Schémas d'induction : de la séparation de langages à la coloration de graphes

Théo Pierron

Encadrants : Marthe Bonamy, Éric Sopena, Marc Zeitoun

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université de BORDEAUX

LaBRI

Induction Schemes: From Languages Separation to Graph Colorings

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July 8, 2019

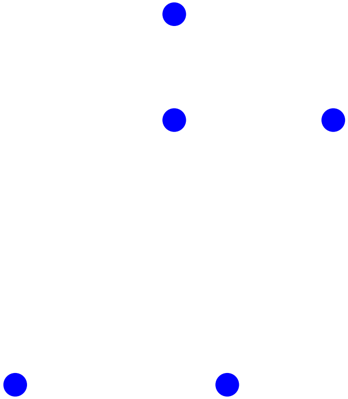
Organization

1. Graph and colorings
2. Separation of languages

Part I: Graphs and colorings

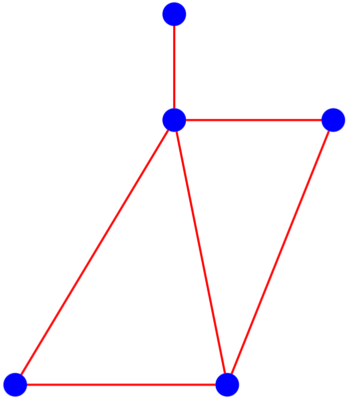
What is a graph?

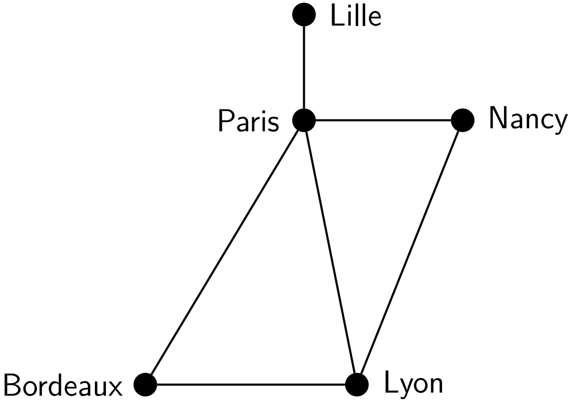
Graph = vertices



What is a graph?

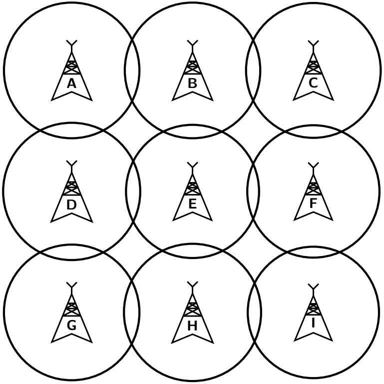
Graph = vertices + edges





Various optimization problems

Coloring problems



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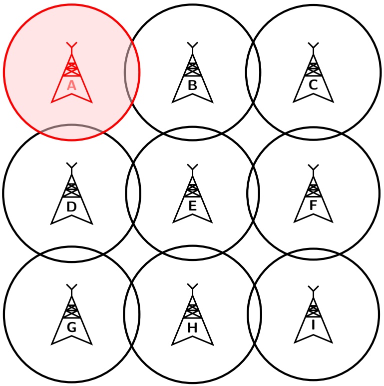

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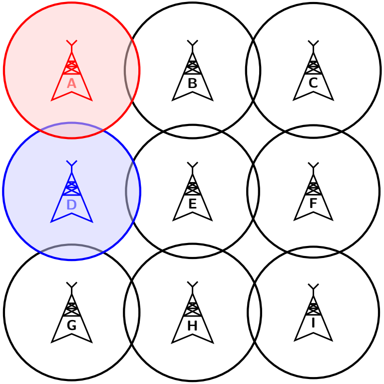

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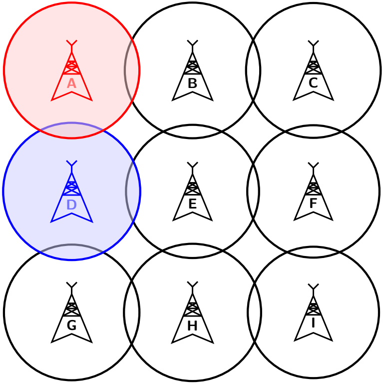

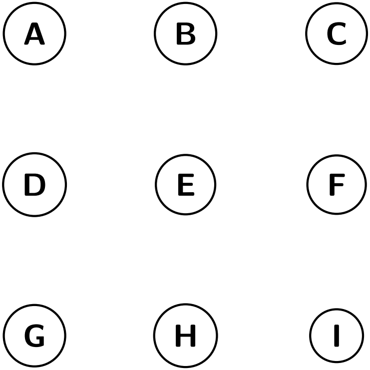

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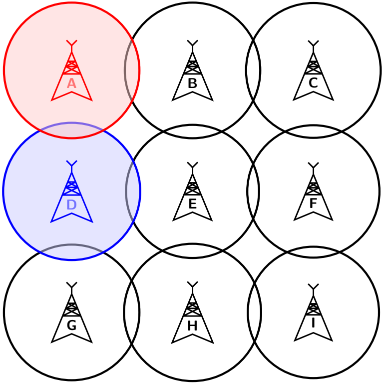

