# Brief announcement: Global certification via perfect hashing

Nicolas Bousquet, Laurent Feuilloley, Sébastien Zeitoun

June 20, 2024

# Distributed certification

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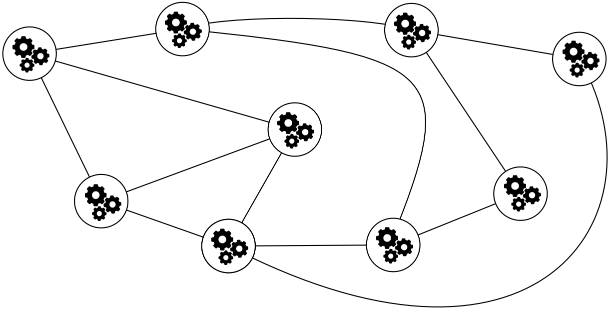
Graph $G$ on $n$ vertices



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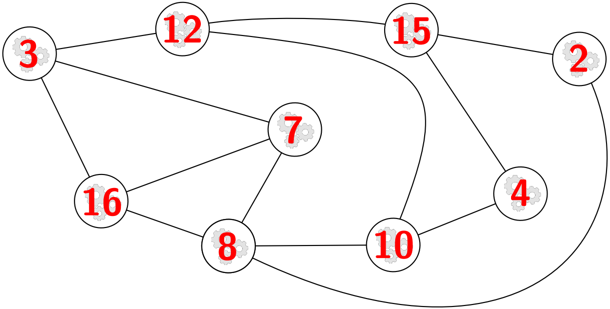

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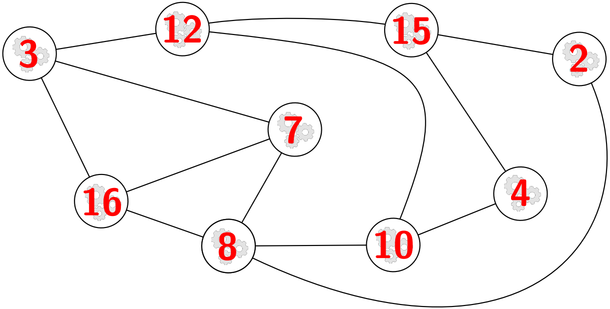

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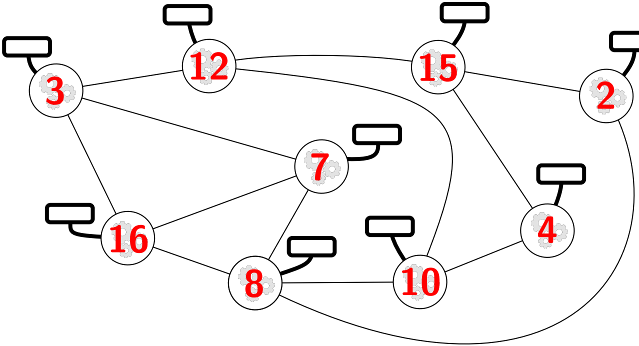

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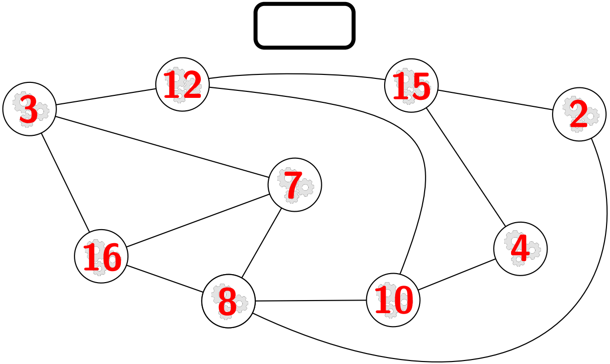

