Local certification of forbidden subgraphs

Nicolas Bousquet, Linda Cook, Laurent Feuilloley, Théo Pierron, Sébastien Zeitoun

November 21, 2024

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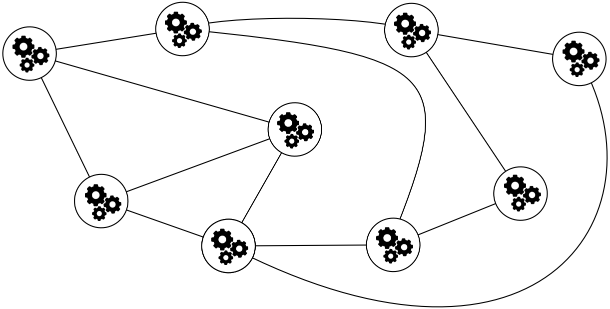
Model: graph, $\begin{cases} \text{vertices} = \text{computation units} \\ \text{edges} = \text{communication channels} \end{cases}$



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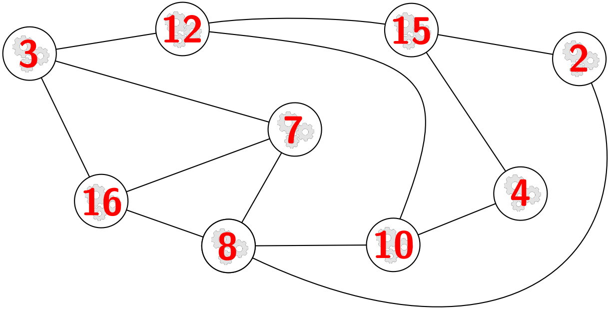

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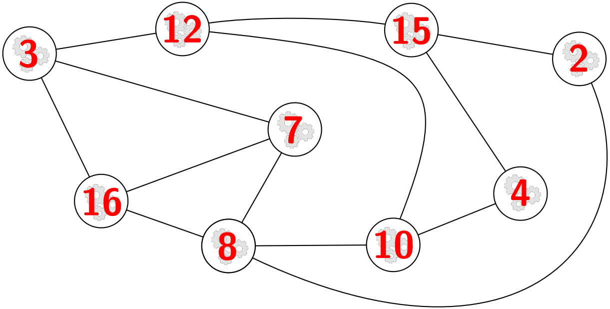

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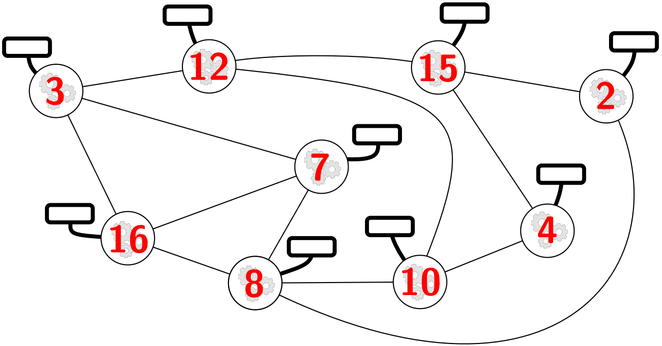

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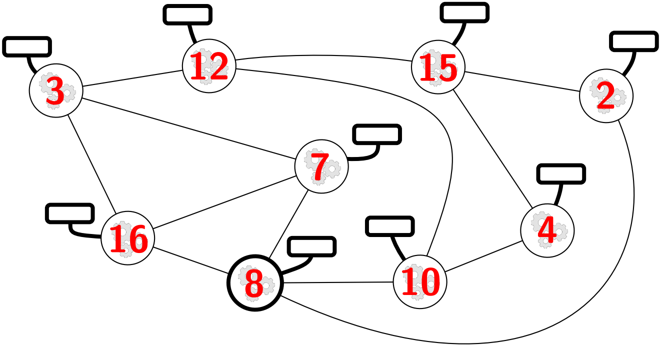

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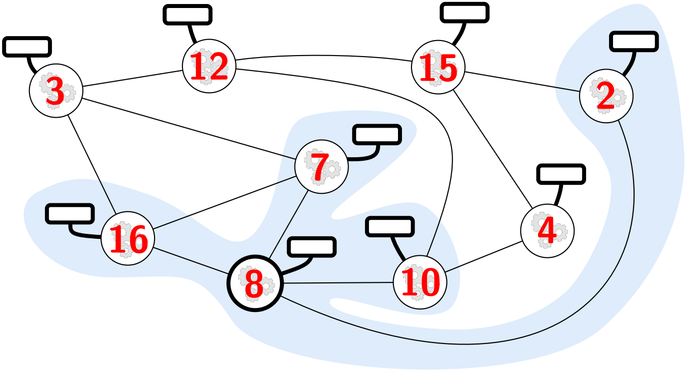

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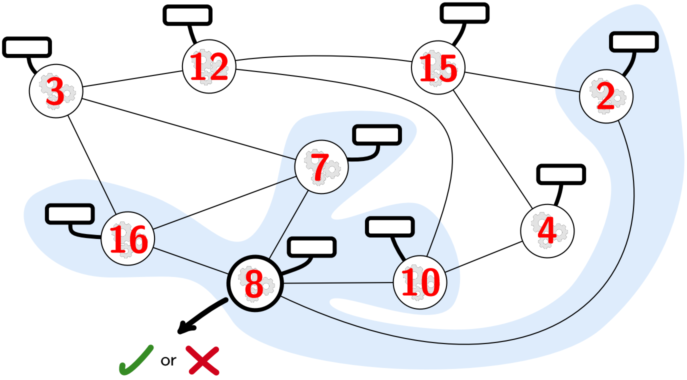

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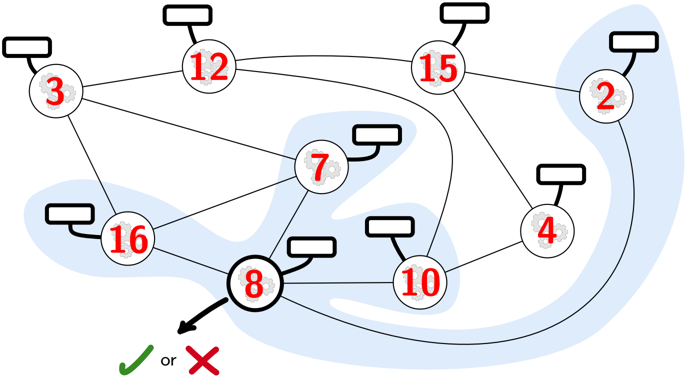

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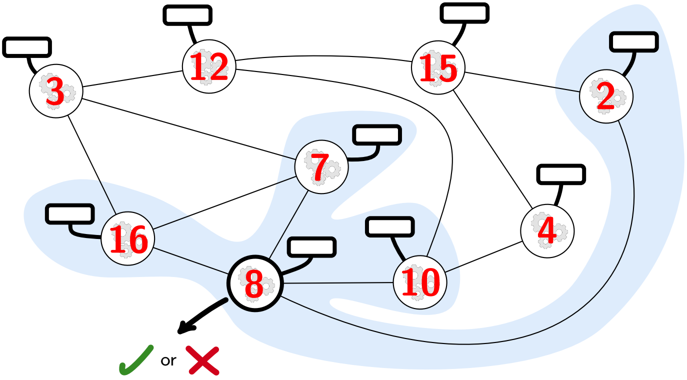