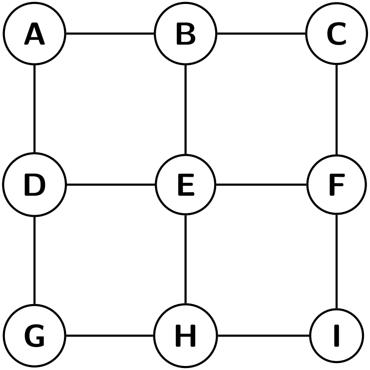

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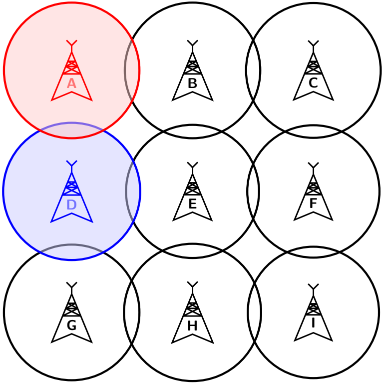

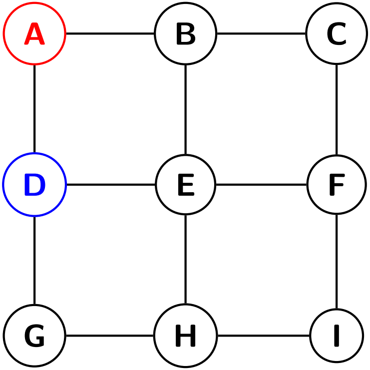

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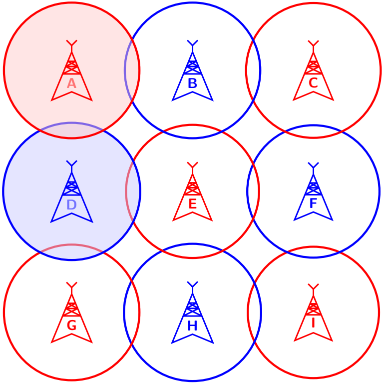

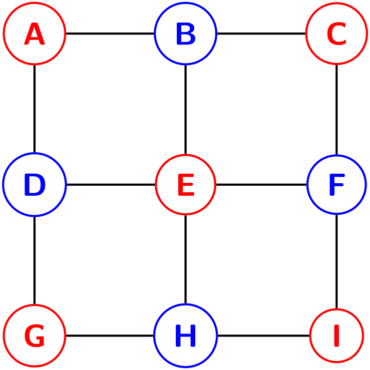

Chromatic number

$\chi(G)$ = minimum number of colors such that:

$$ⓐ\text{—}ⓑ \Rightarrow a \neq b.$$

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Maximum degree

$\Delta(G)$ = maximum number of neighbors of a vertex in G.

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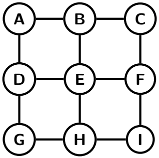

$\chi(G) = 2$
$\Delta(G) = 4$

Greedy upper bound

$\chi(G)$ = minimum number of colors
$\Delta(G)$ = maximum number of neighbors

$$\chi(G) \leqslant \Delta(G) + 1$$

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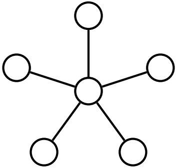
Greedy argument:



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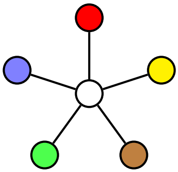
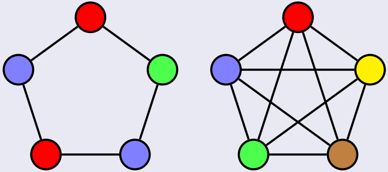


Can we do better?

Theorem (Brooks, 1941)

$\chi(G) \leqslant \Delta(G)$ *unless G is*



Odd cycle *Clique*

Cycle = graph where each vertex is linked only to the previous and next vertices, and first to last.
Clique = graph with all possible edges.

For more specific graphs

A graph is planar when it can be drawn without *crossing edges*.

