Decomposition of Rational Discrete Planes

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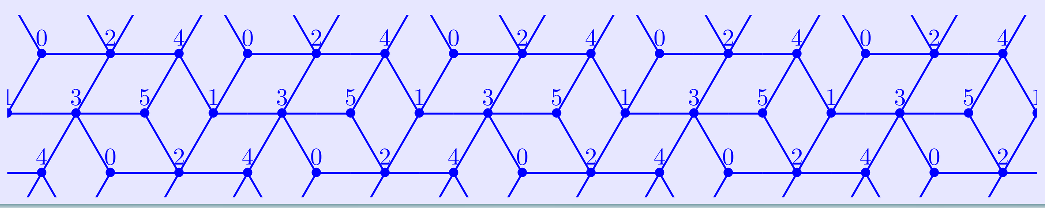
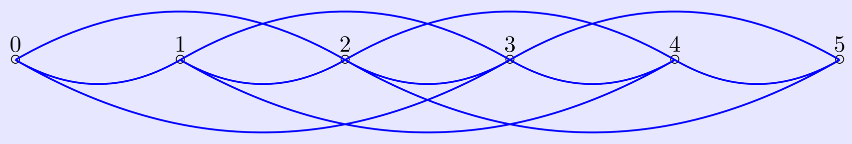
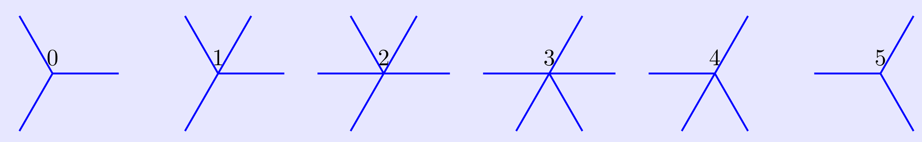
Rational Discrete planes

Given a non-zero normal vector $\mathbf{a} \in \mathbb{N}^3$, a standard arithmetical rational discrete plane is defined as follows: $\mathcal{P}_\mathbf{a} := \{\mathbf{x} \in \mathbb{Z}^3 \mid 0 \leq \mathbf{x} \cdot \mathbf{a} < \|\mathbf{a}\|_1\}$.
It is represented by a graph $\mathcal{G}_\mathbf{a}$ whose nodes are $\{0, \dots, \|\mathbf{a}\|_1 - 1\}$ and whose arcs are such that $\forall i \in \{1,2,3\}$, $(\mathbf{x} \cdot \mathbf{a}, \mathbf{x} \cdot \mathbf{a} \pm a_i) \in \mathcal{G}_\mathbf{a}$ iff $\mathbf{x}, \mathbf{x} \pm e_i \in \mathcal{P}_\mathbf{a}$.

Definitions

- Let $\mathbf{a}, \mathbf{b} \in \mathbb{N}^3 \setminus \{\mathbf{0}\}$, with $\gcd(\mathbf{a}) = \gcd(\mathbf{b}) = 1$, be such that $\mathbf{a} - \mathbf{b} \in \mathbb{N}^3 \setminus \{\mathbf{0}\}$ and δ be in $\{-\|\mathbf{b}\|_1 + 1, \dots, \|\mathbf{a}\|_1 - \|\mathbf{b}\|_1\}$.
- Let $\mathcal{S}(\mathcal{G}_\mathbf{a}, \mathbf{b}, \delta)$ contain the nodes $h \in \{0, \dots, \|\mathbf{a}\|_1 - 1\}$ such that $(h\|\mathbf{b}\|_1 - \delta) \bmod \|\mathbf{a}\|_1 < \|\mathbf{b}\|_1$ and all the arcs emanating from those nodes.
- $\mathcal{S}(\mathcal{G}_\mathbf{a}, \mathbf{b}, \delta) \simeq \mathcal{G}_\mathbf{b}$ if there exists a bijection $f : \{0, \dots, \|\mathbf{a}\|_1 - 1\} \mapsto \{0, \dots, \|\mathbf{b}\|_1 - 1\}$ such that:
  $\forall i \in \{1,2,3\}$, $(h, h + a_i) \in \mathcal{S}(\mathcal{G}_\mathbf{a}, \mathbf{b}, \delta) \Leftrightarrow (f(h), f(h) + b_i) \in \mathcal{G}_\mathbf{b}$ and $(h, h - a_i) \in \mathcal{S}(\mathcal{G}_\mathbf{a}, \mathbf{b}, \delta) \Leftrightarrow (f(h), f(h) - b_i) \in \mathcal{G}_\mathbf{b}$,
  where f is defined as $f(h) := \left\lceil \frac{h\|\mathbf{b}\|_1 - \delta}{\|\mathbf{a}\|_1} \right\rceil$, its inverse being $f^{-1}(h) = \left\lceil \frac{h\|\mathbf{a}\|_1 + \delta}{\|\mathbf{b}\|_1} \right\rceil$.