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G satisfies $\mathcal{P} \iff$ there exists an assignment of the certificates such that G is accepted

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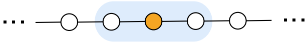

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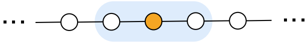

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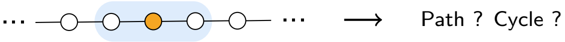

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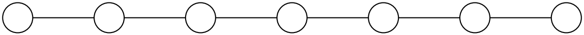

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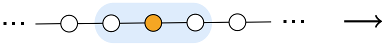

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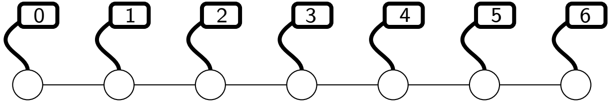

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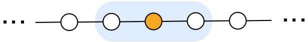

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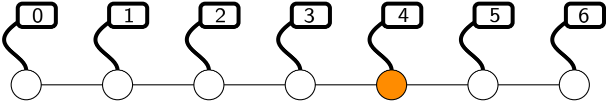

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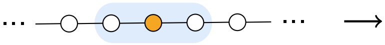

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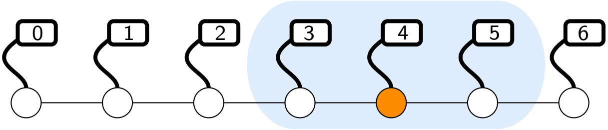

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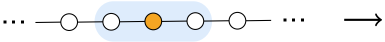

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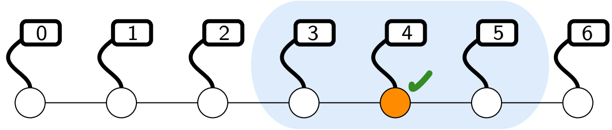

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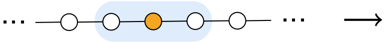

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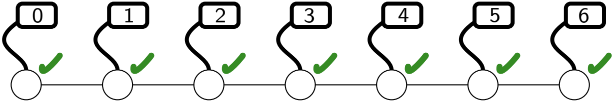

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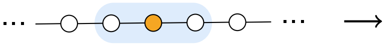

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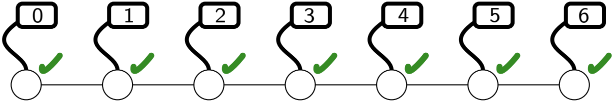

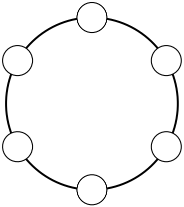

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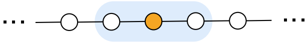

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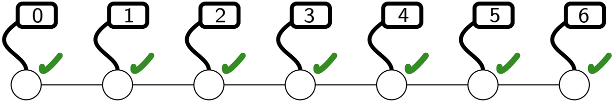

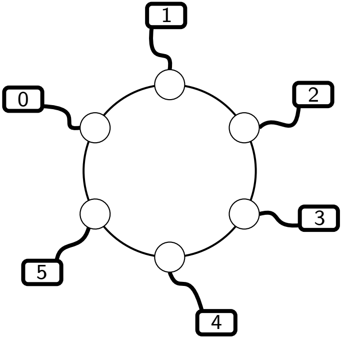

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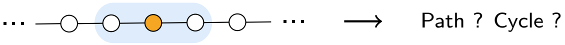

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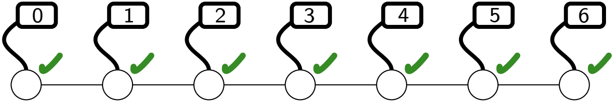

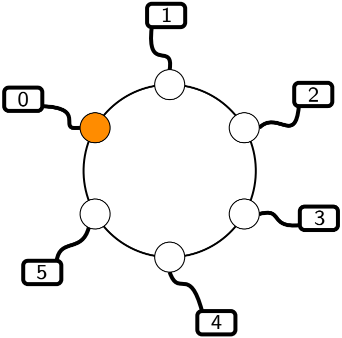

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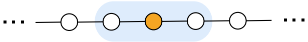

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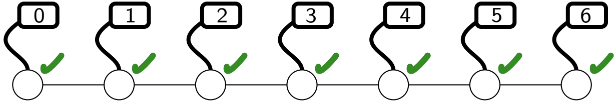

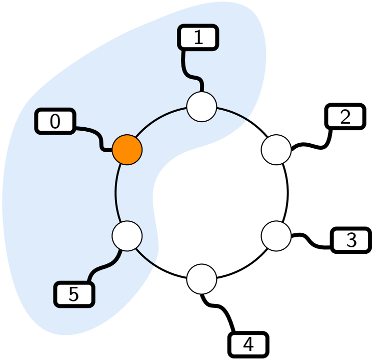

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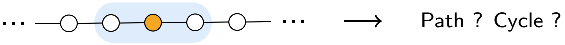

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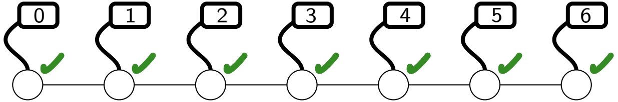

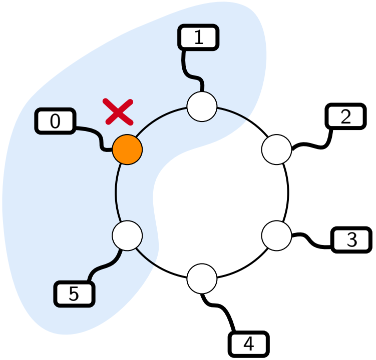

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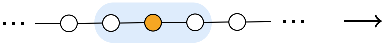

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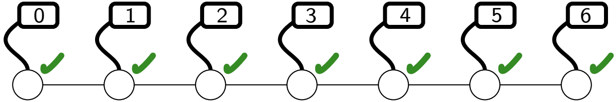

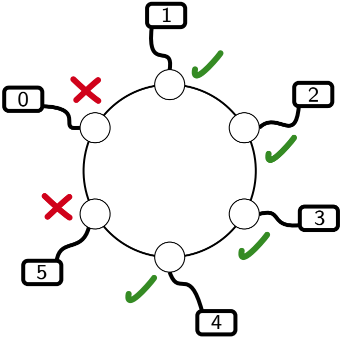

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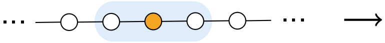

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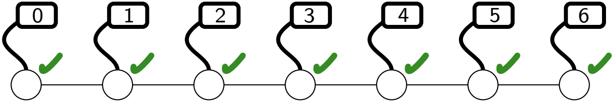

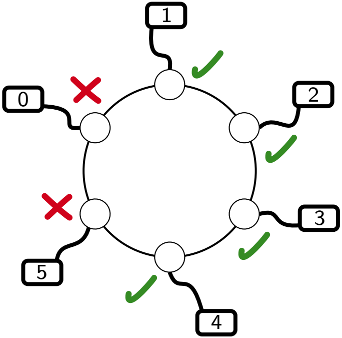

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▪ Non-3-colorability	▪ Paths
▪ Non-trivial automorphism	▪ Trees
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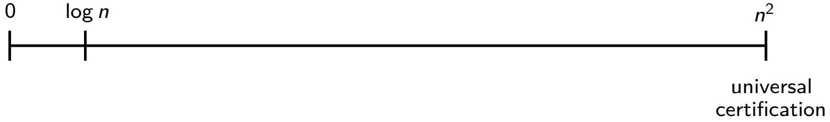