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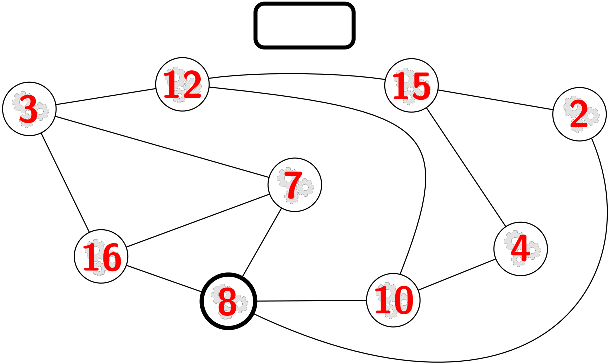

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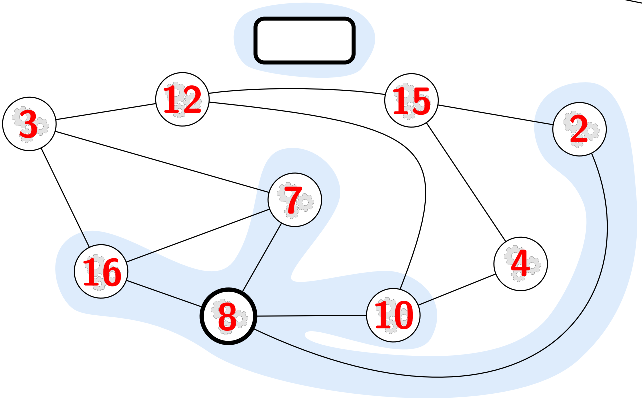

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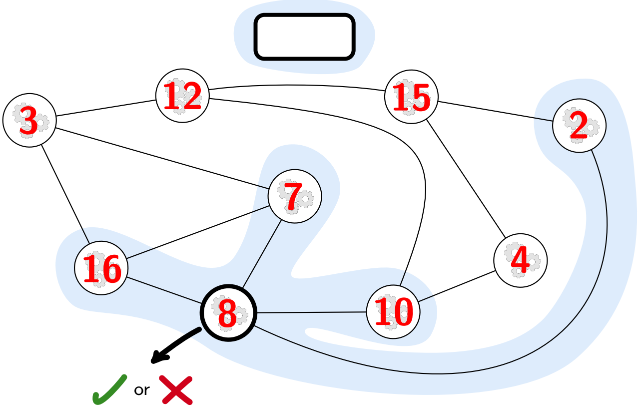

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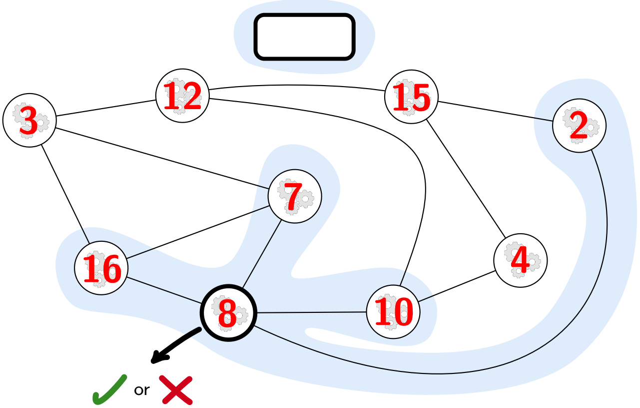


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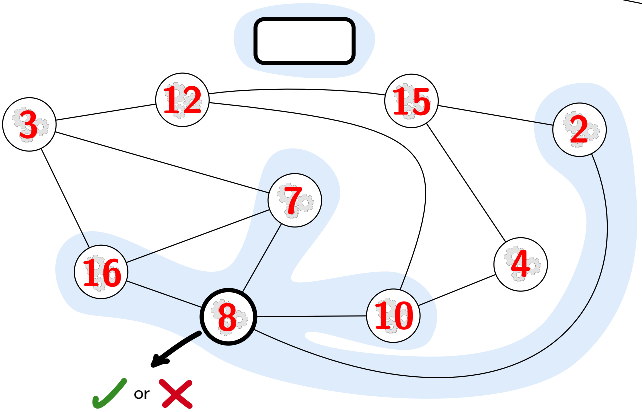


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$G$ satisfies $\mathcal{P} \iff$ there exists a certificate such that $G$ is accepted

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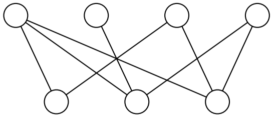
Local certification : 1 bit is sufficient



