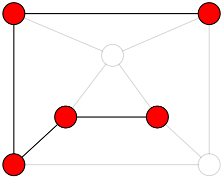
$P_5 =$ 



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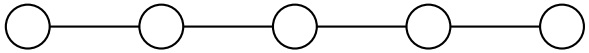
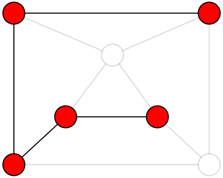
$P_5 =$ 



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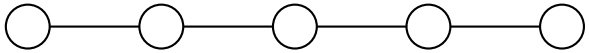
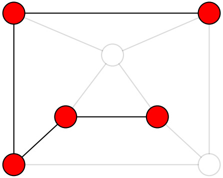
$P_5 =$ 



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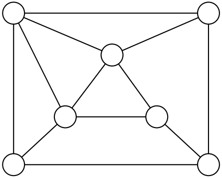

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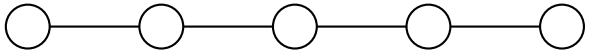
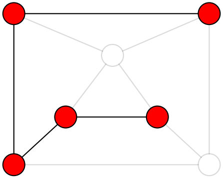
$P_5 =$ 



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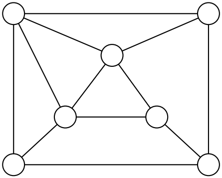

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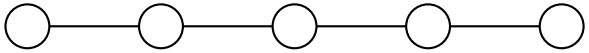

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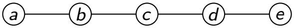

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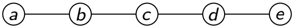

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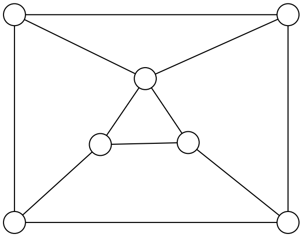

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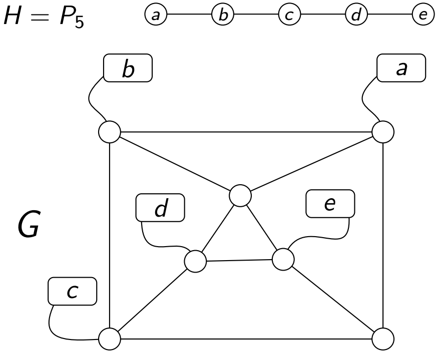

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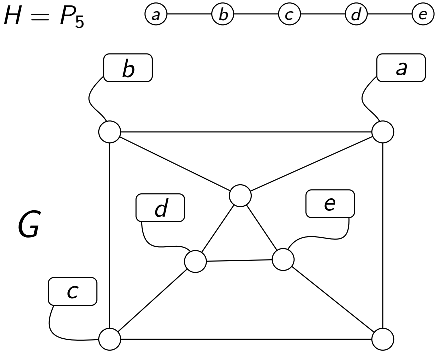

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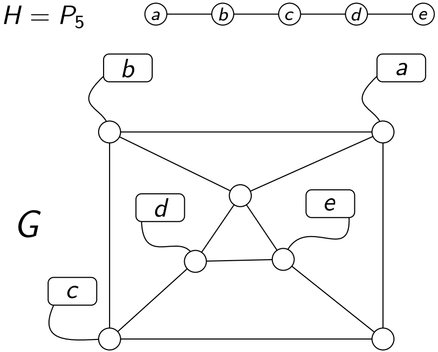

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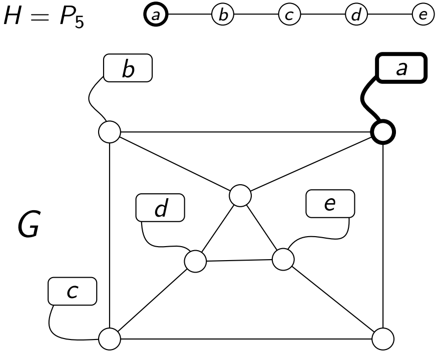

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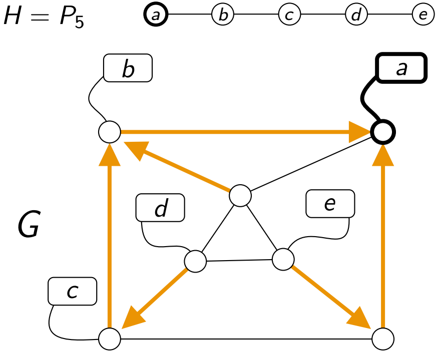

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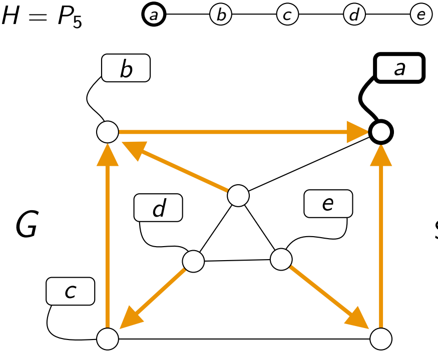

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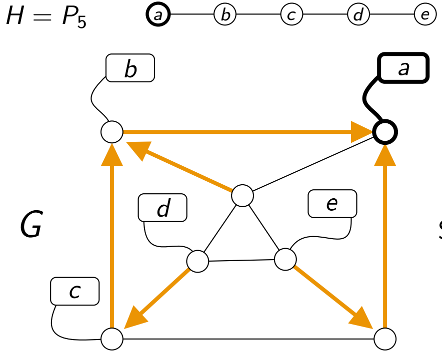

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Question : what about the certification of the complementary property (H-freeness)?