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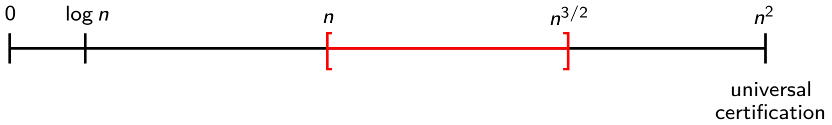

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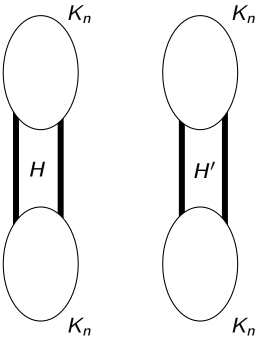

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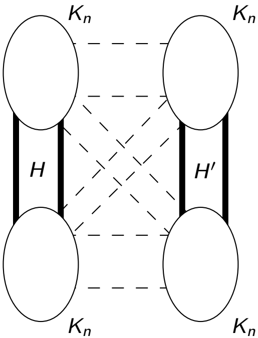

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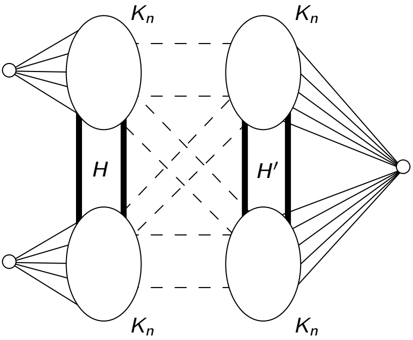

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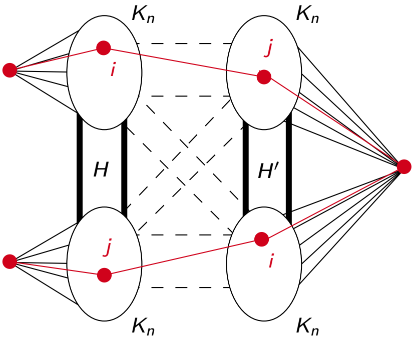

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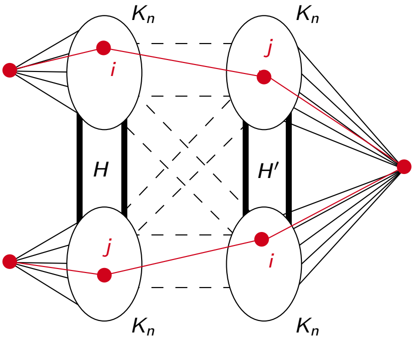

$\exists P_7 \iff H$ and H' have a common non-edge

Certification of P_k-freeness: lower bound

Theorem (Bousquet, Cook, Feuilloley, Pierron, Z.)

$\Omega(n)$ bits are necessary to certify that a graph is P_7-free.

H, H' bipartite graphs with n vertices on each side

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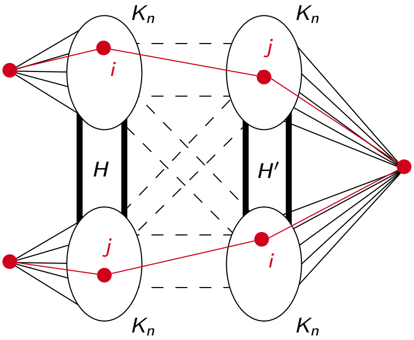

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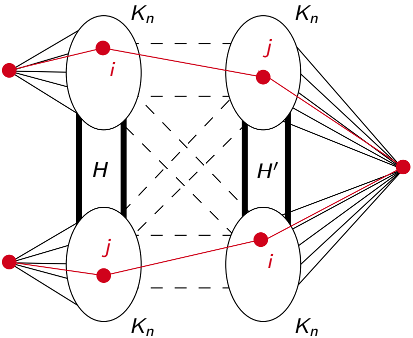

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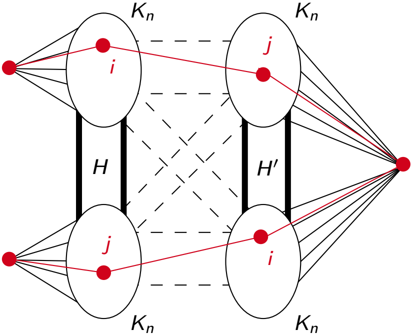

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- vertices (and their identifiers)
- edges
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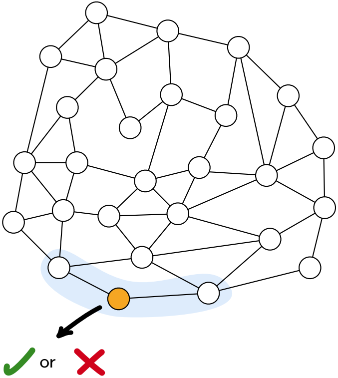

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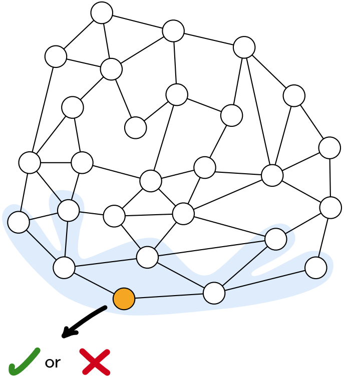

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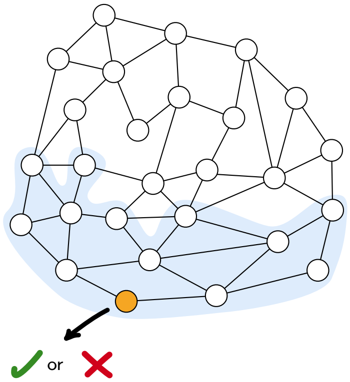

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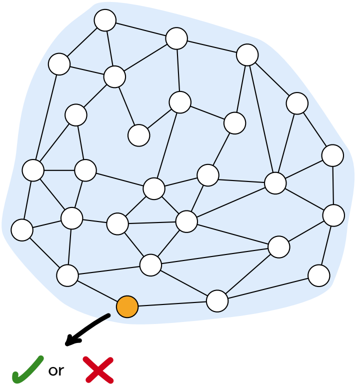

Upper bounds

Certification in graphs of minimum degree $O(n^\delta)$

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Theorem (Bousquet, Cook, Feuilloley, Pierron, Z.)

Let $\delta < 1$. Any property can be certified with certificates of size $O(n^{2-\delta} \log n)$ in graphs of minimum degree n^δ, if vertices can see at distance 2.