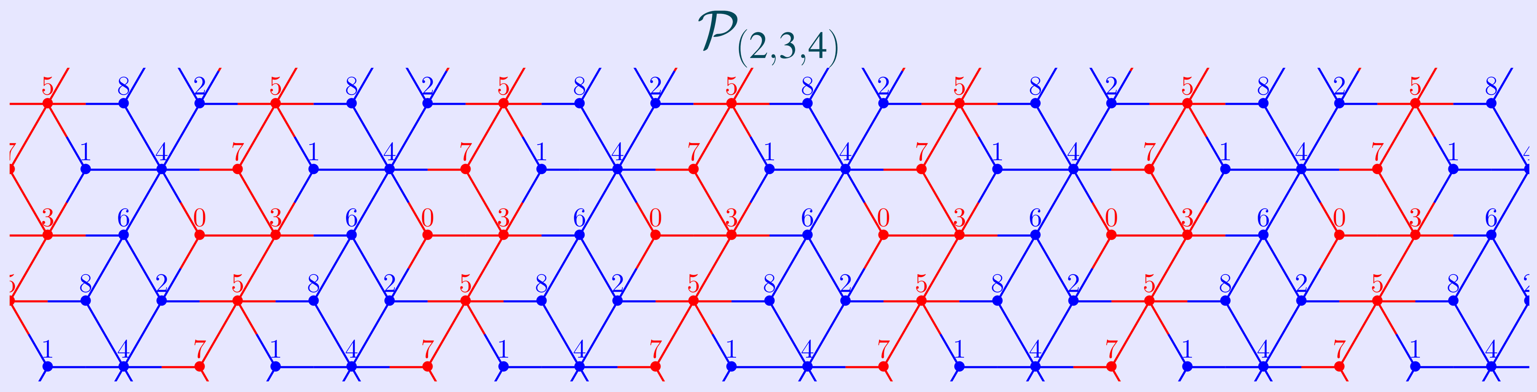
Main theorem

Let $\mathbf{a} \in \mathbb{N}^3 \setminus \{\mathbf{0}\}$ be such that $\gcd(\mathbf{a}) = 1$. If $\mathbf{a}$ is not a permutation of one of the vectors $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ and $(1,1,2)$, then there exist $\mathbf{b}$ and δ such that

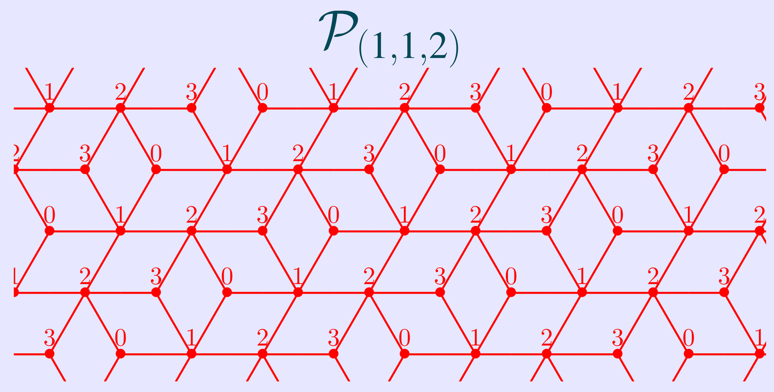
$$\mathcal{G}_\mathbf{a} = \mathcal{S}(\mathcal{G}_\mathbf{a}, \mathbf{b}, \delta) \cup \mathcal{S}(\mathcal{G}_\mathbf{a}, \mathbf{a} - \mathbf{b}, -\delta + 1), \quad \mathcal{S}(\mathcal{G}_\mathbf{a}, \mathbf{b}, \delta) \simeq \mathcal{G}_\mathbf{b} \quad \text{and} \quad \mathcal{S}(\mathcal{G}_\mathbf{a}, \mathbf{a} - \mathbf{b}, -\delta + 1) \simeq \mathcal{G}_{(\mathbf{a}-\mathbf{b})}.$$