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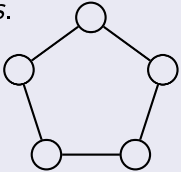

Planar

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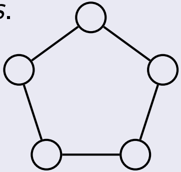

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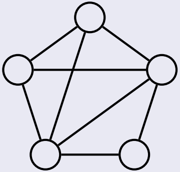

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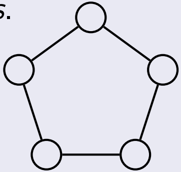

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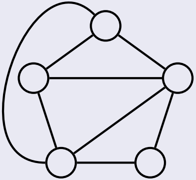

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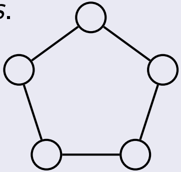

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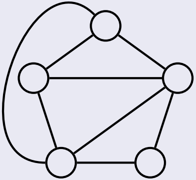

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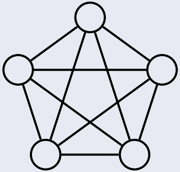

Not planar

For more specific graphs

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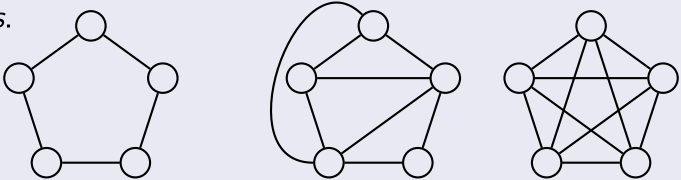

Planar Planar Not planar

Question (Guthrie, 1852)

How many colors are needed to color a planar graph?

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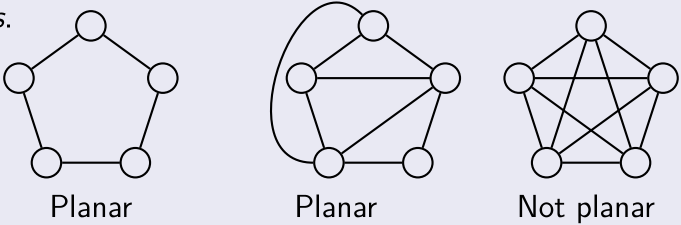

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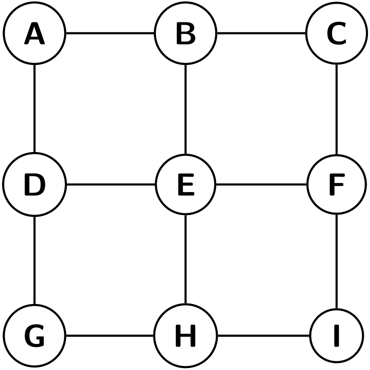
How many colors are needed to color a planar graph?

Theorem (Appel, Haken, 1976)

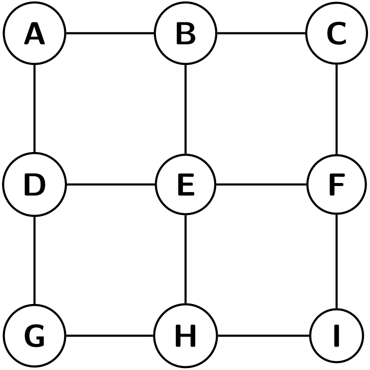
If G is planar, $\chi(G) \leqslant 4$.

Powers of graphs



G

Powers of graphs



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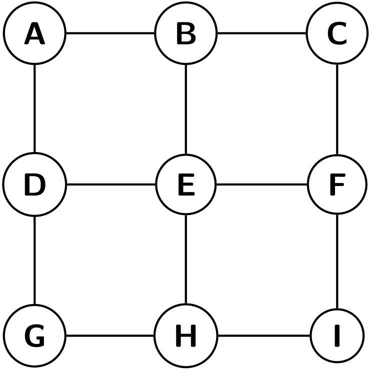

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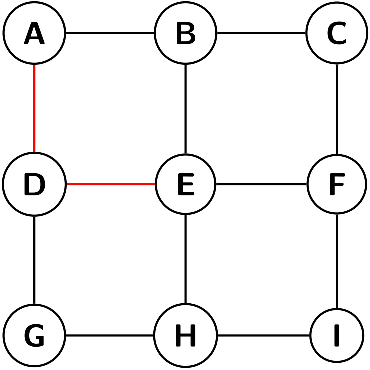

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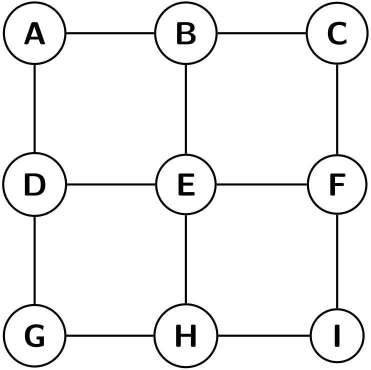

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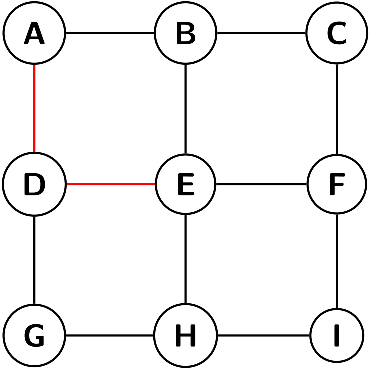