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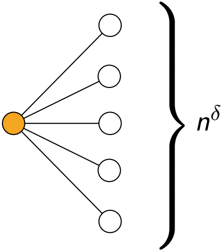

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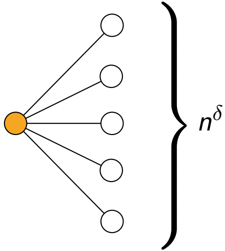

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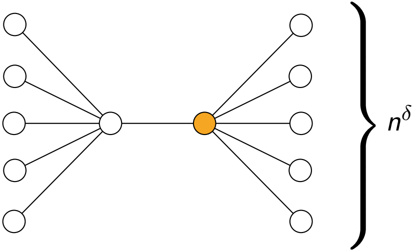

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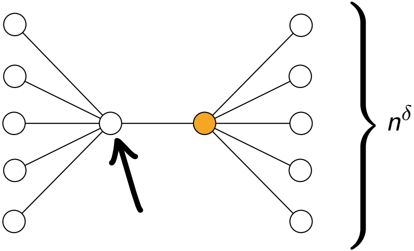

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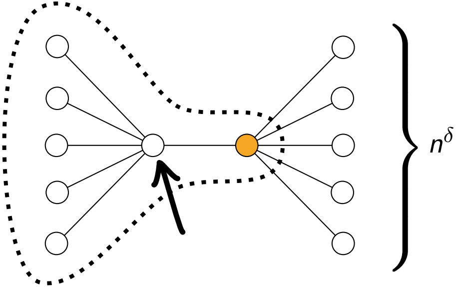

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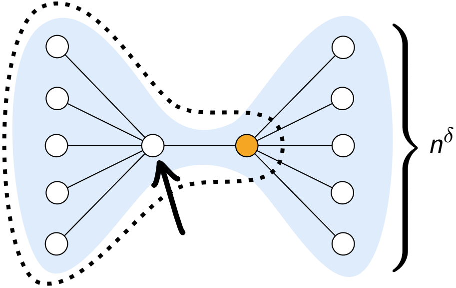

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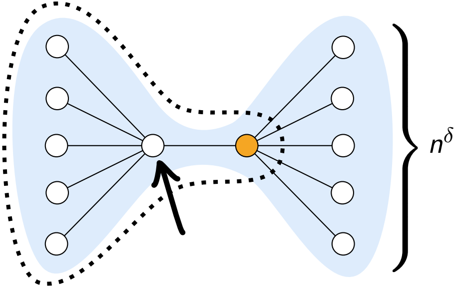

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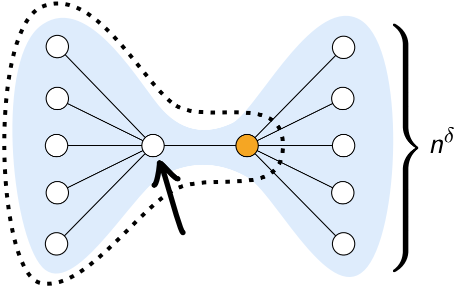

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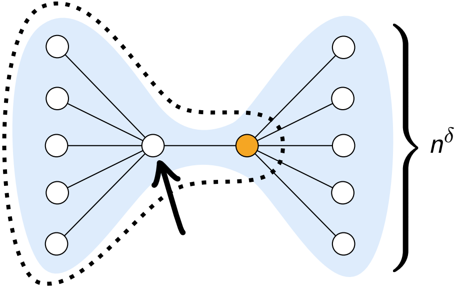
$\Longrightarrow$ every vertex knows G



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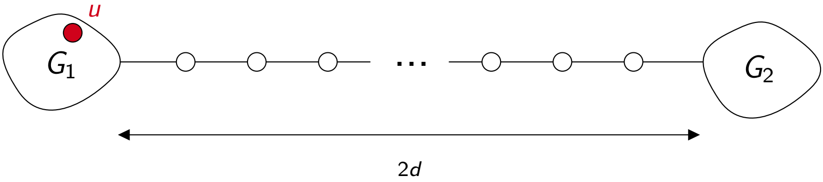

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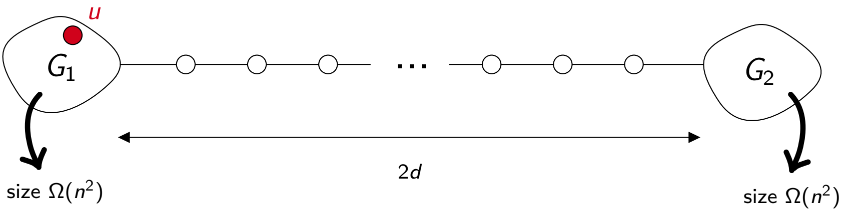

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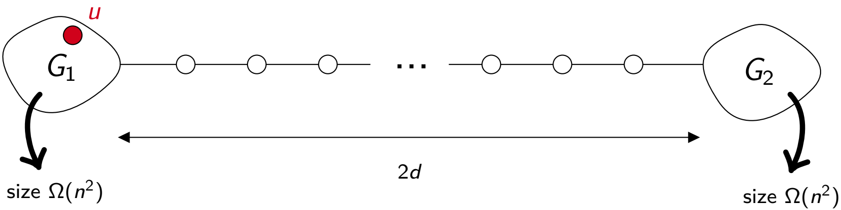

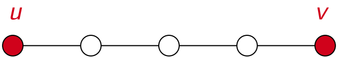

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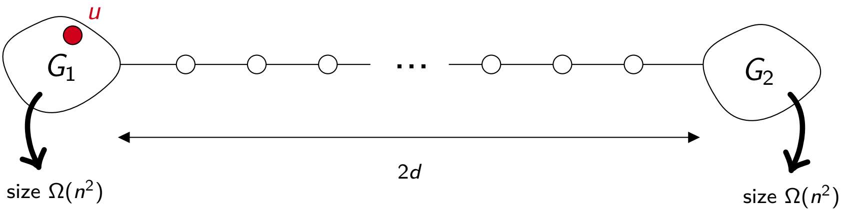

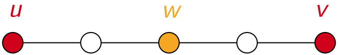

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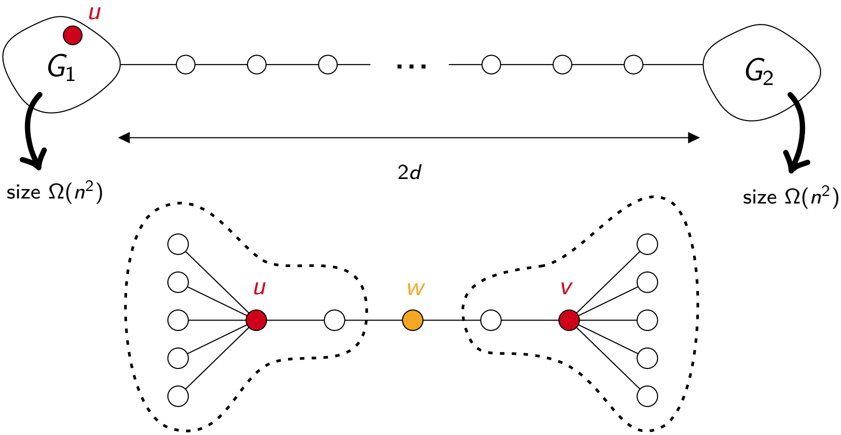

