Heat kernel Laplace-Beltrami operator on digital surfaces

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Abstract. Many problems in image analysis, digital processing and shape optimization are expressed as variational problems and involve the discretization of the Laplace-Beltrami operator. Discretization of the Laplace-Beltrami operator has been widely studied for meshes or polyhedral surfaces. On digital surfaces, trivial applications of classical operators are usually not satisfactory (lack of multigrid convergence, lack of precision...). In this paper, we first evaluate previous alternatives and propose a new digital Laplace-Beltrami operator showing interesting properties. This new operator adapts Belkin *et al.* [1] for digital surfaces embedded in 3D. The core of the method relies on an accurate estimation of measures associated to digital surface elements. We experimentally evaluate the interest of such operator for digital geometry processing tasks.

1 Introduction

Objectives In geometry processing, Partial Differential Equations (PDEs) containing Laplace-Beltrami operator arise in surface fairing, mesh smoothing, mesh parametrization, remeshing, feature extraction, shape matching, etc [2]. Prior work on a robust and convergent operator is mandatory: for example, in applications such as [3], the discrete laplacian controls the shape of isolines of the distance maps and therefore the quality of the reconstruction.

Contributions We propose a discrete Laplace-Beltrami operator on digital surfaces (subsets of $\mathbb{Z}^2$ embedded in 3D). This new operator adapts Belkin *et al.* [1] on our complex. The method uses a precise estimation of areas associated with digital elements. This estimation is achieved through a convergent digital normal estimator described in [4, 5]. We show experimental convergence of our operator, and compare it with various discretizations of the literature adapted on digital surfaces. We compare the behavior of the heat diffusion associated with the heat equation [6] between our approximation and the laplacian constructed through the Discrete Exterior Calculus (DEC) framework.

Related Works First works on discrete calculus may be found in the Regge Calculus [7] for quantum physics, where tetrahedra in combination edge lengths are used. Works on geometric aquisition devices and models drived studies toward calculus working on meshes and mor generally on simplicial complexes. Early work include the famous cotangent formula [8] for solving the problem of minimal surfaces, which is an analog of the standart finite element method [2].
After that, the framework of Discrete Exterior Calculus (DEC) was developped in the computational mathematics and geometry processing community focusing their work on triangular meshes. Exact calculus generalizing the cotan discretization in 2D based on finite elements [9] emerged from the "German school" but with a restriction to triangular complexes. Another more recent formulation of the DEC comes from Hirani's thesis [10] and later by the monograph [11]. Their construction works on simplicial complexes, but they do not proove convergence toward the smooth counterparts, focusing their work on local operators and validity of the generalized Stokes' theorem.
In parallel, another discrete calculus emerges in the image, graph, electric circuits and network analysis community, summed up in [12]. Although intrinsic measures of quantities can be incorporated, it has no relation with the ambient space, leading to a calculus designed to analyse data without knowledge of an embedding.
Finally, we can see an aliked discrete calculus on "chainlets" in geometric measure theory, for the mathematical analysis of general compact shapes like fractals [13, 14]. The Laplace-Beltrami operator is defined here for very general spaces, but computational aspects are unclear.

Outline Many problems in image analysis, digital processing and shape optimization can be expressed as variationnal problems and involve the discretization of the Laplace-Beltrami operator (see for example [2].
An important objective when proposing discretization of the operator is to give convergence results: as meshes refine and tend toward the underlying manifold under certain properties, the approximated Laplace-Beltrami operator should tends toward the usual one on the manifold. On arbitrary triangular meshes, it is shown that the discrete operator cannot recover all the properties of the smooth manifold one [15]. Regarding DEC, Hildebrandt *et al.* [16] provided convergence results when the triangulated meshes tend toward the manifold with those properties: Hausdorff distance tends to zero, mesh normals tend to surface normals and the mesh is projected one-to-one on the continuous surface. Similar proofs exist in the context of finite element methode [17, 18] and for chainlet discrete calculus [19]. Recent work [20] shows a laplacian that have all the desired properties discribed in [15] with an extension to polygon meshes. Their method rely on modifying the embedding of the complex by moving vertices inside the mesh.