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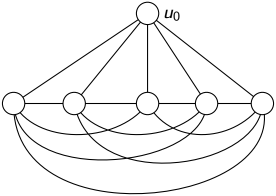

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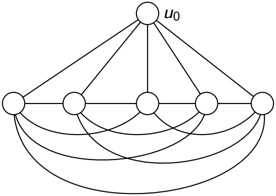

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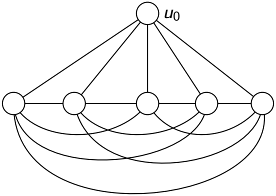

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Lower bounds

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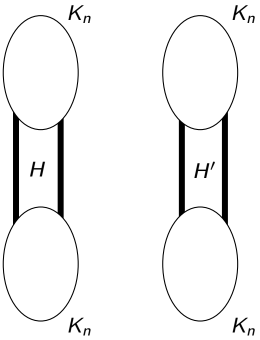

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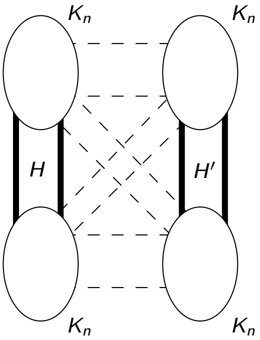

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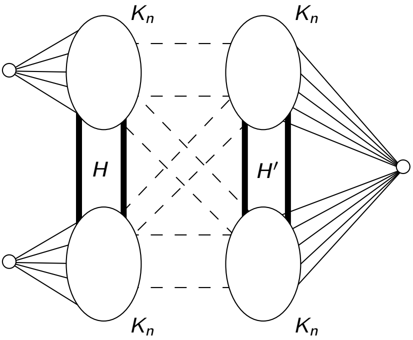

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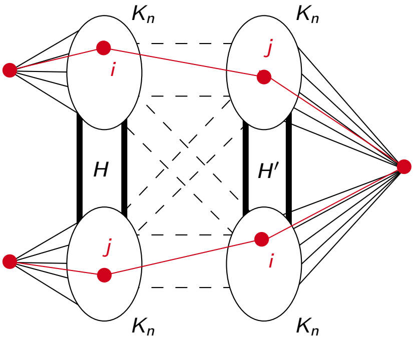

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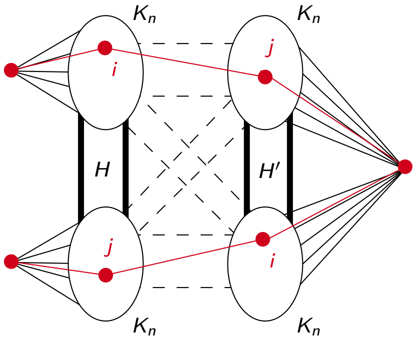

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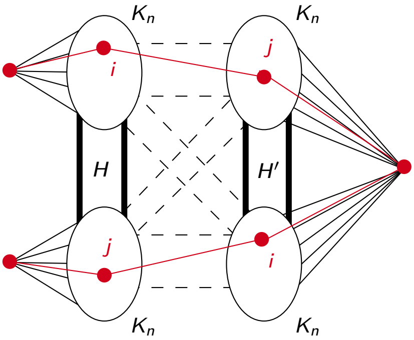

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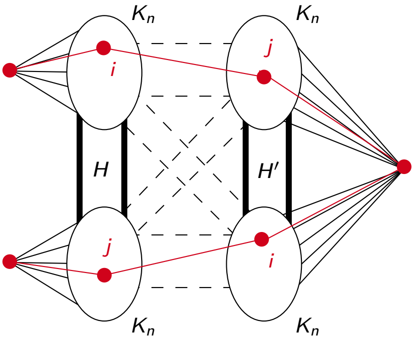

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In the certificates, $\Theta(n^2)$ bits of information have to be transmitted through $O(n)$ vertices

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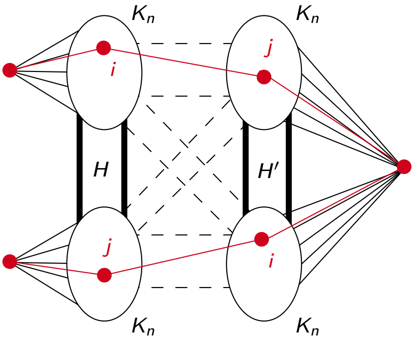

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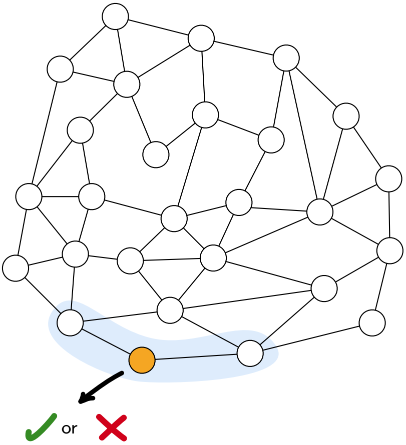

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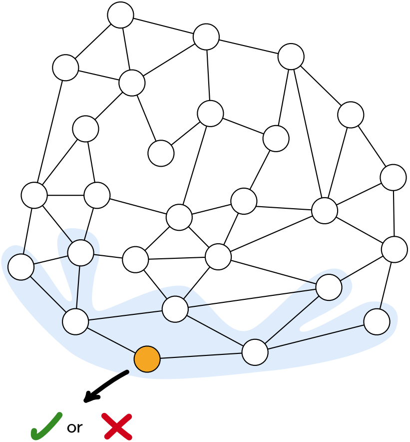

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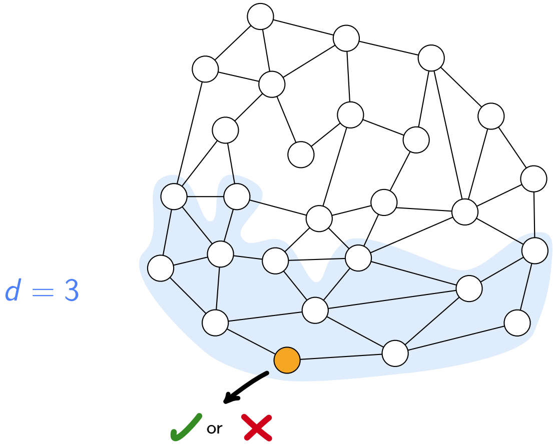

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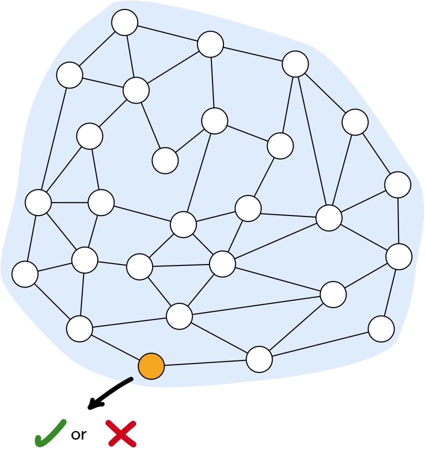

Certification of P_k-freeness: lower bound

Theorem (Bousquet, Cook, Feuilloley, Pierron, Z.)