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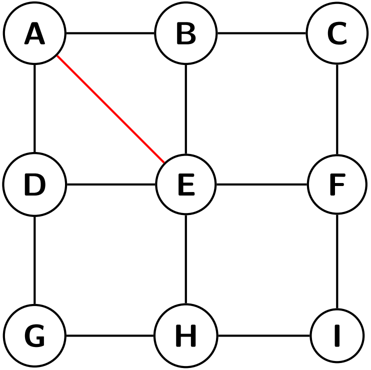

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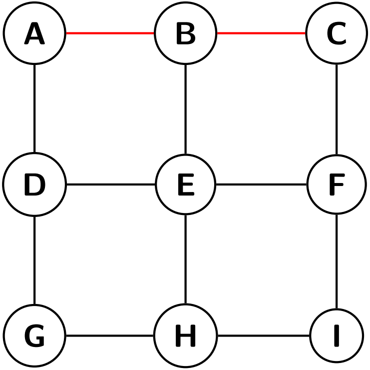

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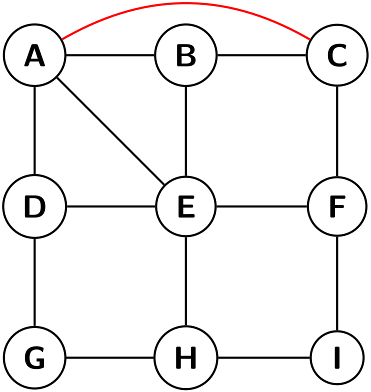

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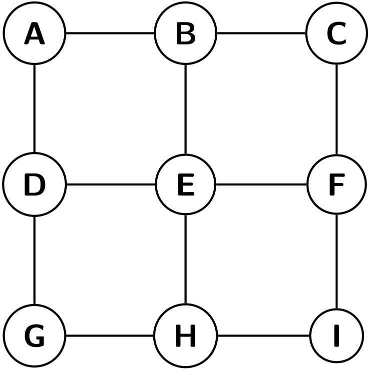

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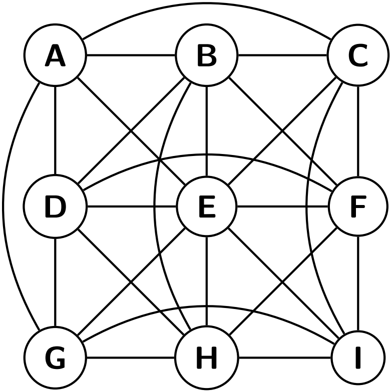
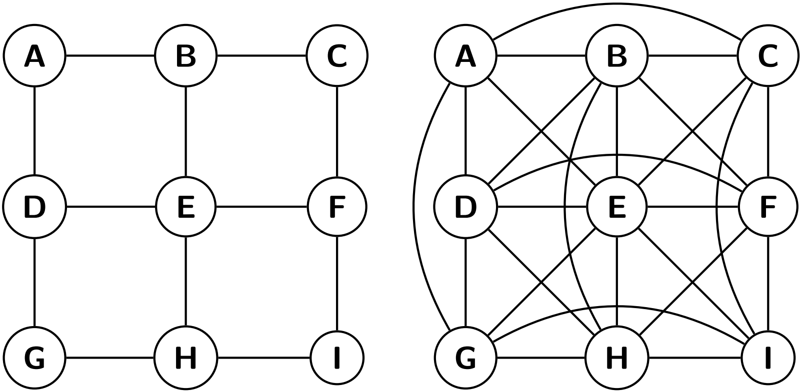


G^2

Powers of graphs



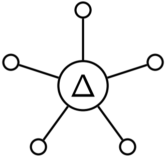
G G^2

$G^k = G +$ edges between vertices at distance $\leqslant k$.

The case of squares ($k = 2$)

For every graph G,

$$\Delta(G) + 1 \leqslant \chi(G^2)$$



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For every graph G,

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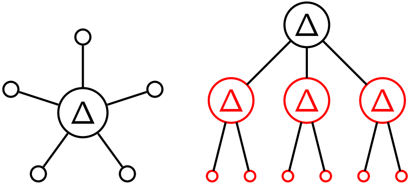