Examples

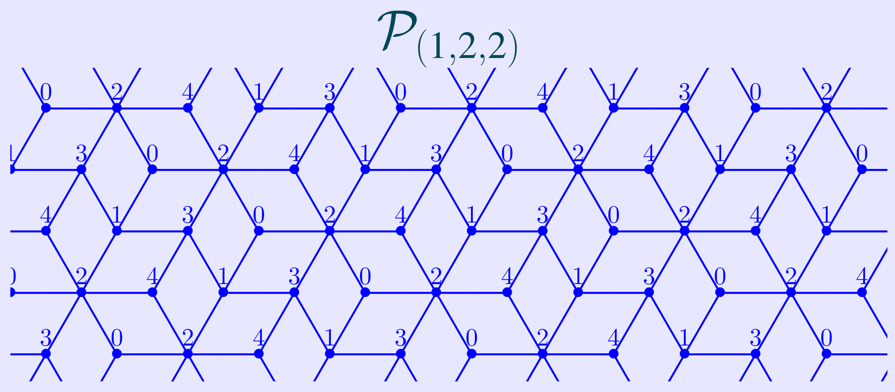
$\mathcal{P}_{(2,3,4)}$



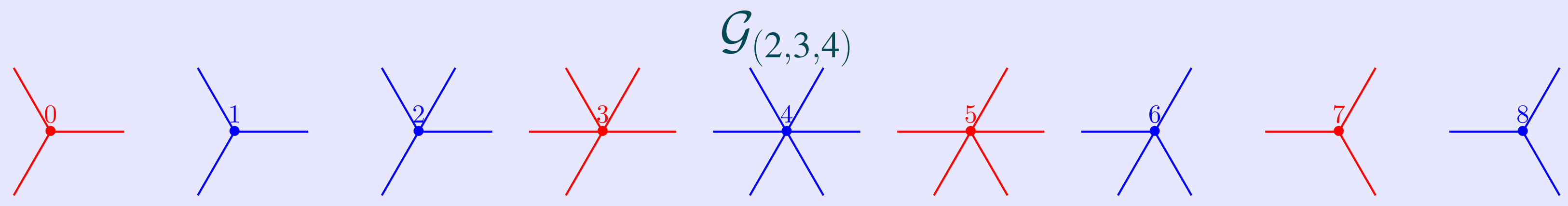
$\mathcal{P}_{(1,1,2)}$



$\mathcal{P}_{(1,2,2)}$

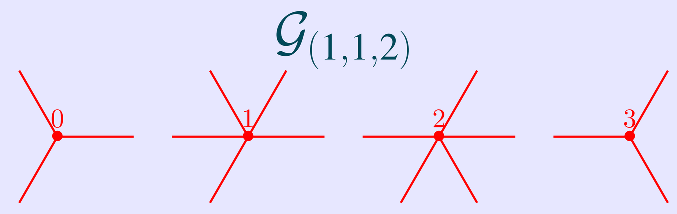


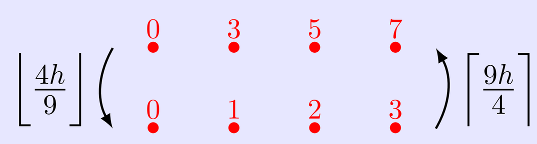
$\mathcal{G}_{(2,3,4)}$



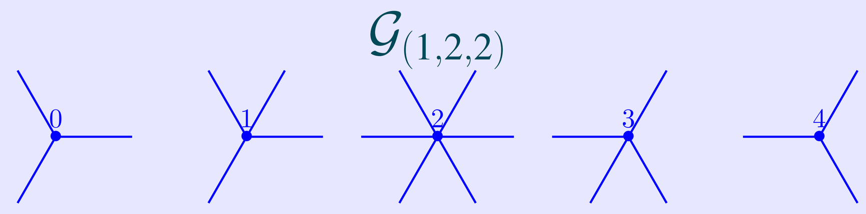
The partionning depends on $\|\mathbf{a}\|_1$, $\|\mathbf{b}\|_1$, $\|\mathbf{c}\|_1$ and an integer $\delta = 0$:
if $4h \mod 9 < 4$ then • else • $\Leftrightarrow$ if $(5h - 1) \mod 9 < 5$ then • else •