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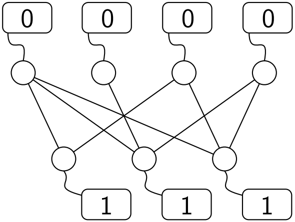

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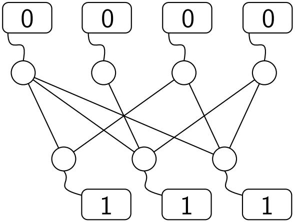


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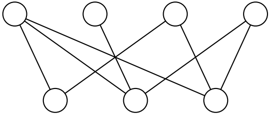

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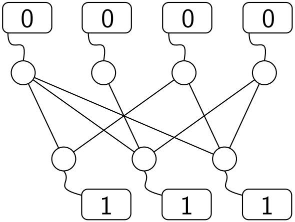


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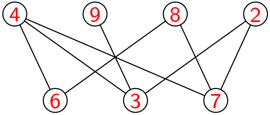

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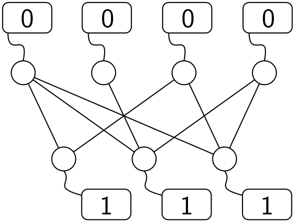


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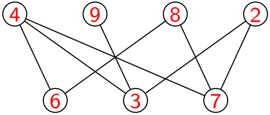


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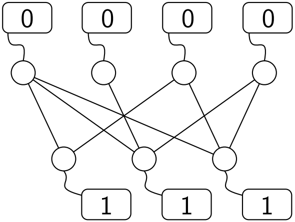


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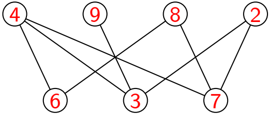


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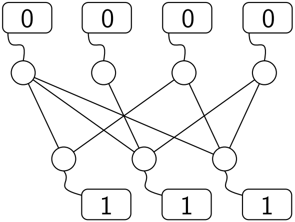


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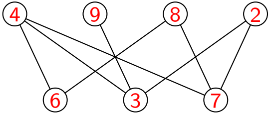


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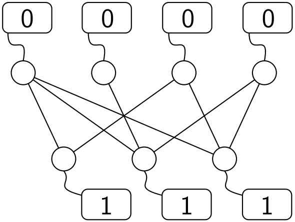


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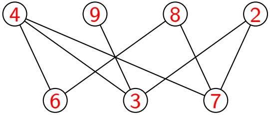


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This certification : size $O(n \log M(n))$. Is it optimal ? $\longrightarrow$ **No.**

Known lower bound: $\Omega(n + \log\log M(n))$

What we proved: $O(n + \log\log M(n))$

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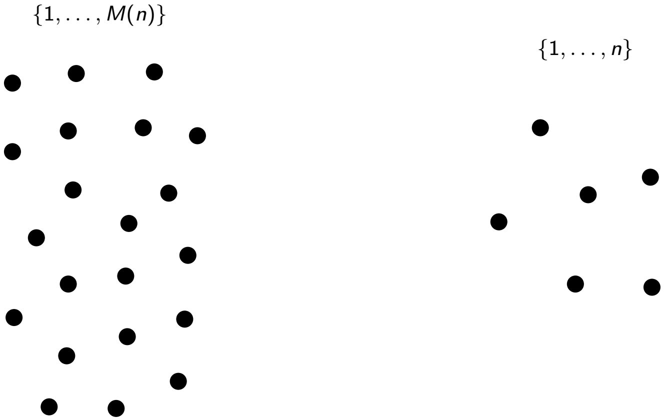