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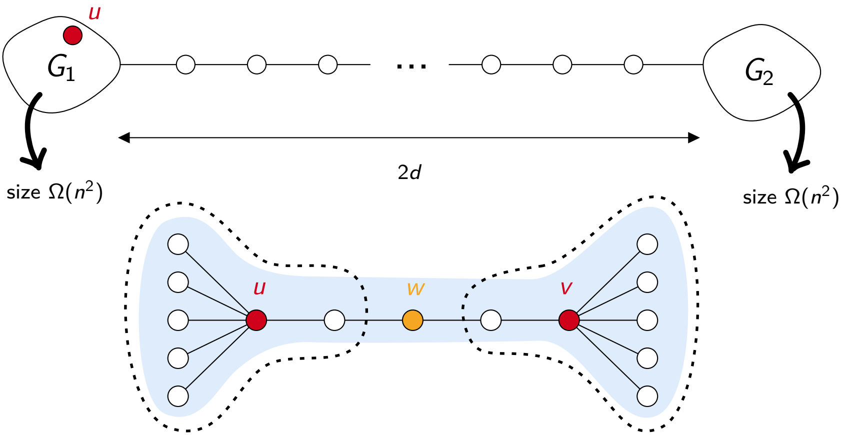

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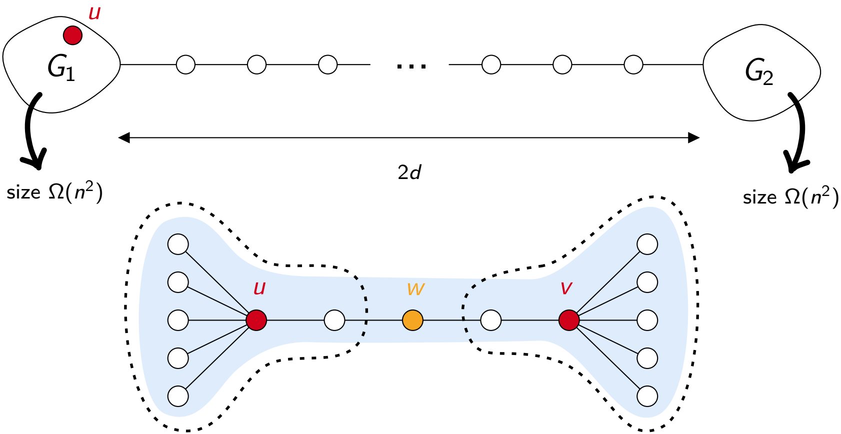

$d(u,v) \leqslant 2d-2 \Longrightarrow u$ and v reconstruct the same graph

Upper bound for path-freeness certification

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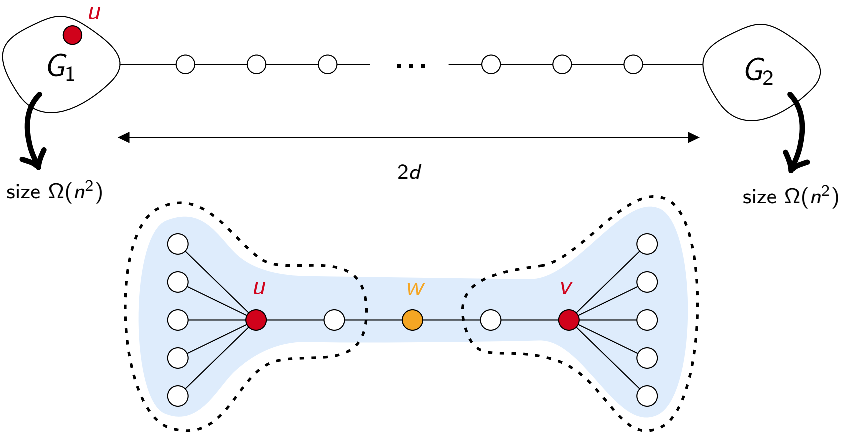

$d(u,v) \leqslant 2d-2 \Longrightarrow u$ and v reconstruct the same graph

Partition V^+ into components: set of vertices which reconstruct the same graph

Upper bound for path-freeness certification

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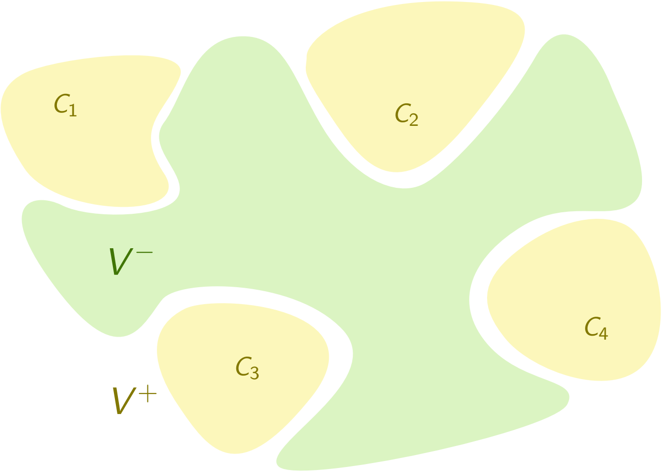
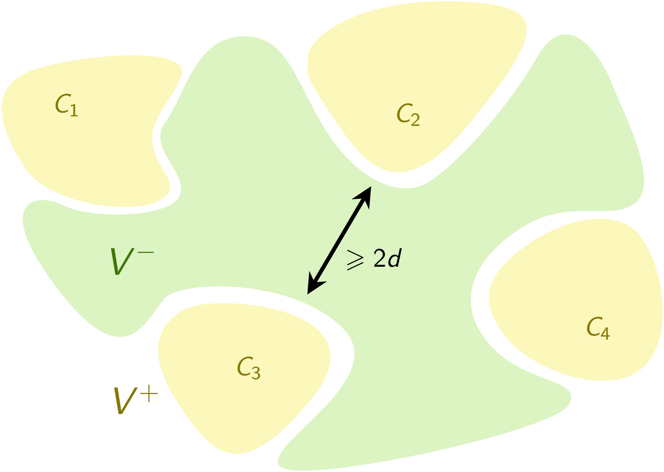


Upper bound for path-freeness certification



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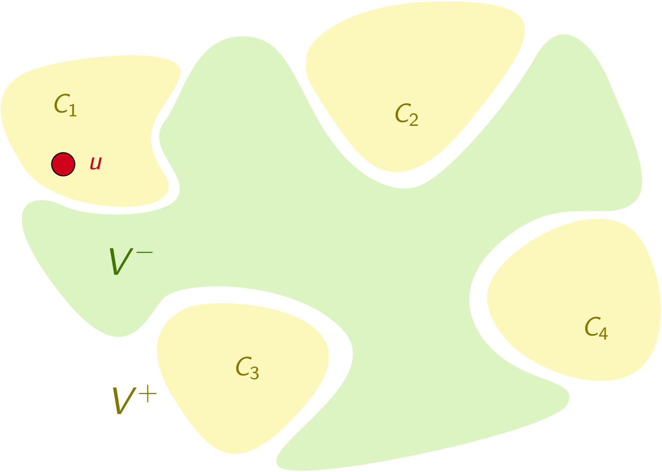