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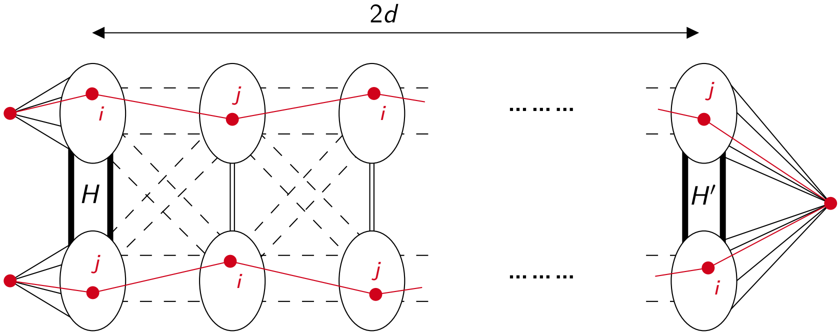

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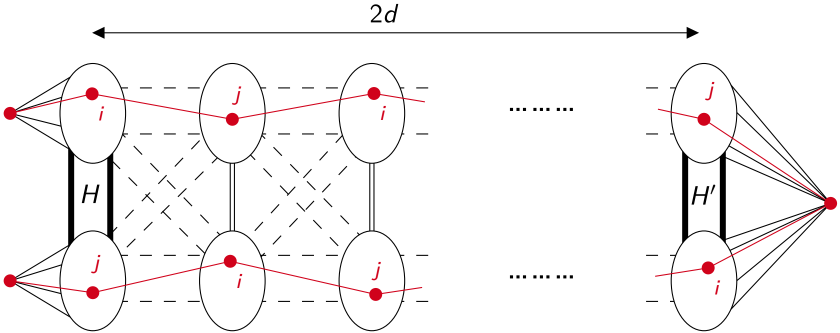

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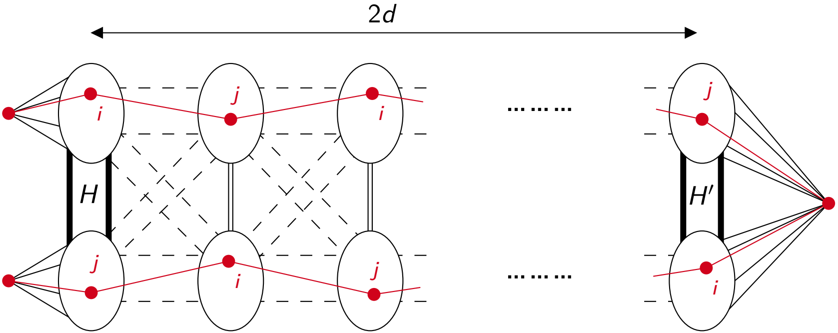

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What about upper bounds ?

Upper bounds

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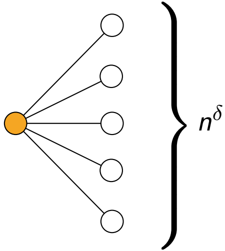

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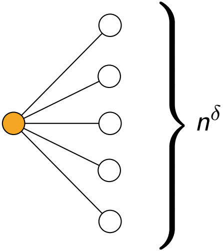

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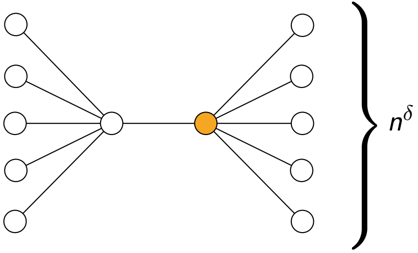

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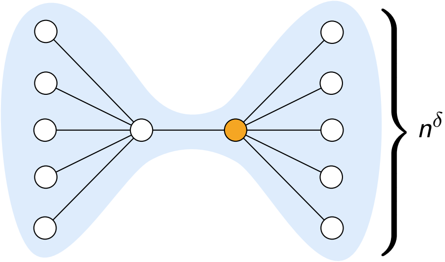

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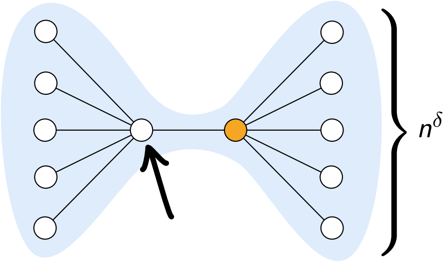

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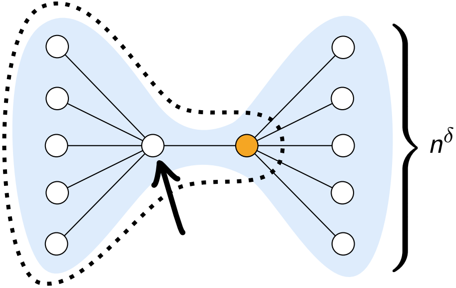

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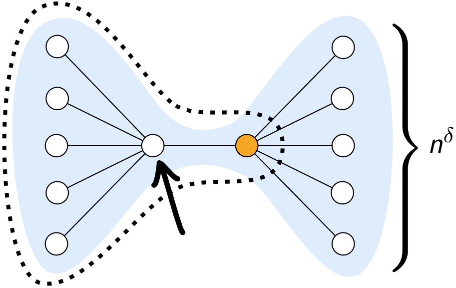

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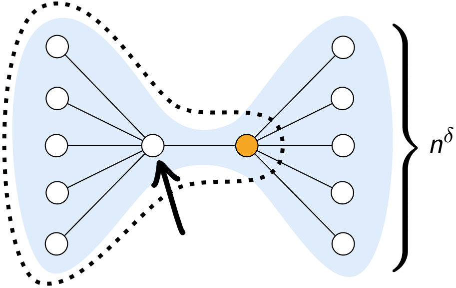

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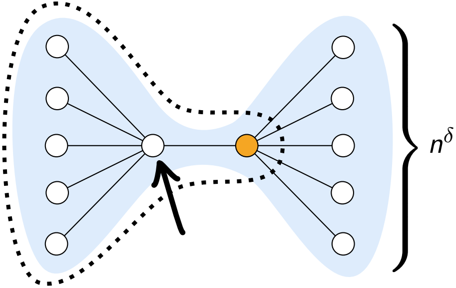
$\Longrightarrow$ every vertex knows G