Greedy upper bound for graph powers

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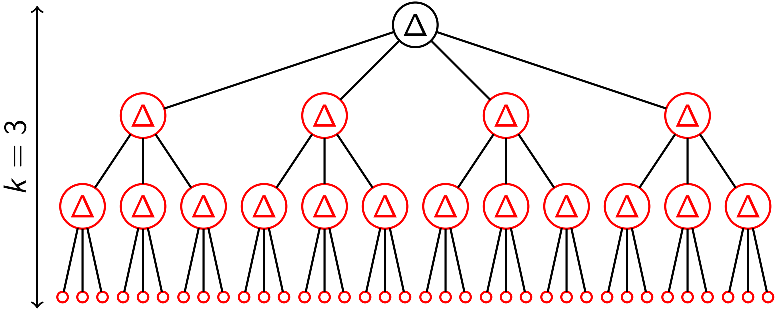

$$f(k, \Delta) = \Delta \cdot (1 + (\Delta - 1) + \cdots + (\Delta - 1)^{k-1}).$$

Theorem (Brooks, revisited)

For every graph G with $k \geqslant 2$,

$$\chi(G^k) \leqslant f(k, \Delta(G)) + 1 - 1$$

unless G^k is a clique or an odd cycle.

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Theorem (Hoffman, Singleton, 1960)

For every graph G with $k \geqslant 2$ and $\Delta(G) \geqslant 3$,

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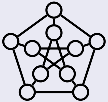
unless $k = 2$ and G is a Moore graph:



$+$ *finitely many others*

Can we do better?

Theorem (Bonamy, Bousquet, 2014, Cranston, Rabern, 2016)

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Theorem (P., 2019)

Gap is at least $k - 2$, except for “few” graphs.

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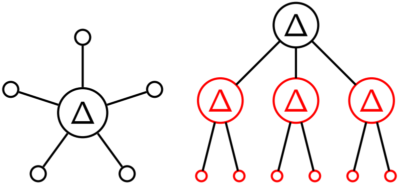

and

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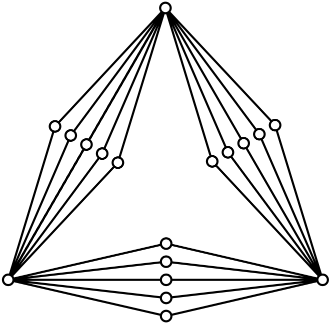
What about planar graphs?



$\Rightarrow \frac{3\Delta}{2}$ colors needed

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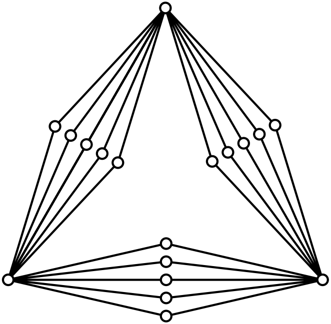

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Conjecture (Wegner, 1977)

If G is planar with $\Delta \geqslant 8$,

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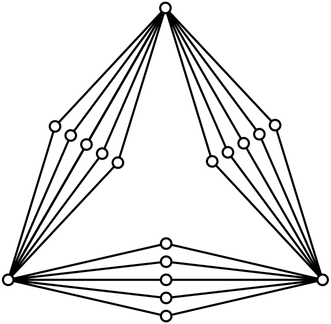

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Theorem (Amini et al., 2007)

If G is planar with large Δ,

$$\chi(G^2) \leqslant \frac{3\Delta(G)}{2} + o(\Delta)$$

Can we do better than $\frac{3\Delta}{2}$ for large Δ?

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Girth = length of smallest cycle

Can we do better than $\frac{3\Delta}{2}$ for large Δ?

- ☹ Girth 4: not sufficient (Wegner, 1977).
- ☺ Girth $g \geqslant 7$: $\chi(G^2) \leqslant \Delta(G) + 1$ (Borodin et al., 2004).
- ☺ Girth $g \geqslant 6$: $\chi(G^2) \leqslant \Delta(G) + 2$ (Borodin et al., 2004).
- ☺ Girth $g \geqslant 5$: $\chi(G^2) \leqslant \Delta(G) + 2$ (Bonamy et al., 2015).

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- ☺ No 4 nor 5-cycles: $\chi(G^2) \leqslant \Delta(G) + 2$ (Dong and Xu, 2017).

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Which cycles to forbid for obtaining $\Delta + O(1)$ for large Δ?

Theorem (Choi, Cranston, P., 2019)

- *C_4 has to be forbidden.*
- *If G is C_4-free, planar and $\Delta(G)$ is large,*

$$\chi(G^2) \leqslant \Delta(G) + 2.$$

Girth = length of smallest cycle

Idea of the proof

Theorem (Choi, Cranston, P., 2019)

- *C_4 has to be forbidden.*
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$G =$ minimum counterexample.