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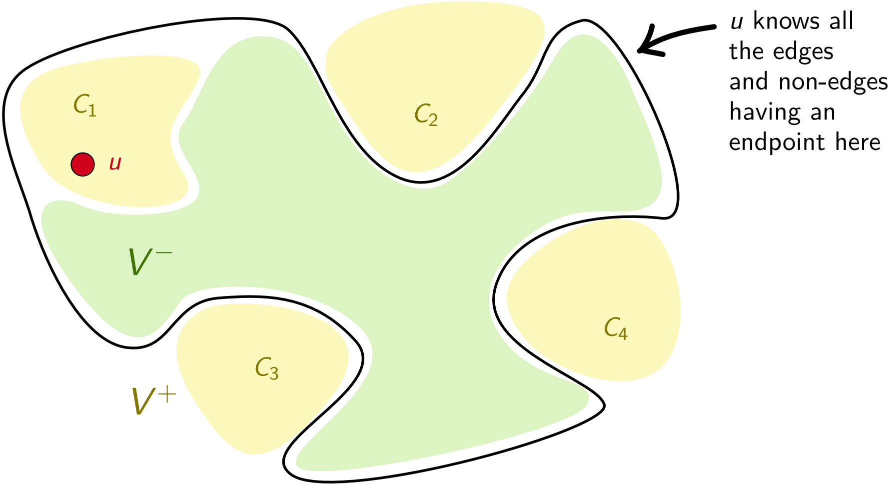
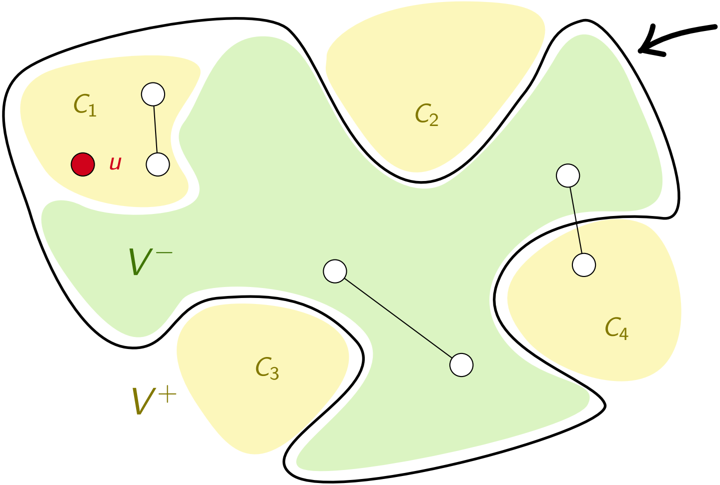
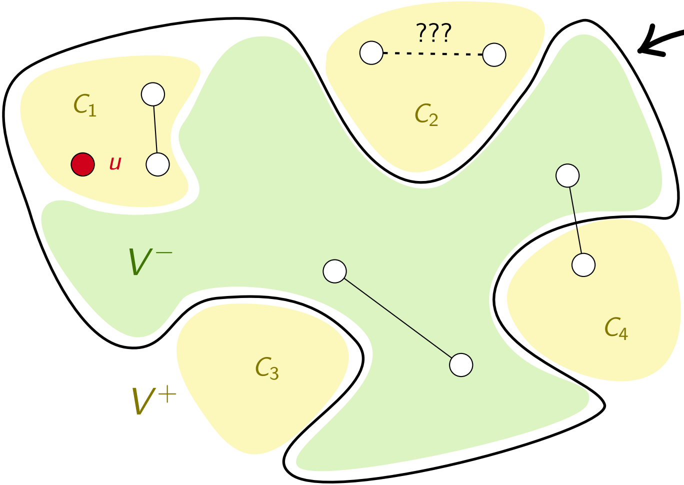


Upper bound for path-freeness certification



u knows all the edges and non-edges having an endpoint here

Upper bound for path-freeness certification



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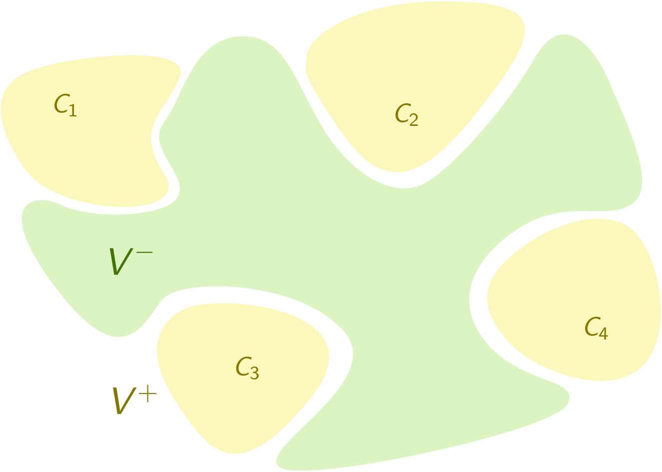
Upper bound for path-freeness certification

If there is a P_{4d-1}, which vertex detects it ?



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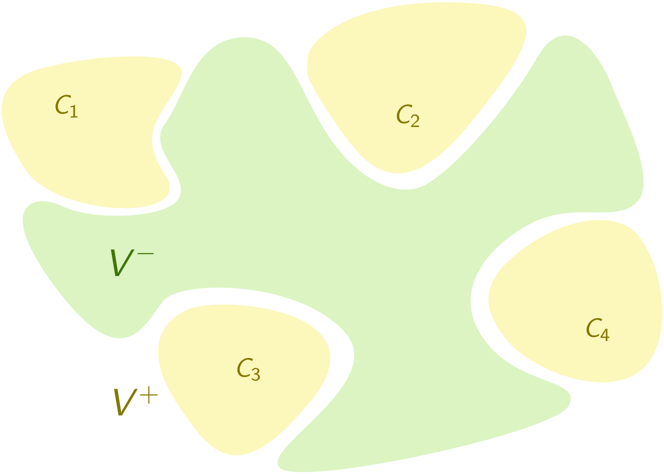

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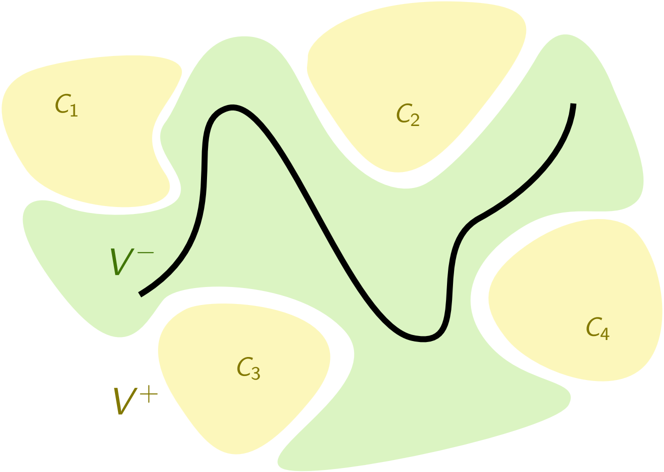