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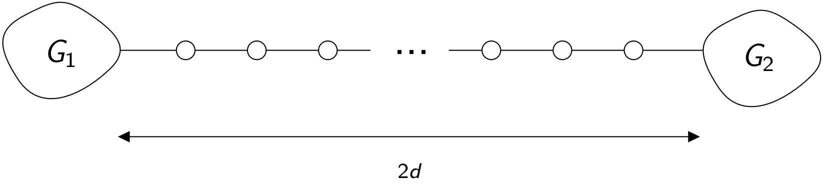

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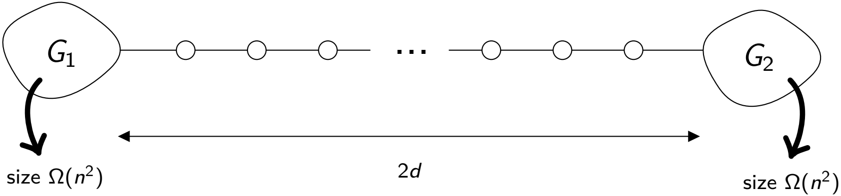

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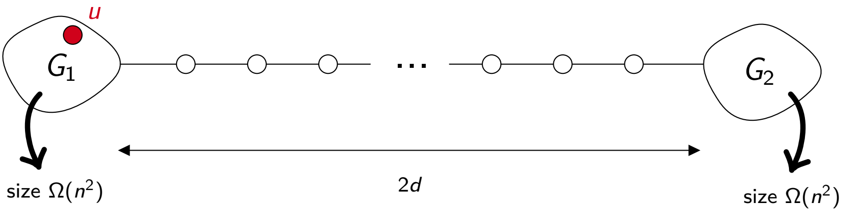

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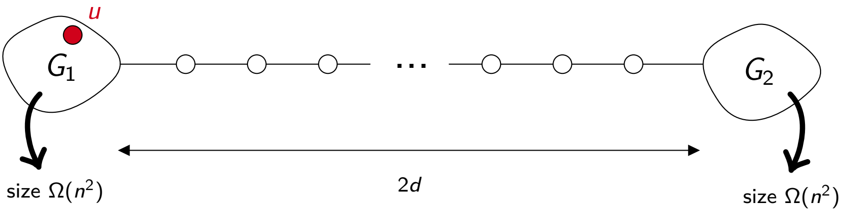

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u can't check that G_2 is correct unless the middle vertices carry n^2 bits
$\Longrightarrow$ it would need certificates of size $\Omega(n^2)$

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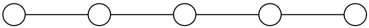

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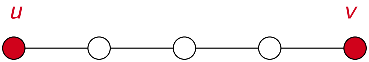

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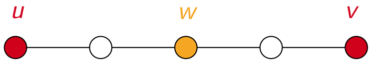

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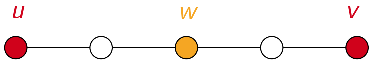

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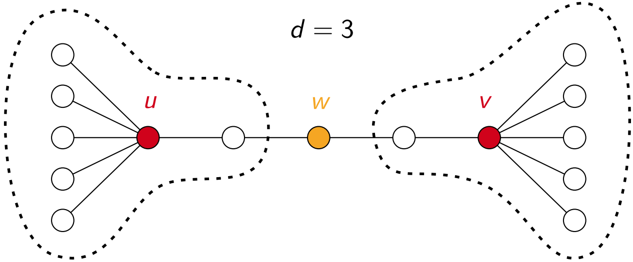

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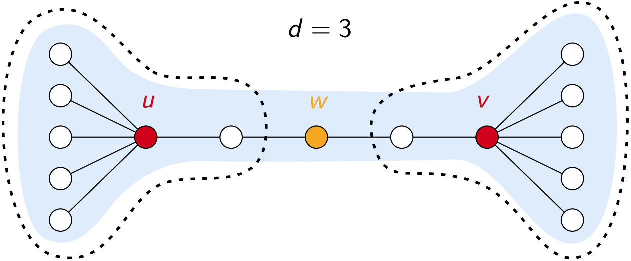

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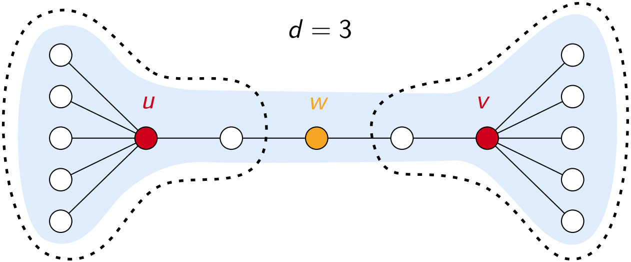

Partition V^+ in components = sets of vertices which reconstruct the same graph.

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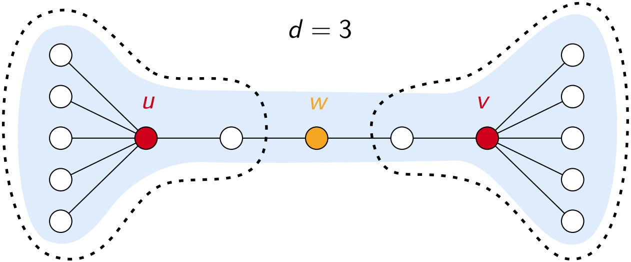
If $u, v \in V^+$ are at distance $\leqslant 2d-2$, there exists $w \in V$ at distance at most $d-1$ from both.

w can check that u and v reconstruct the same graph.

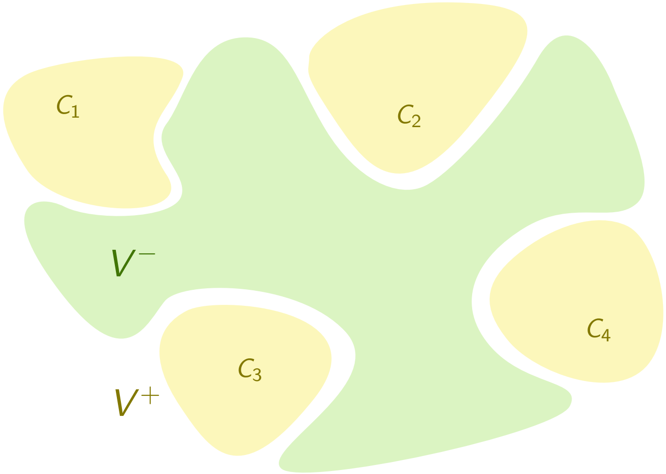


Partition V^+ in components = sets of vertices which reconstruct the same graph.