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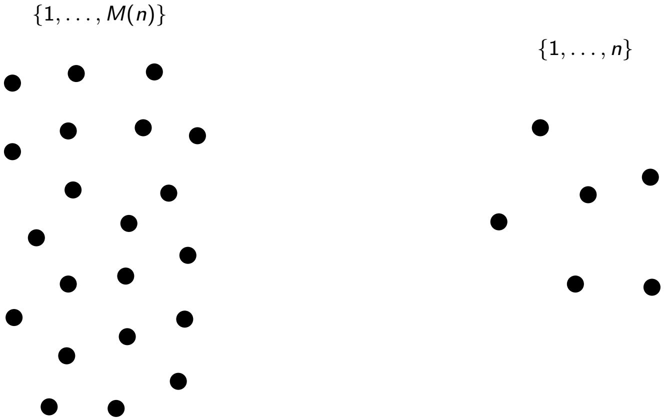


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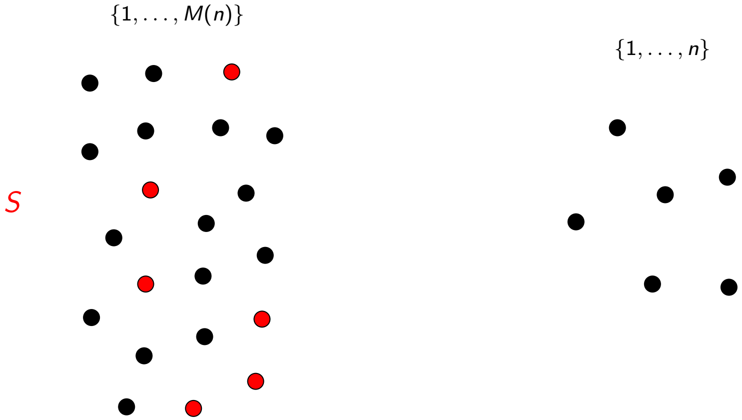


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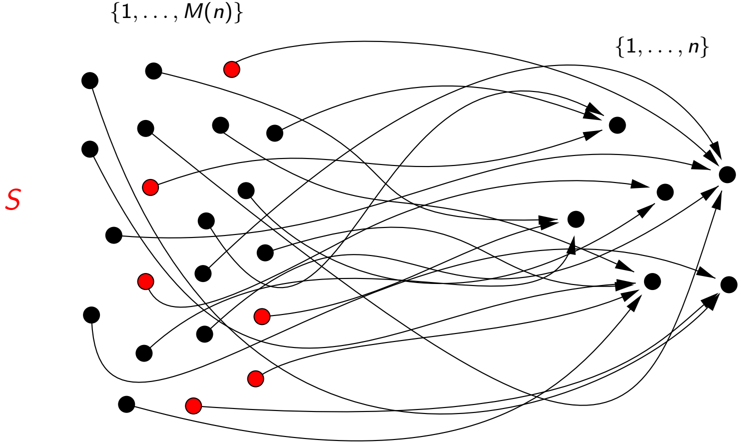


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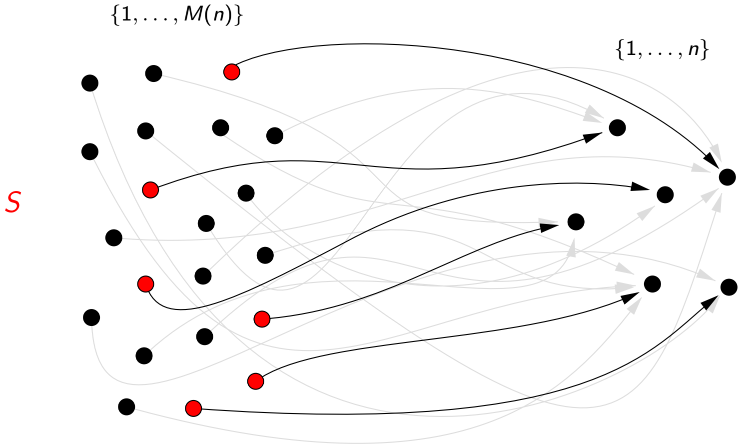