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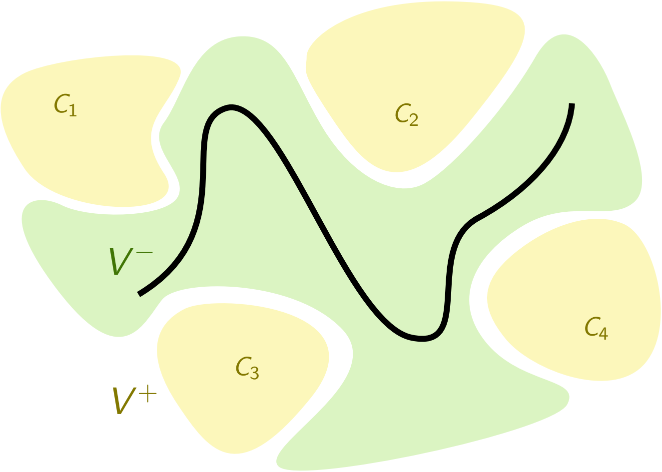
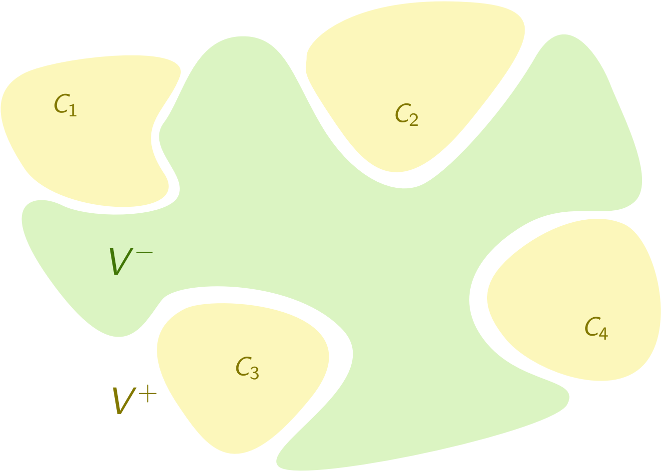


Every vertex detects it !

Upper bound for path-freeness certification

If there is a P_{4d-1}, which vertex detects it ?

Case 2: exactly one component intersects P_{4d-1}.



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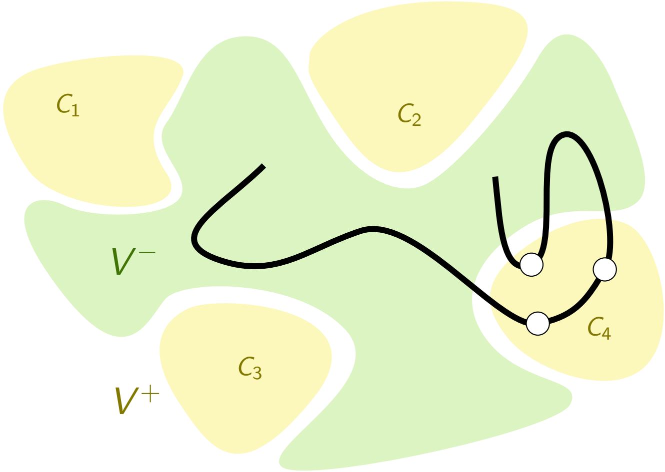

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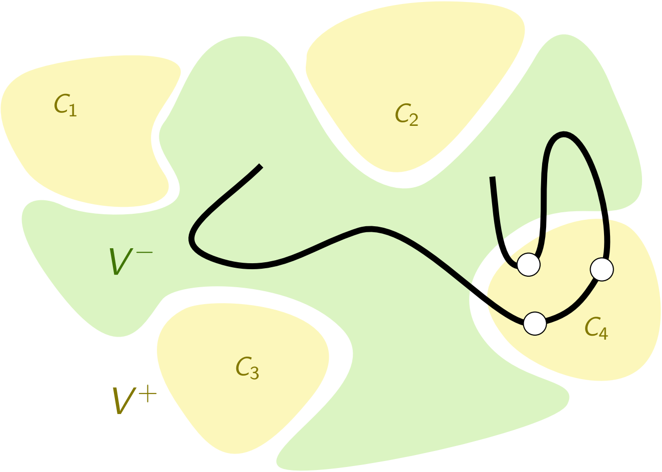
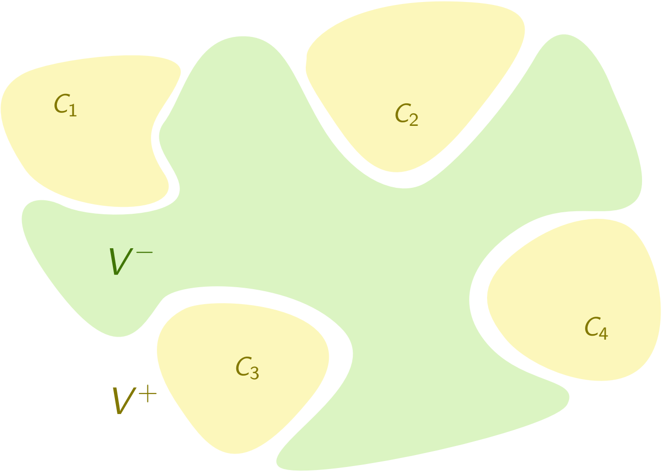


Every vertex in C_4 detects it !

Upper bound for path-freeness certification

If there is a P_{4d-1}, which vertex detects it ?

Case 3: at least two components intersect P_{4d-1}.



Upper bound for path-freeness certification

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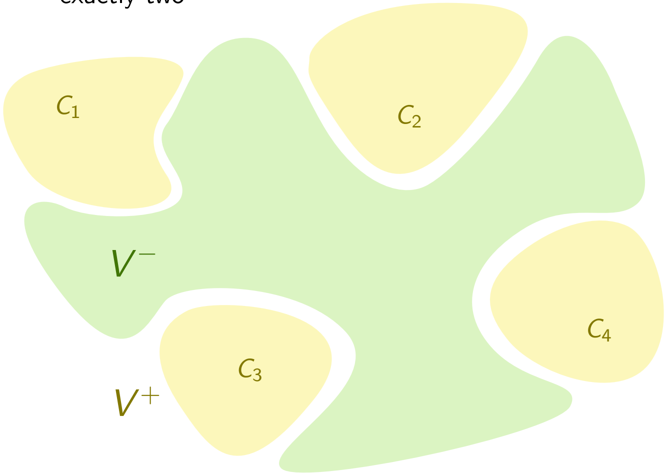
Case 3: ~~at least two~~ components intersect P_{4d-1}.
exactly two



Upper bound for path-freeness certification

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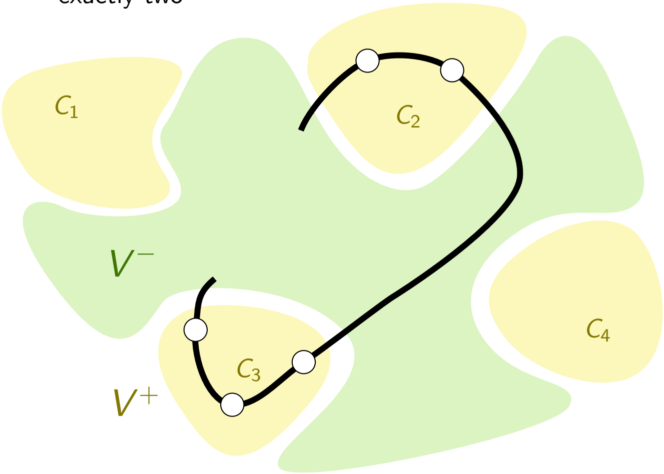