1.1 argumentaires

- $L_h^\star$converge experimentallement (+cut)

- $\mathcal{L}_{DEC}$et L_{DEC}convergent pas
- Mieux dans des applis geom processing
  - smoothing?
  - Crane : + isotropique
  - Reco : Plus précis geom. à nombre vect propre egal

2 Discretizations of the Laplace-Beltrami operator

Let M be a 2 Riemannian manifold with or without boundary embedded in $\mathbb{R}^3$, that is a pair (M, g) where M is a smooth manifold and g is a Riemannian metric on M (ie with know an intrinsic notion of distances). Let

$$\begin{aligned}\Delta : C^\infty(M) &\to C^\infty(M)\\ u &\mapsto -\mathrm{div}(\mathrm{grad}\, u),\end{aligned}$$

be the intrinsic smooth Laplace-Beltrami operator [6] where C^∞ is the set of smooths function of M.
Discretizations of such operator comes in many flavour for meshes or polyhedral surfaces. Let Γ be a mesh (a triangular one for example), $V(\Gamma)$ its set of vertices and $E(\Gamma)$ its edges. Let $\tilde{u} : V(\Gamma) \to \mathbb{R}$ be a twice differentiable function. We recall first the definition of the cotan operator [8] denoted $\mathcal{L}_{COT}$:

$$\mathcal{L}_{COT}\, \tilde{u}(w) = \frac{1}{2A_w} \sum_{p \sim w} (\cot(\alpha_{wp}) + \cot(\beta_{wp}))(\tilde{u}(p) - \tilde{u}(w)), \quad (1)$$

where $p \sim w$ are the one-ring points from w and A_w is one third the area of all triangles incident on vertex w. α_{wp} and β_{wp} are the angles opposing the corresponding edge wp (see Fig. 1).

2.1 Notations

We wish to compare discritizations of the Laplace-Beltrami operator on triangular meshes with our discritization on digital surfaces. Given a triangular surface Γ, We denote by $\mathcal{L}_{COT}$ the famous cotan operator [8], by $\mathcal{L}_{DEC}$ the laplace operator related to the Discrete Exterior Calculus [10, 11] and by $\mathcal{L}_{MESH}$ the mesh laplacian presented in [1]. For a digital surface D, operators are called L_{COT}, L_{DEC} and L_{MESH}. We call our discritized operator acting on digital surfaces $L_h^\star$ where h is the grid step. We recall some desired properties of the discrete laplacian described in [15]:
Symmetry(SYM): $\omega_{ij} = \omega_{ji}$. The symmetry property ensures both real eigenvalues and orthogonal eigenvectors.
Locality (LOC): ω_{ij} is different of 0 if and only if i and j shares a common edge.
Linear Precision (LIN): $Lu = 0$ whenever u is a linear function restricted to a plane.

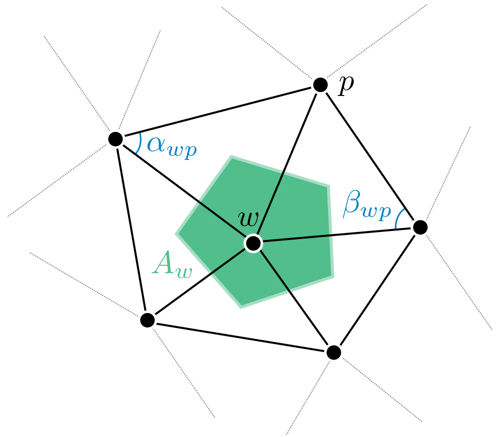

Fig. 1. Illustration of $\mathcal{L}_{COT}$ on triangular meshes. Points in the one-ring around w are in black, he area of integration A_w is in green (one third the area of all triangles incident on vertex w), the angles α_{wp} and β_{wp} opposing the corresponding edge wp are in blue.