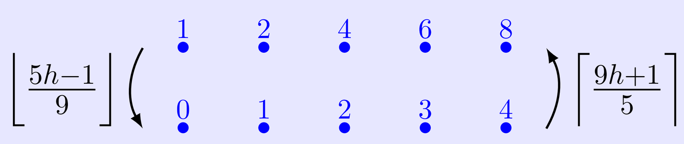
$\mathcal{G}_{(1,1,2)}$



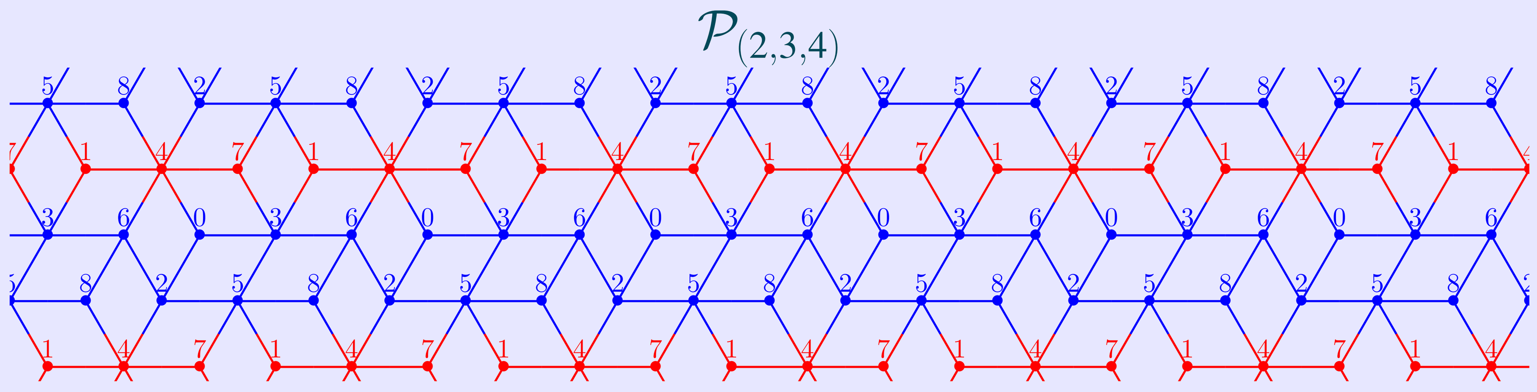


$\mathcal{G}_{(1,2,2)}$

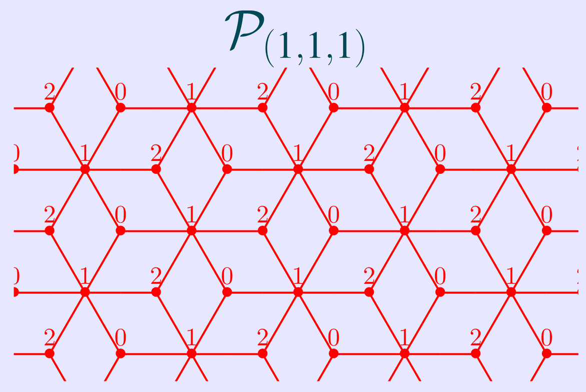




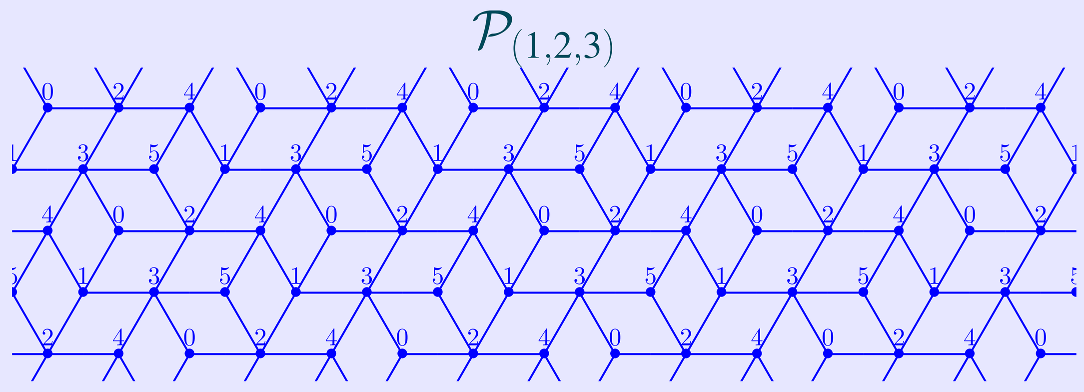
$\mathcal{P}_{(2,3,4)}$



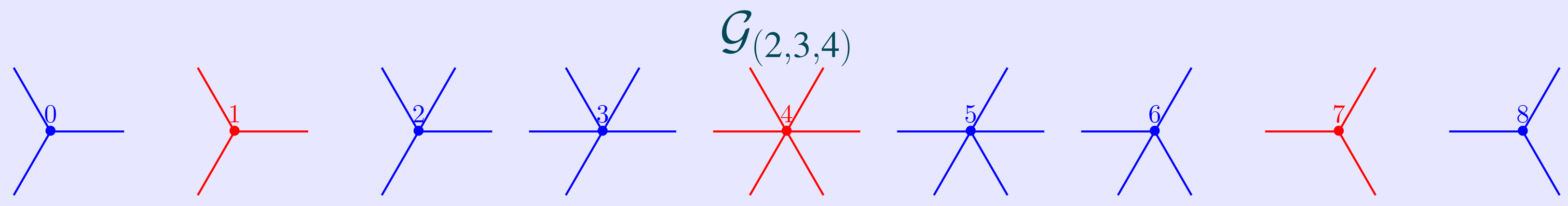
$\mathcal{P}_{(1,1,1)}$



$\mathcal{P}_{(1,2,3)}$