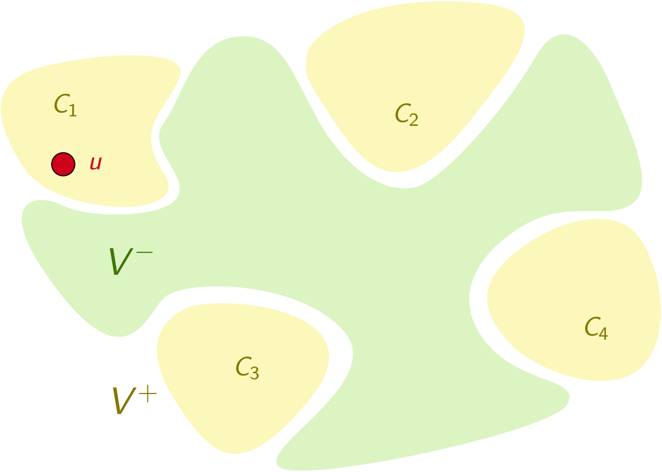
Two different components are far from each other.

Upper bound for path-freeness certification

Upper bound for path-freeness certification



Upper bound for path-freeness certification



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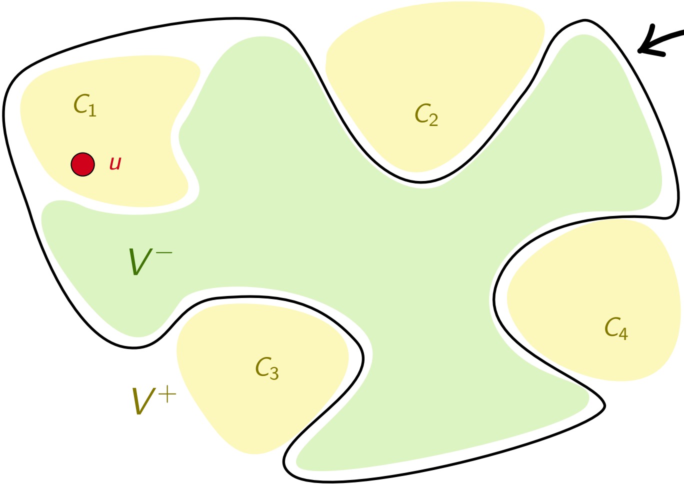
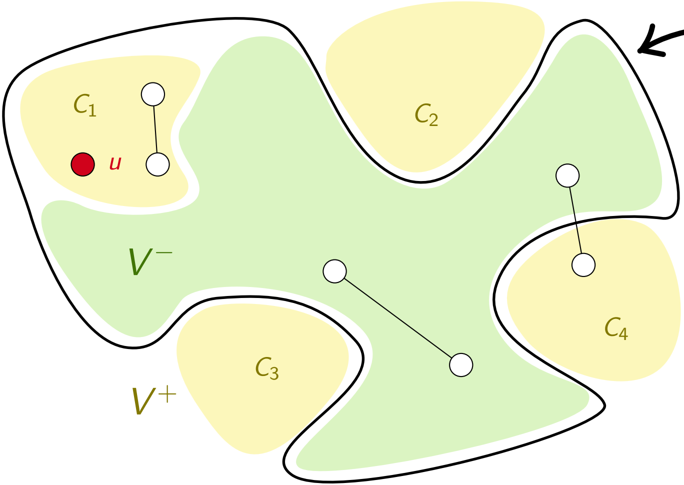


u knows all the edges and non-edges having an endpoint here

Upper bound for path-freeness certification



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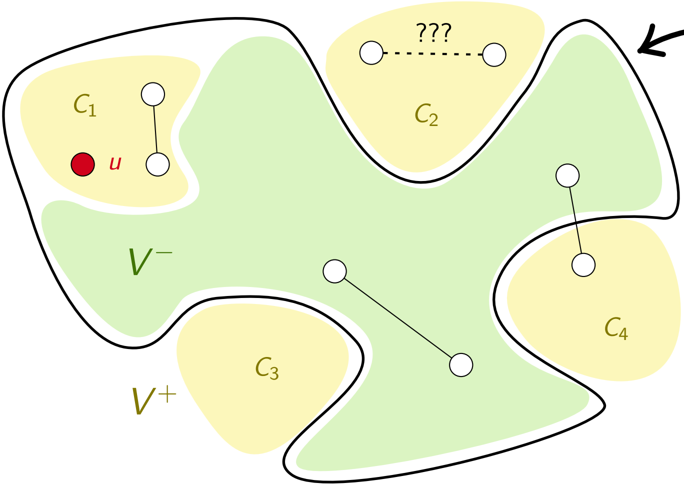
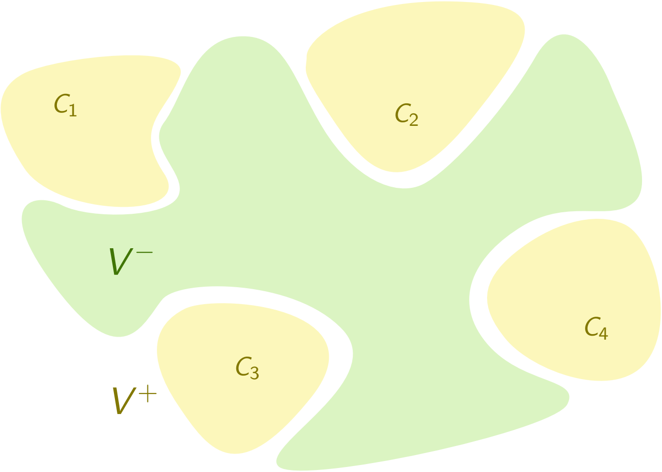


u knows all the edges and non-edges having an endpoint here

Upper bound for path-freeness certification

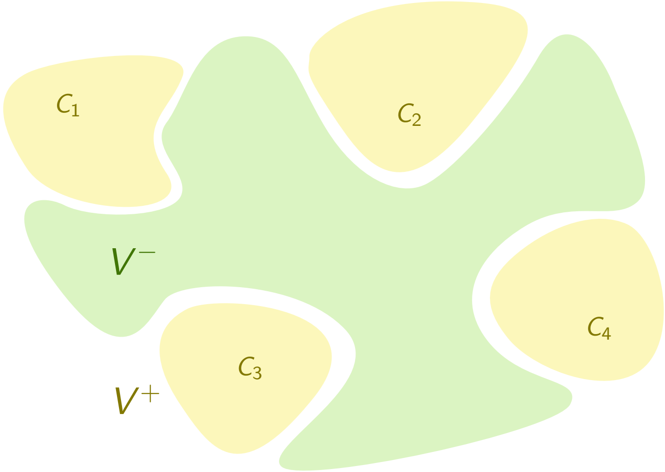
If there is a P_{4d-1}, which vertex detects it ?