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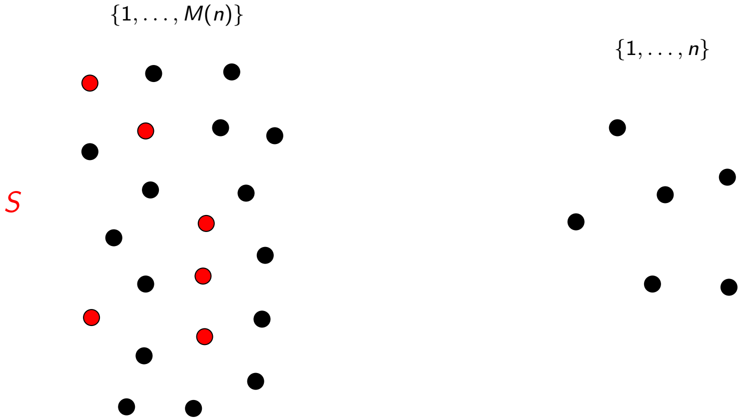


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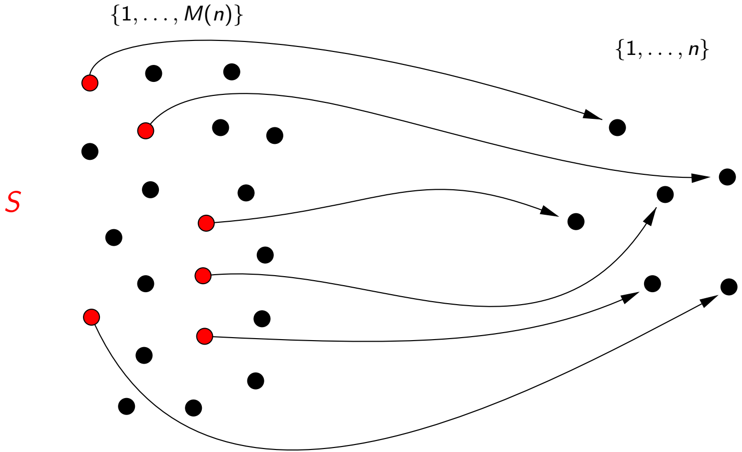


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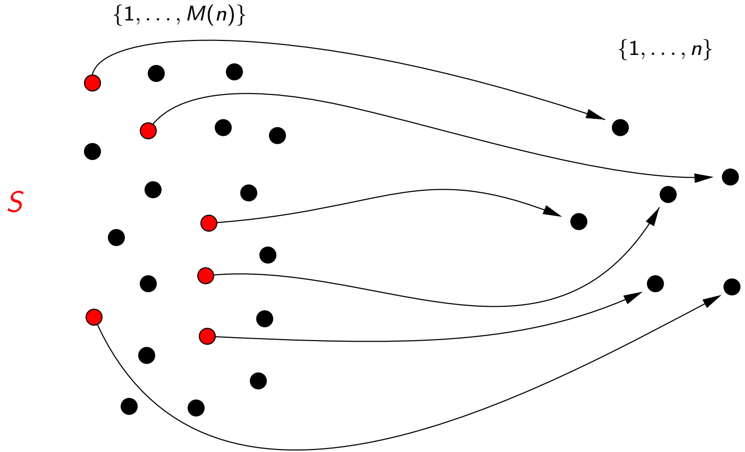


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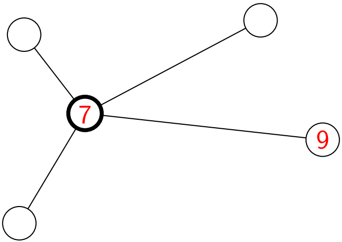

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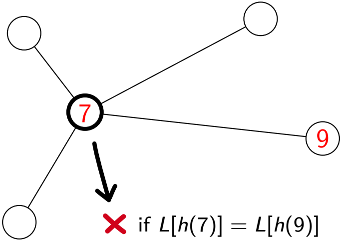

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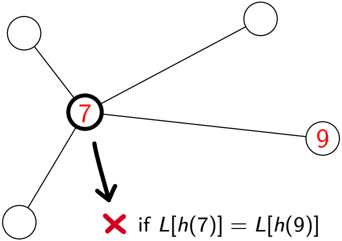


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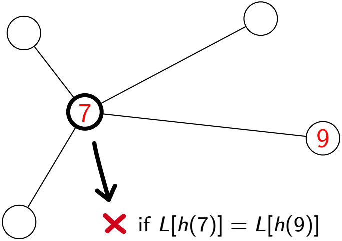


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It is not necessary to check that $h$ is injective !