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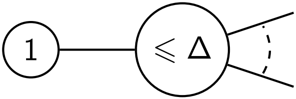

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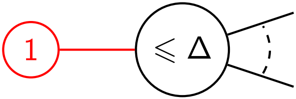
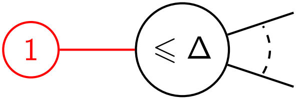


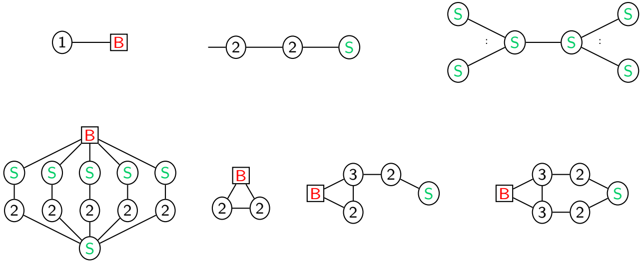
An example for step 1

1. G does not contain some configurations, otherwise we can find a smaller counterexample H.

By contrapositive, extend a coloring with $\Delta + 2$ colors to the red vertex.



The configurations



+ 1 other "dense" configuration

S = small = degree $\leqslant \sqrt{\Delta}$
B = big = degree $> \sqrt{\Delta}$

Ideas for step 2

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- Decomposition into regions.
- Find a dense region.
- Auxiliary (**multi**)graph: find a vertex with large degree and few neighbors.

To sum up

1. Graph colorings:
   - $\Omega(1)$ gap for coloring graph powers.
   - Definitive answer for cycle obstructions in square coloring of planar graphs.
2. Language separation problem:
   - Complexity does not depend on the representation.

Part II: Separation of regular languages

Word = sequence of letters

$$ab \quad ababb \quad \varepsilon$$

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Language = set of words

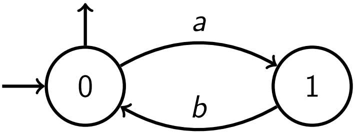
$$\{a, ab\} \quad \{a^n, n \in \mathbb{N}\} \quad \{(ab)^n, n \in \mathbb{N}\}$$

Regular languages

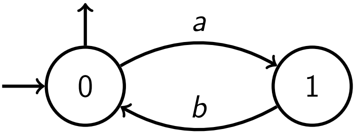
Three representations:

- Automata
- Monoids
- Expressions

Automata

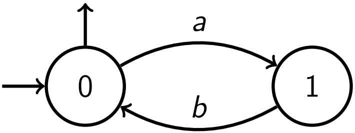


Automata



ab

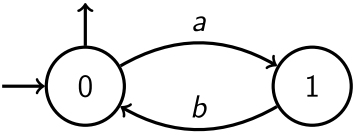
Automata



ab

accept

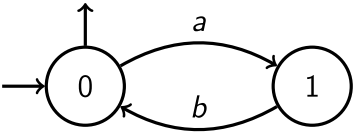
Automata



ab *aba*

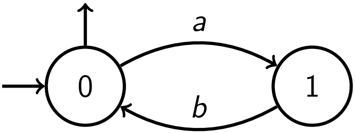
accept

Automata



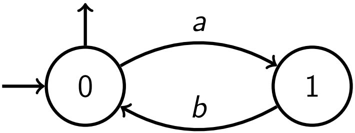
ab	aba
accept	reject

Automata



ab	*aba*	*abb*
accept	reject	

Automata



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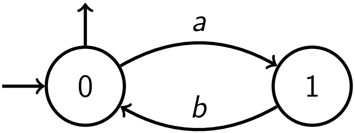

ab	aba	abb
accept	reject	reject

Accepted language $= \{(ab)^n, n \in \mathbb{N}\}$.

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1	+	1	+	1	+	1	+	1

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Recognition by an automaton $\Leftrightarrow$ Recognition by a monoid.

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Constructed from letters with three operations:

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Without star?

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  - No known language of star-height 2.
  - Star-height 0 already challenging.

Star-height 0 languages

$(ab)^*$ has star-height 0:

$$(ab)^* = \overline{b\overline{\emptyset} \cup \overline{\emptyset}a \cup \overline{\emptyset}aa\overline{\emptyset} \cup \overline{\emptyset}bb\overline{\emptyset}}.$$

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Theorem (Schützenberger, 1965)

One can decide whether a given regular language has star-height 0.

The membership problem

$\mathcal{C}$ = class of languages.

$\mathcal{C}$-membership

- Input: a regular language L
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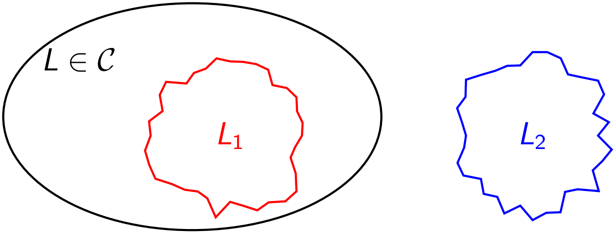
Deciding membership $\Leftrightarrow$ understanding expressiveness of $\mathcal{C}$.

The separation problem

$\mathcal{C}$-separation

- Input: L_1, L_2 regular
- Output: does there exist $L \in \mathcal{C}$ such that $L_1 \subset L$ and $L_2 \cap L = \varnothing$?