Upper bound for path-freeness certification

If there is a P_{4d-1}, which vertex detects it ?

Case 1: no ECC contains at least two vertices of P_{4d-1}.



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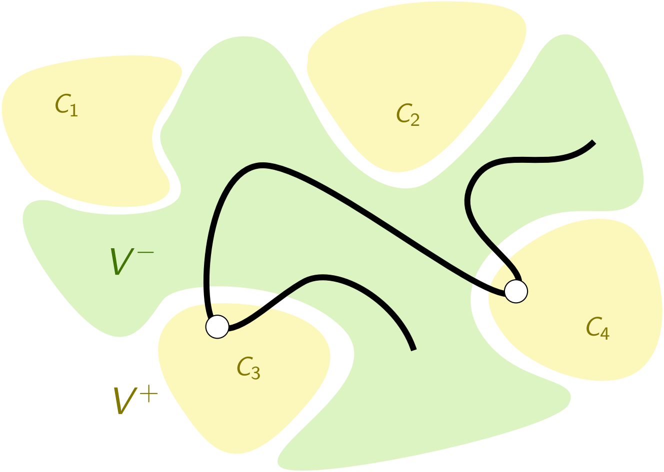

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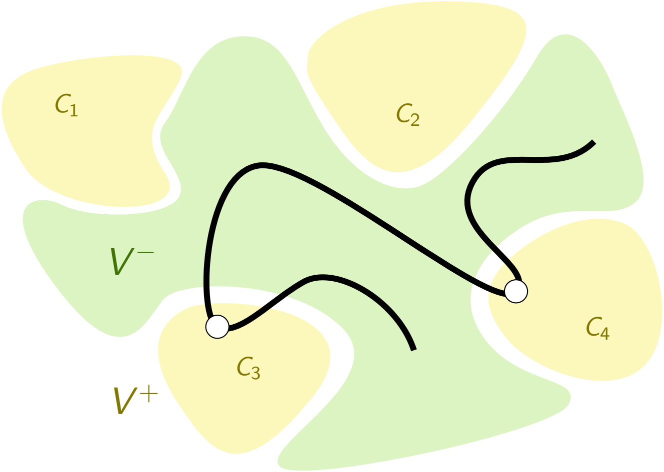
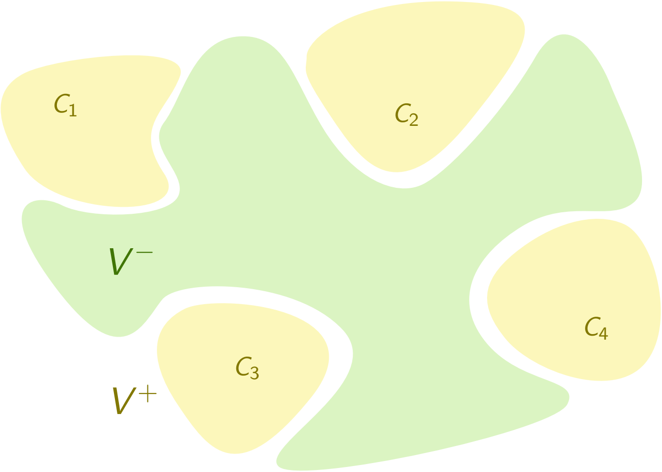


Every vertex detects it !

Upper bound for path-freeness certification

If there is a P_{4d-1}, which vertex detects it ?

Case 2: exactly one ECC contains at least two vertices of P_{4d-1}.



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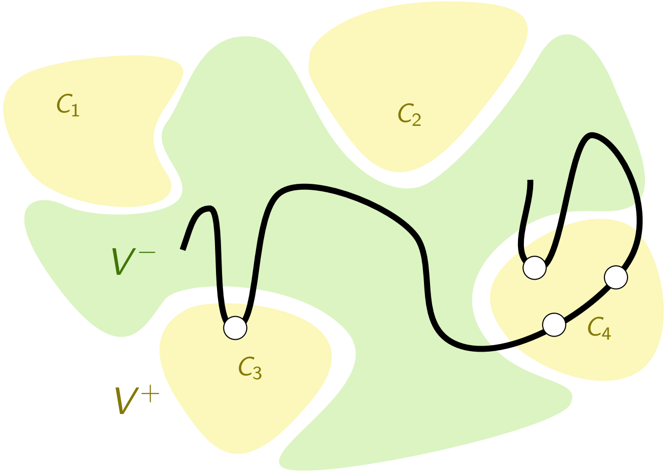

Upper bound for path-freeness certification

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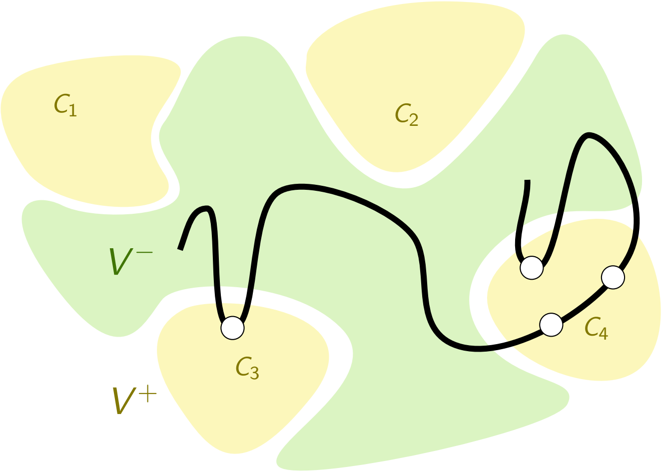
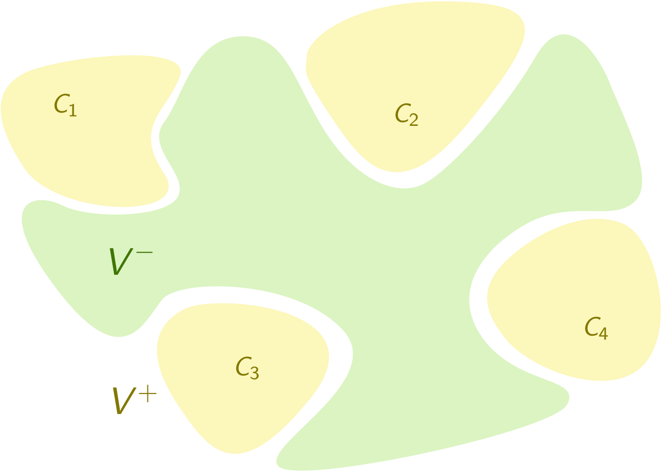


Every vertex in C_4 detects it !

