Decomposition of Rational Discrete Planes

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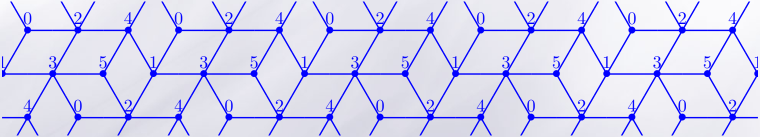
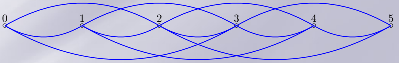
Given a non-zero vector $\mathbf{a} \in \mathbb{N}^3$,

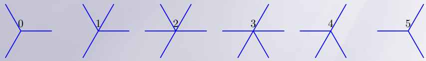
$$\mathcal{P}_{\mathbf{a}} := \{\mathbf{x} \in \mathbb{Z}^3 \mid 0 \leq \mathbf{x} \cdot \mathbf{a} < \|\mathbf{a}\|_1\}.$$



Representation by a graph $\mathcal{G}_{\mathbf{a}}$

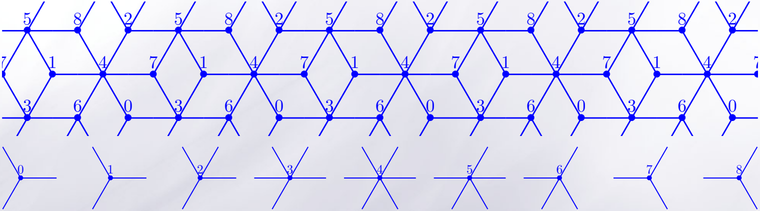
- whose nodes are $\{0, \ldots, \|\mathbf{a}\|_1 - 1\}$
- and whose arcs are such that $\forall i \in \{1, 2, 3\}$, $(\mathbf{x} \cdot \mathbf{a}, \mathbf{x} \cdot \mathbf{a} \pm a_i) \in \mathcal{G}_{\mathbf{a}}$ iff $\mathbf{x}, \mathbf{x} \pm e_i \in \mathcal{P}_{\mathbf{a}}$.





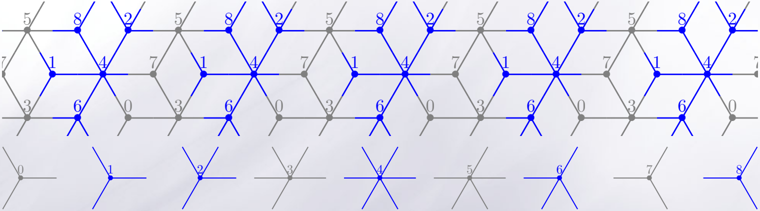
A key observation

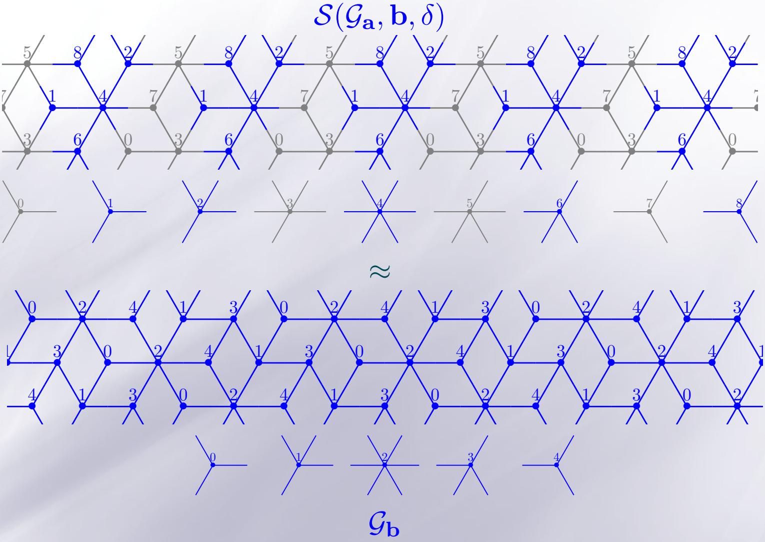
$\mathcal{G}_{\mathbf{a}}$



A key observation

$\mathcal{S}(\mathcal{G}_{\mathbf{a}}, \mathbf{b}, \delta)$





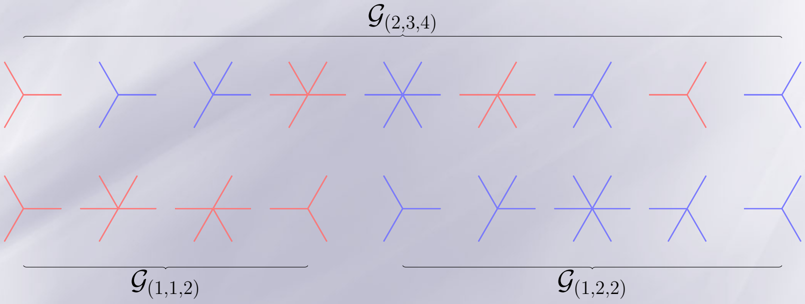
$\mathcal{S}(\mathcal{G}_{\mathbf{a}}, \mathbf{b}, \delta)$
$\approx$
$\mathcal{G}_{\mathbf{b}}$

Main theorem

Let $\mathbf{a} \in \mathbb{N}^3 \setminus \{\mathbf{0}\}$ be such that $\gcd(\mathbf{a}) = 1$. If $\mathbf{a}$ is not a permutation of one of the vectors $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ and $(1,1,2)$, then there exist $\mathbf{b} \in \mathbb{N}^3 \setminus \{\mathbf{0}\}$, $\delta \in \mathbb{Z}$ such that $\mathbf{a} - \mathbf{b} \in \mathbb{N}^3 \setminus \{\mathbf{0}\}$,

$$\mathcal{G}_{\mathbf{a}} = \mathcal{S}(\mathcal{G}_{\mathbf{a}}, \mathbf{b}, \delta) \ \cup \ \mathcal{S}(\mathcal{G}_{\mathbf{a}}, \mathbf{a} - \mathbf{b}, -\delta + 1),$$

$$\mathcal{S}(\mathcal{G}_{\mathbf{a}}, \mathbf{b}, \delta) \simeq \mathcal{G}_{\mathbf{b}} \text{ and } \mathcal{S}(\mathcal{G}_{\mathbf{a}}, \mathbf{a} - \mathbf{b}, -\delta + 1) \simeq \mathcal{G}_{(\mathbf{a}-\mathbf{b})}.$$



- do we really need δ?
- for a given $\mathbf{a}$, is the pair $\mathbf{b}, \delta$ unique?
- why $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ and $(1,1,2)$ are special cases?
- ...