Positive Weights (POS): $\omega_{ij} \geq 0$ whenever i is not equal to j.

Positive Semi-Definiteness (PSD): the matrix is symmetric positive semi-difinite regarding the standart inner product and has a one-dimensionnal kernel. (SYM) and (POS) imply (PDS), but (PSD) does not implies (POS).

Dirichlet Convergence (CON): $L_n \to \Delta$ such that solutions to the discrete Dirichlet problem using L_n converge to the solution of the smooth one.

We also add our own convergence setting:

Digital Convergence (DCON): Given an digital surface sampled with grid step h, we have

$$|L_t \tilde{u} - \Delta u| \leq \sigma(h), \tag{2}$$

where $\lim_{h \to 0} \sigma(h) = 0$ and the function σ is called the convergence speed.

DC: ajouter Belkin dans le tableau

	SYM	*LOC*	*LIN*	*POS*	*PSD*	*CON*	DCON
MEAN VALUE	✗	✓	✓	✓	✗	✗	?
INTRINSINC DEL	✓	✗	✓	✓	✓	?	?
$\mathcal{L}_{DEC}$	✓	✓	✗	✓	✓	✗	?
$\mathcal{L}_{COT}$	✓	✓	✓	✗	✓	✓	?
$L_h^\star$	✗	✗	✗	✓	✗	?	✓

Table 1. Properties of various laplacians

3 Experiments

3.1 Experimental Convergence

ajouter $h^1/3$ $h^2/3$ dans les graphes

3.2 Shape approximation using eigenvectors decomposition

We use in this section the framework of *spectral analysis* for geometry. Given a symmetric matrix $\boldsymbol{L}$, we know from linear algebra theory that is has real eigenvalues and a set of real and orthogonal eigenvectors thus giving us a basis. Given any laplacian square matrix L, we denote by $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ its normalized eigenvectors and the matrix $\boldsymbol{E}$ whose columns are those eigenvectors. By $\lambda_1, \lambda_2, \ldots, \lambda_n$ we denote the associated eigenvalues where n is the size of $\boldsymbol{L}$.
Given an input vector $\boldsymbol{X}$ in the standard $\mathbb{R}^3$ basis, we want to rewrite it onto the basis formed by the eigenvectors of $\boldsymbol{L}$:

$$\boldsymbol{X} = \sum_{i=1}^{n} \boldsymbol{e}_i \tilde{x}_i = \begin{pmatrix} \boldsymbol{E}_{11} & \ldots & \boldsymbol{E}_{1n} \\ \boldsymbol{E}_{21} & \ldots & \boldsymbol{E}_{2n} \\ \vdots & \vdots & \vdots \\ \boldsymbol{E}_{n1} & \ldots & \boldsymbol{E}_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \boldsymbol{E}\tilde{\boldsymbol{X}}.$$

This expression represents a transform of $\boldsymbol{X}$ to $\tilde{\boldsymbol{X}}$ in terms of the basis given by the eigenvectors of $\boldsymbol{L}$. This is called a *spectral transform* and we have:

$$\tilde{\boldsymbol{X}} = \boldsymbol{E}^T \boldsymbol{X},$$

where $\boldsymbol{E}^T$ is the transpose of $\boldsymbol{E}$. Now we can approximate the input signal $\boldsymbol{X}$ by using a fixed number k of eigenvectors:

$$\boldsymbol{X}^{(k)} = \boldsymbol{E}^{(k)} (\boldsymbol{E}^{(k)})^T \boldsymbol{X},$$

where $\boldsymbol{E}^{(k)}$ is a matrix of size $n \times k$ containing the first k eigenvectors columnwise.