Upper bound for path-freeness certification

If there is a P_{4d-1}, which vertex detects it ?

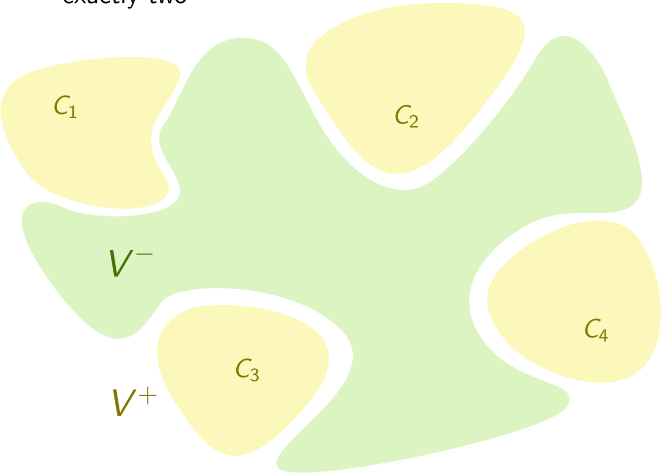
Case 3: at least two ECCs contain at least two vertices of P_{4d-1}.



Upper bound for path-freeness certification

If there is a P_{4d-1}, which vertex detects it ?

Case 3: ~~at least two~~ exactly two ECCs contain at least two vertices of P_{4d-1}.



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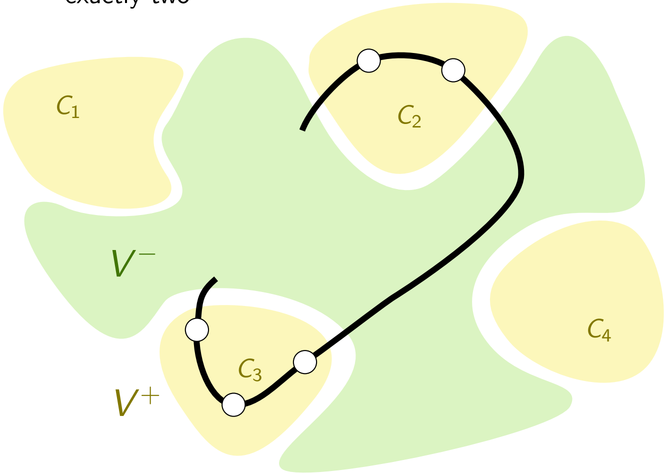

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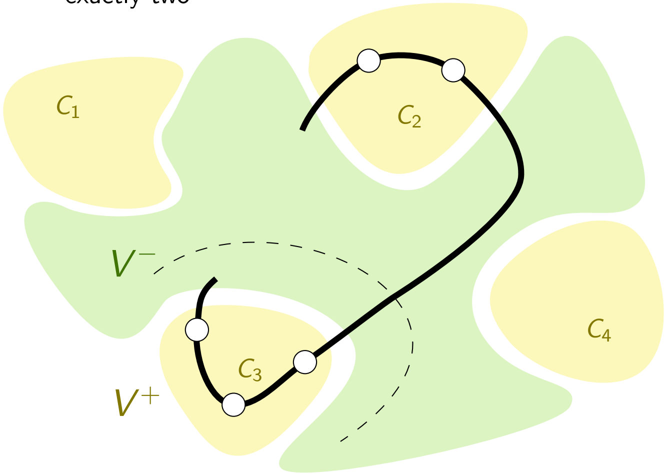

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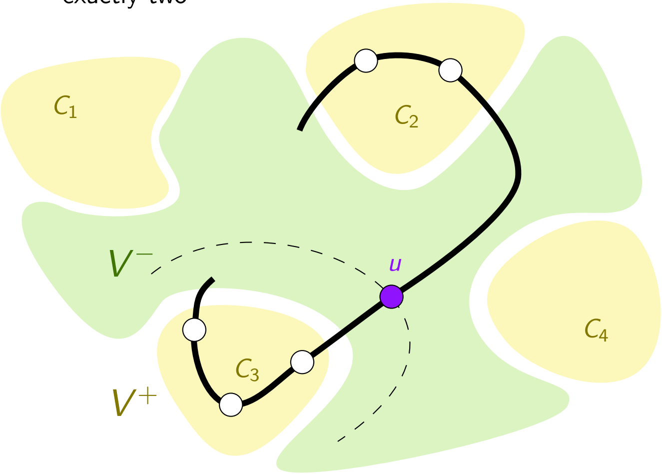

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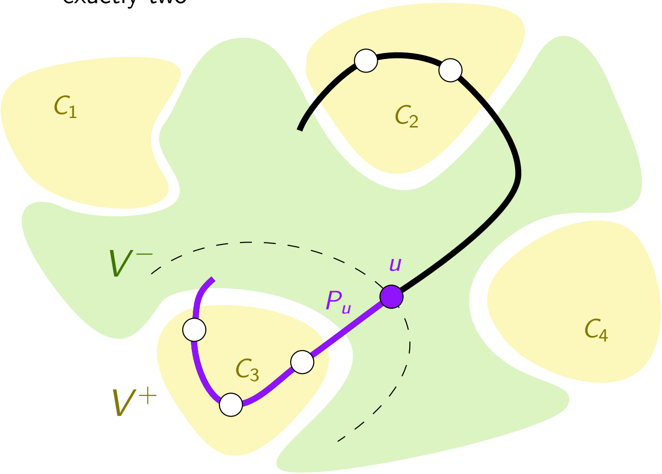

Upper bound for path-freeness certification

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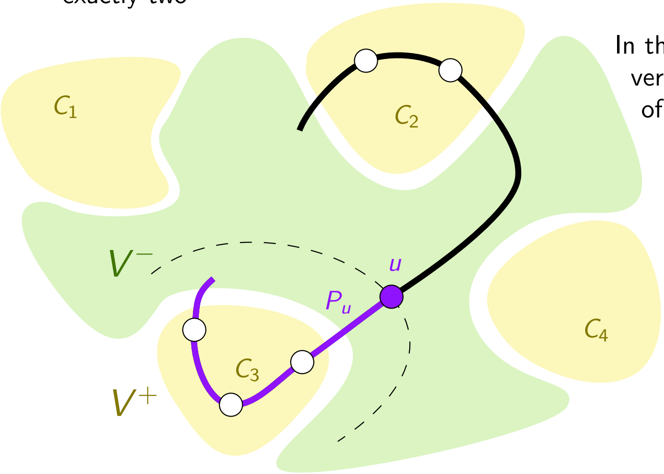

In the certificate of every vertex, add the length of P_u.