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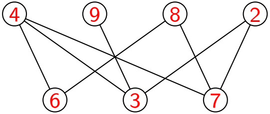

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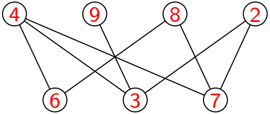


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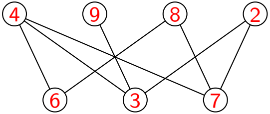


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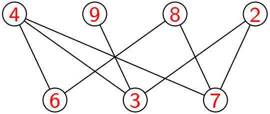

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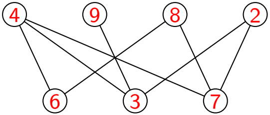


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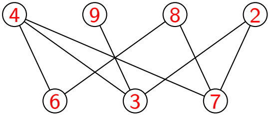


$$h : \begin{array}{l} 2 \longrightarrow L[4] \\ 3 \longrightarrow L[2] \\ 4 \longrightarrow L[1] \\ 6 \longrightarrow L[5] \\ 7 \longrightarrow L[6] \\ 8 \longrightarrow L[7] \\ 9 \longrightarrow L[3] \end{array} \quad , \quad L = [0, 1, 0, 0, 1, 1, 0]$$

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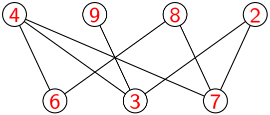


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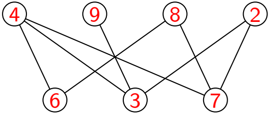


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**Theorem (Bousquet, Feuilloley, Z.)**

In global certification, $O(n \log k + \log \log M(n))$ bits are sufficient to certify that a graph is $k$-colorable.

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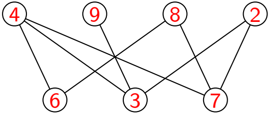


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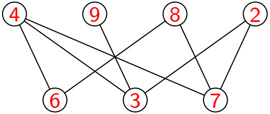



$$h : \begin{array}{l} 2 \longrightarrow L[4] \\ 3 \longrightarrow L[2] \\ 4 \longrightarrow L[1] \\ 6 \longrightarrow L[5] \\ 7 \longrightarrow L[6] \\ 8 \longrightarrow L[7] \\ 9 \longrightarrow L[3] \end{array} , \quad L = [0, 1, 0, 0, 1, 1, 0]$$

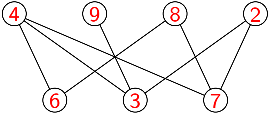
**Theorem (Bousquet, Feuilloley, Z.)**

In global certification, $O(n \log k + \log \log M(n))$ bits are sufficient to certify that a graph is $k$-colorable.

↪ this can be generalized to graph homomorphism

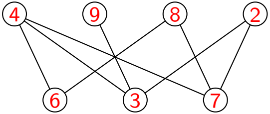
# Conclusion

Before:




$$\begin{array}{l} 2 \longrightarrow 0 \\ 3 \longrightarrow 1 \\ 4 \longrightarrow 0 \\ 6 \longrightarrow 1 \\ 7 \longrightarrow 1 \\ 8 \longrightarrow 0 \\ 9 \longrightarrow 0 \end{array}$$

Now:




$$h : \begin{array}{l} 2 \longrightarrow L[4] \\ 3 \longrightarrow L[2] \\ 4 \longrightarrow L[1] \\ 6 \longrightarrow L[5] \\ 7 \longrightarrow L[6] \\ 8 \longrightarrow L[7] \\ 9 \longrightarrow L[3] \end{array} \quad , \quad L = [0, 1, 0, 0, 1, 1, 0]$$

**Theorem (Bousquet, Feuilloley, Z.)**

In global certification, $O(n \log k + \log \log M(n))$ bits are sufficient to certify that a graph is $k$-colorable.

→ this can be generalized to graph homomorphism

Question: are there other problems for which this hashing technique can be used ?

Thanks for your attention !