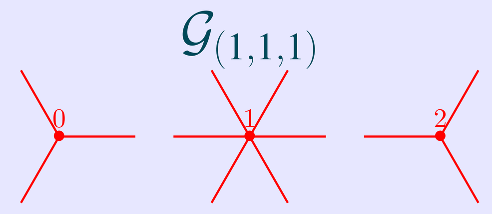


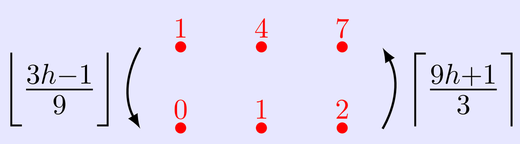
$\mathcal{G}_{(2,3,4)}$



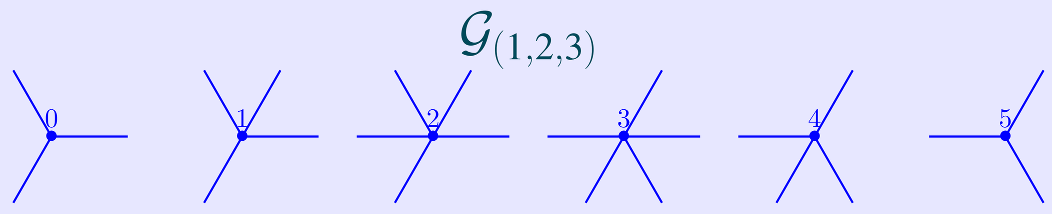
The partionning depends on $\|\mathbf{a}\|_1$, $\|\mathbf{b}\|_1$, $\|\mathbf{c}\|_1$ and an integer $\delta = 1$:
if $(3h - 1) \mod 9 < 3$ then • else • $\Leftrightarrow$ if $6h \mod 9 < 6$ then • else •