Upper bound for path-freeness certification

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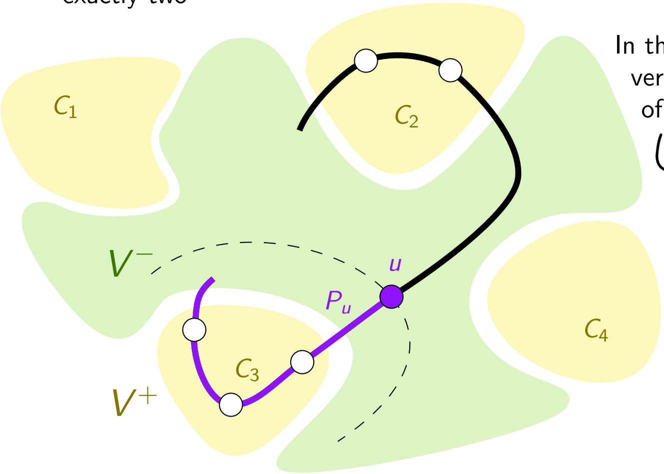
Case 3: ~~at least two~~ exactly two ECCs contain at least two vertices of P_{4d-1}.



In the certificate of every vertex, add the length of P_u.

→ size $O(n \log n)$ in total.

Upper bound for path-freeness certification

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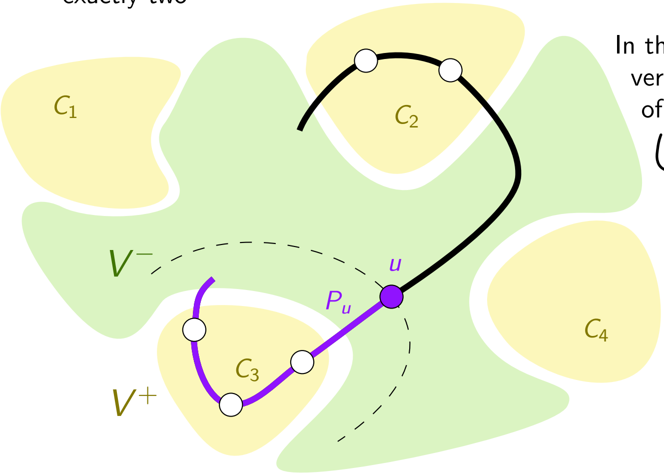

In the certificate of every vertex, add the length of P_u.

↳ size $O(n \log n)$ in total.

Every vertex in C_2 detects it !

Conclusion: overview of our results for *H*-freeness and open questions

Conclusion: overview of our results for H-freeness and open questions

Graph H	Bound

Conclusion: overview of our results for H-freeness and open questions

Graph H	Bound
P_{4d+3}	$\Omega(n)$

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Graph H	Bound
P_{4d+3}	$\Omega(n)$
P_{4d-1}	$\tilde{O}(n^{3/2})$

Conclusion: overview of our results for H-freeness and open questions

Graph H	Bound
P_{4d+3}	$\Omega(n)$
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$\lvert V(H)\rvert \leqslant 4d-1$	$\tilde{O}(n^{3/2})$

Conclusion: overview of our results for H-freeness and open questions

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P_{4d+3}	$\Omega(n)$
P_{4d-1}	$\tilde{O}(n^{3/2})$
$\lvert V(H)\rvert \leqslant 4d-1$	$\tilde{O}(n^{3/2})$
$P_{\lceil 14d/3 \rceil - 1}$	$\tilde{O}(n^{3/2})$

Conclusion: overview of our results for H-freeness and open questions

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P_{4d+3}	$\Omega(n)$
P_{4d-1}	$\tilde{O}(n^{3/2})$
$\lvert V(H)\rvert \leqslant 4d-1$	$\tilde{O}(n^{3/2})$
$P_{\lceil 14d/3\rceil-1}$	$\tilde{O}(n^{3/2})$
P_{3d-1}	$\tilde{O}(n)$

Conclusion: overview of our results for H-freeness and open questions

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P_{4d+3}	$\Omega(n)$
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$\lvert V(H)\rvert \leqslant 4d-1$	$\tilde{O}(n^{3/2})$
$P_{\lceil 14d/3\rceil -1}$	$\tilde{O}(n^{3/2})$
P_{3d-1}	$\tilde{O}(n)$

Open questions:

- what if $d = 1$?

Conclusion: overview of our results for H-freeness and open questions

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Open questions:

- what if $d = 1$? $\longrightarrow \tilde{O}(n^{3/2})$ for P_5

Conclusion: overview of our results for H-freeness and open questions

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$P_{\lceil 14d/3\rceil -1}$	$\tilde{O}(n^{3/2})$
P_{3d-1}	$\tilde{O}(n)$

Open questions:

- what if $d = 1$? $\longrightarrow \tilde{O}(n^{3/2})$ for P_5
- can we get subquadratic upper-bounds for $P_{\alpha d}$ if $\alpha > \frac{14}{3}$?

Conclusion: overview of our results for H-freeness and open questions

Graph H	Bound
P_{4d+3}	$\Omega(n)$
P_{4d-1}	$\tilde{O}(n^{3/2})$
$\|V(H)\| \leqslant 4d-1$	$\tilde{O}(n^{3/2})$
$P_{\lceil 14d/3 \rceil -1}$	$\tilde{O}(n^{3/2})$
P_{3d-1}	$\tilde{O}(n)$

Open questions:

- what if $d = 1$? $\longrightarrow \tilde{O}(n^{3/2})$ for P_5
- can we get subquadratic upper-bounds for $P_{\alpha d}$ if $\alpha > \frac{14}{3}$?
- can we get a superlinear lower-bound for $P_{10^{1000}d}$?

Thanks for your attention !