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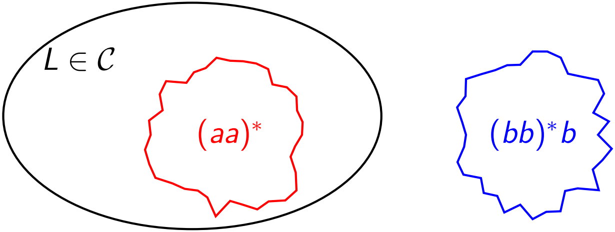

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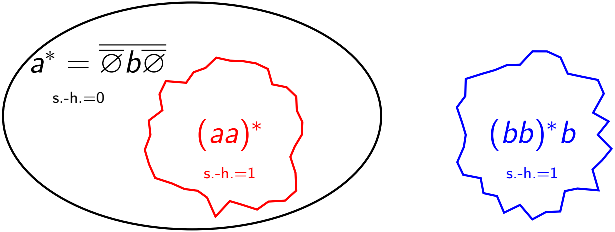
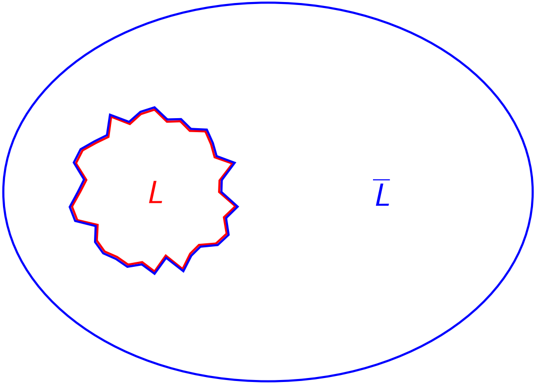


Separation is harder than membership



$\mathcal{C}$-separation for $(L, \overline{L})$ $\Leftrightarrow$ $\mathcal{C}$-membership for L.

A generic complexity result for separation

Membership for star-height 0:

- PSpace-complete on automata
- LogSpace on monoids.

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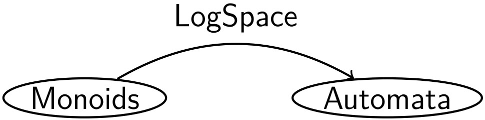
Theorem (P., Place, Zeitoun, 2017)

The complexity of $\mathcal{C}$-separation does not depend on whether inputs are automata or monoids when $\mathcal{C}$ is reasonable.

Ideas of the proof



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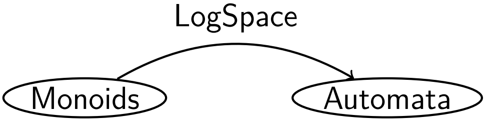

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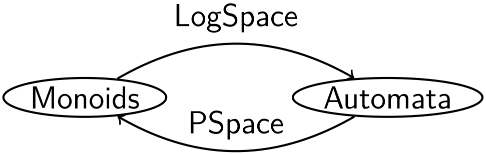

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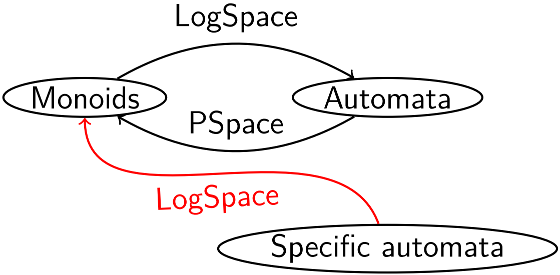

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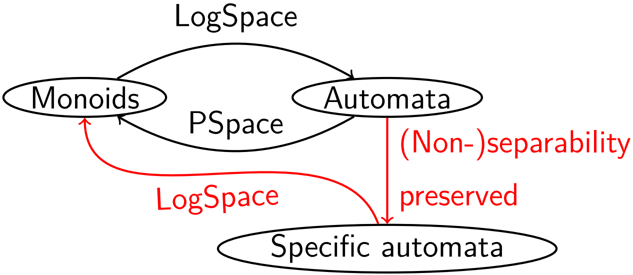

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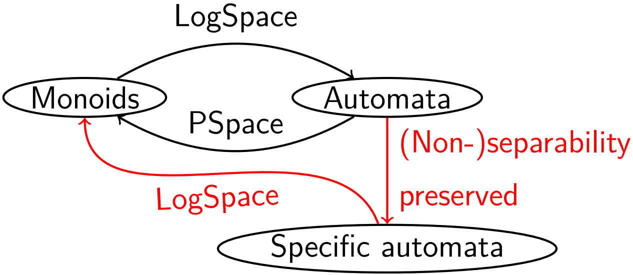

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To sum up

1. Graph colorings:
   - $\Omega(1)$ gap for coloring graph powers.
   - Definitive answer for cycle obstructions in square coloring of planar graphs.
   - If G is planar with $\Delta = 8$, $\chi''_\ell(G) \leqslant 10 = \Delta(G) + 2$.
2. Language separation problem:
   - Complexity does not depend on the representation.
   - PSpace lower bound for $\mathrm{Pol}(\mathcal{C})$-separation.
   - Extension to infinite words.