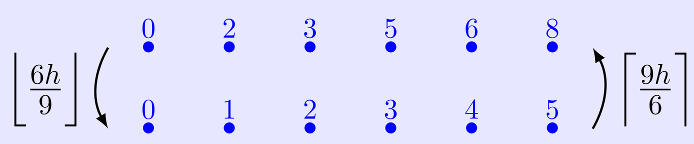
$\mathcal{G}_{(1,1,1)}$





$\mathcal{G}_{(1,2,3)}$



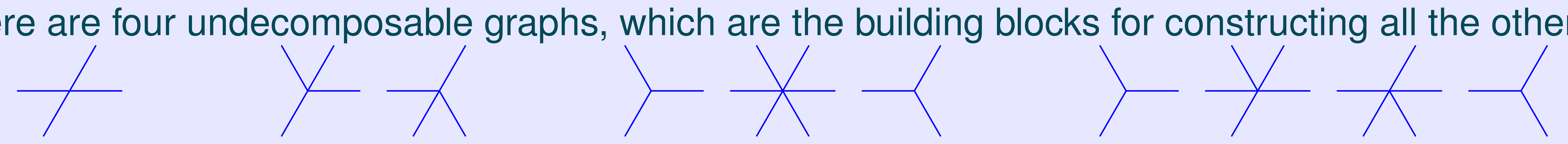


Sketch of the proof

- Existence of $\mathbf{b}$:
  There exists at least one vector $\mathbf{b} \in \mathbb{N}^3 \setminus \{\mathbf{0}\}$ such that $\gcd(\mathbf{b}) = 1$ and $\left\| \|\mathbf{b}\|_1 \mathbf{a} - \|\mathbf{a}\|_1 \mathbf{b} \right\|_\infty < \frac{\|\mathbf{a}\|_1}{2}$ if and only if $\mathbf{a}$ is not a permutation of $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ or $(1,1,2)$.
  (Based on a variant of the notion of approximation of a rational vector and Minkowski's theorem)
- Existence of δ:
  There exists $\delta \in \{-\|\mathbf{b}\|_1 + 1, \dots, \|\mathbf{a}\|_1 - \|\mathbf{b}\|_1\}$ such that $\left\| \|\mathbf{b}\|_1 \mathbf{a} - \|\mathbf{a}\|_1 \mathbf{b} \right\|_\infty < \frac{\|\mathbf{a}\|_1}{2} \Leftrightarrow \left\| \|\mathbf{b}\|_1 \mathbf{a} - \|\mathbf{a}\|_1 \mathbf{b} \right\|_\infty \leq \min\left(|\|\mathbf{b}\|_1 - 1 + \delta|, |\|\mathbf{a}\|_1 - \|\mathbf{b}\|_1 - \delta|\right)$.
  (Simple calculations)
- Criterion for bijection:
  $\mathcal{S}(\mathcal{G}_\mathbf{a}, \mathbf{b}, \delta) \simeq \mathcal{G}_\mathbf{b}$ if $\left\| \|\mathbf{b}\|_1 \mathbf{a} - \|\mathbf{a}\|_1 \mathbf{b} \right\|_\infty \leq \min\left(|\|\mathbf{b}\|_1 - 1 + \delta|, |\|\mathbf{a}\|_1 - \|\mathbf{b}\|_1 - \delta|\right)$.
  (Technical part, see paper)

Remark

There are four undecomposable graphs, which are the building blocks for constructing all the others:



$\mathcal{G}_{(0,0,1)}$ $\mathcal{G}_{(0,1,1)}$ $\mathcal{G}_{(1,1,1)}$ $\mathcal{G}_{(1,1,2)}$