Upper bound for path-freeness certification

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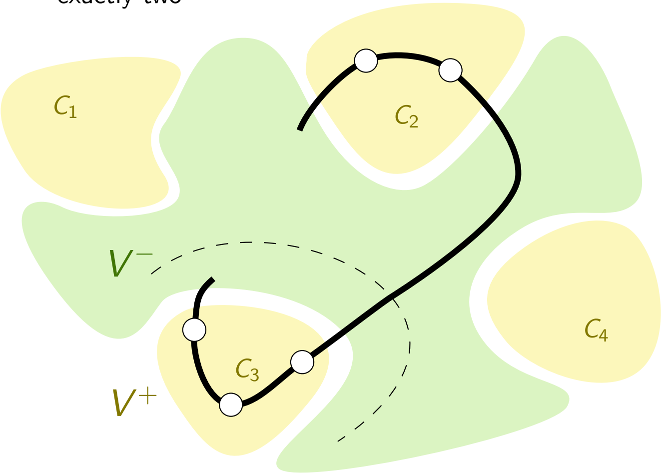

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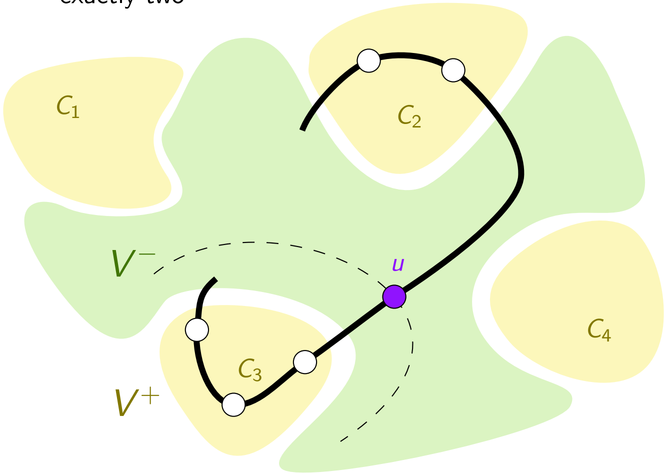

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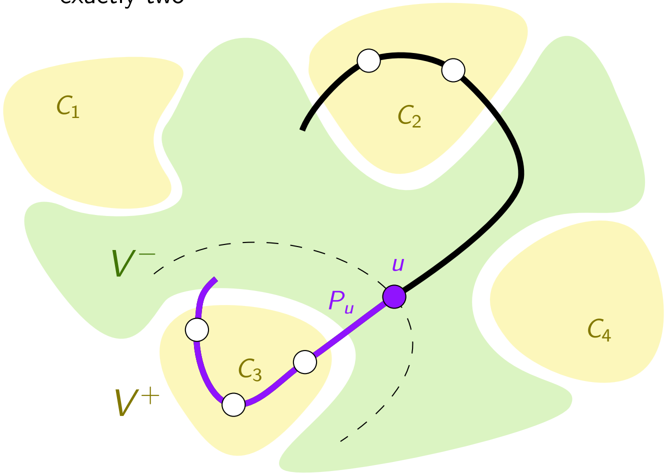

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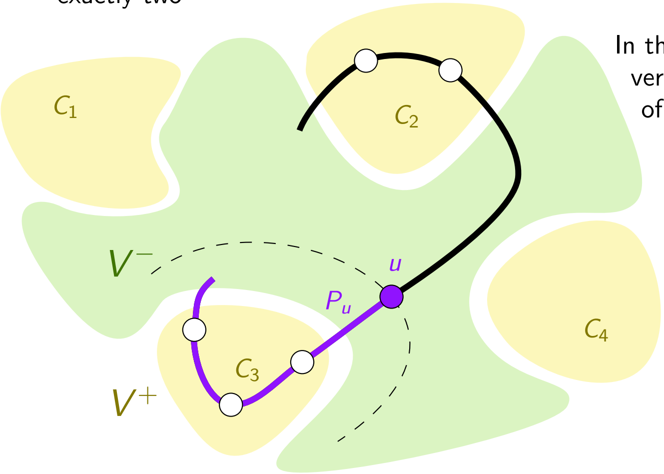

In the certificate of every vertex, add the length of P_u.

Upper bound for path-freeness certification

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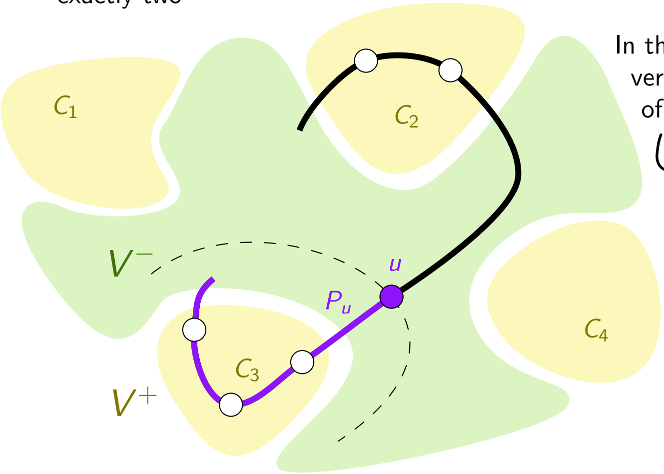

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↳ size $O(n \log n)$ in total.

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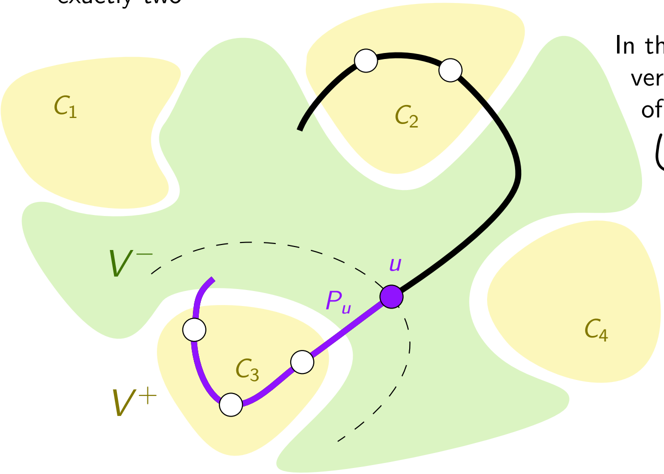

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↳ size $O(n \log n)$ in total.

Every vertex in C_2 detects it !

Conclusion: overview of our results for H-freeness and open questions

Conclusion: overview of our results for *H*-freeness and open questions

Graph H	Bound

Conclusion: overview of our results for H-freeness and open questions

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Conclusion: overview of our results for H-freeness and open questions

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$\|V(H)\| \leqslant 4d-1$	$\tilde{O}(n^{3/2})$
$P_{\lceil 14d/3 \rceil - 1}$	$\tilde{O}(n^{3/2})$

Conclusion: overview of our results for H-freeness and open questions

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$P_{\lceil 14d/3 \rceil -1}$	$\tilde{O}(n^{3/2})$
P_{3d-1}	$\tilde{O}(n)$

Conclusion: overview of our results for H-freeness and open questions

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Open questions:

- what if $d = 1$?

Conclusion: overview of our results for H-freeness and open questions

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Open questions:

- what if $d = 1$? $\longrightarrow \tilde{O}(n^{3/2})$ for P_5

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Open questions:

- what if $d = 1$? $\longrightarrow \tilde{O}(n^{3/2})$ for P_5
- can we get subquadratic upper-bounds for $P_{\alpha d}$ if $\alpha > \frac{14}{3}$?
- Conjecture: for every $\alpha > 0$, there exists $\varepsilon > 0$ such that we can certify $P_{\alpha d}$-free graphs with certificates of size $O(n^{2-\varepsilon})$.

Thanks for your attention !