Perspectives

1. Graph colorings:
    - Forbidding infinitely many cycle lengths
    - Bounds on the gap
    - Use similar methods for other coloring problems
2. Language separation problem:
    - Decidability and complexity for specific classes
    - Extensions of separation
    - Other structures than finite words

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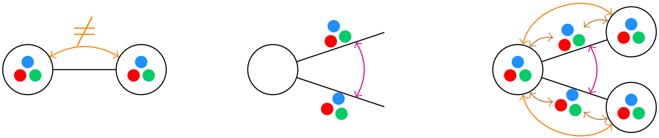
Thanks for your attention.

1. Vertex coloring $\rightarrow \chi, \chi_\ell$
2. Edge coloring $\rightarrow \chi', \chi'_\ell$
3. Total coloring: vertices + edges $\rightarrow \chi'', \chi''_\ell$



Theorem (Bonamy, P., Sopena, 2018)

If G is a planar graph with $\Delta(G) = 8$, then $\chi''_\ell(G) \leqslant 10 = \Delta(G) + 2$.

$$
\begin{array}{llllll}
\chi' & \leqslant & \Delta + 1 & & & \text{(Vizing, 1964)} \\
\chi' & = & \Delta & \text{if} & \Delta \geqslant 8 & \text{(Vizing, 1965)} \\
\chi'' & \leqslant & \Delta + 2 & \text{if} & \Delta \neq 6 & \text{(Kostochka, Sanders, Zhao, \ldots)} \\
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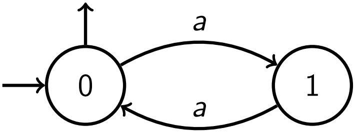
Automaton → monoid



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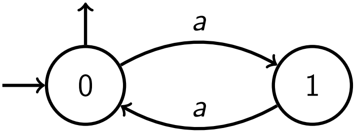

a	0	1
0	X	✓
1	✓	X

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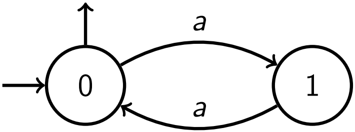
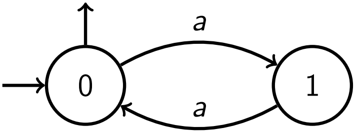


a	0	1
0	✗	✓
1	✓	✗

aa	0	1
0	✓	✗
1	✗	✓

Automaton → monoid



a	0	1
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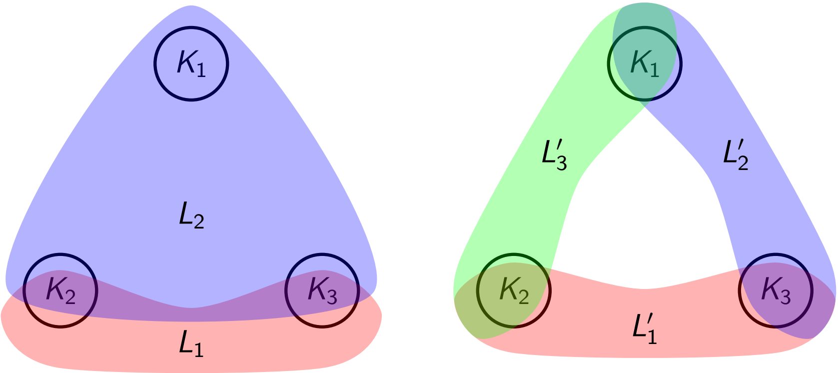
1	X	✓

φ: word $\mapsto$ matrix.

Covering



$\{K_1, K_2, K_3\}$ is covered by $L'_1 \cup L'_2 \cup L'_3$, but not by $L_1 \cup L_2$.

A complexity result

$\mathrm{Pol}(\mathcal{C})$ is the smallest class containing $\mathcal{C}$ and closed under:

- $\cup$ and $\cap$
- marked concatenation: $K, L, a \mapsto KaL$

Theorem

$\mathrm{Pol}(\mathcal{C})$-separation is PSpace-hard when $\mathcal{C}$ is large enough.

Infinite words

Theorem (Place, Zeitoun, 2014)

$\mathrm{Pol}(\mathcal{C})$*-separation is decidable when* $\mathcal{C}$ *is finite and reasonable.*

Theorem (P., Place, Zeitoun, 2016/2018)

$\mathrm{Pol}(\mathcal{C})$*-separation is decidable for infinite words when* $\mathcal{C}$ *is finite and reasonable.*