3.3 Smoothing

Ressortir figures pour $L_h^\star$

- Graphes de convergences des différents laplaciens : convolution, combinatoire, cotangentes et Belkin sur la trigulation du complexe cubique
- Laplacian smoothing
- Approximation de formes avec les valeurs propres du laplacien
- Distance géodésiques (papier de Crane) : comparaison laplacien combinatoire et laplacien de convolution

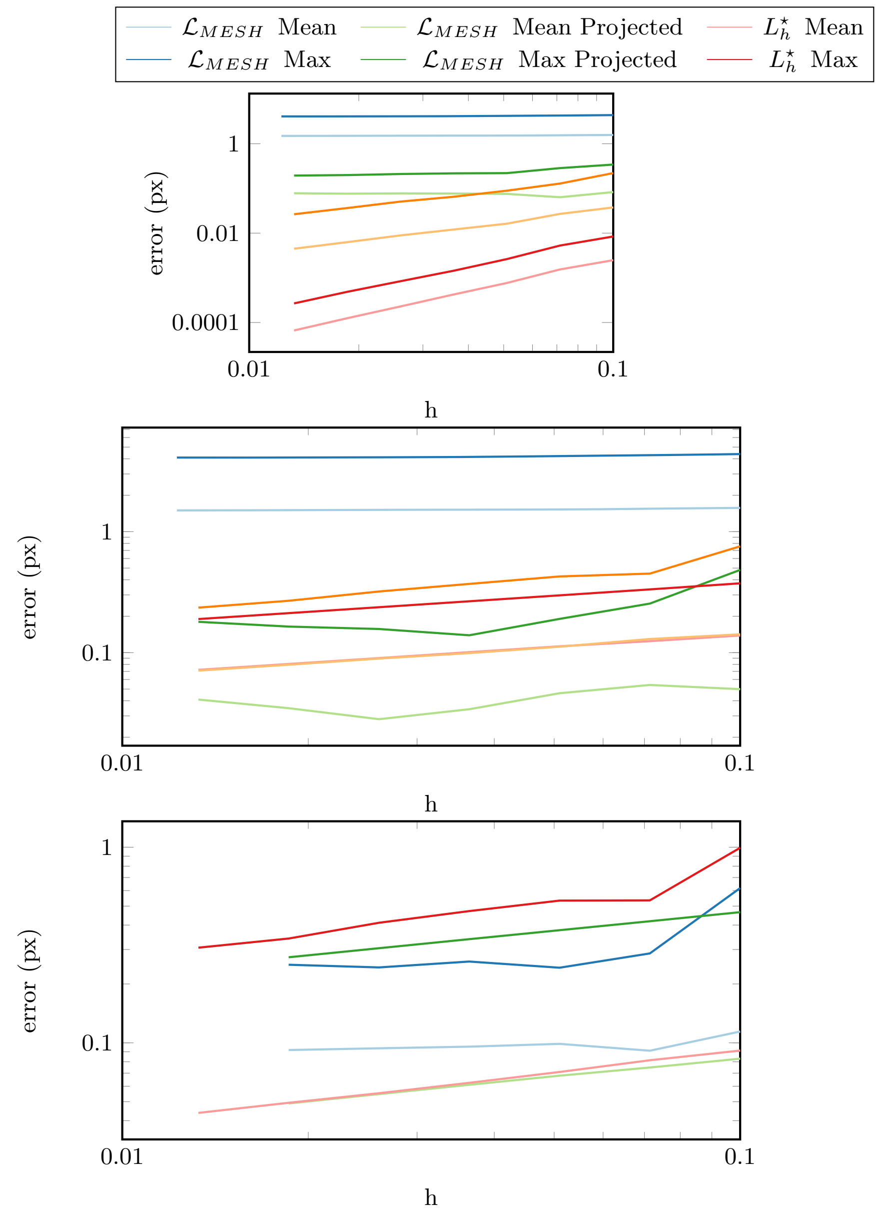

Fig. 2. Multigrid convergence graphs with the $cos(x)$ function with $t = 0.1 \times h^{\frac{1}{3}}$.

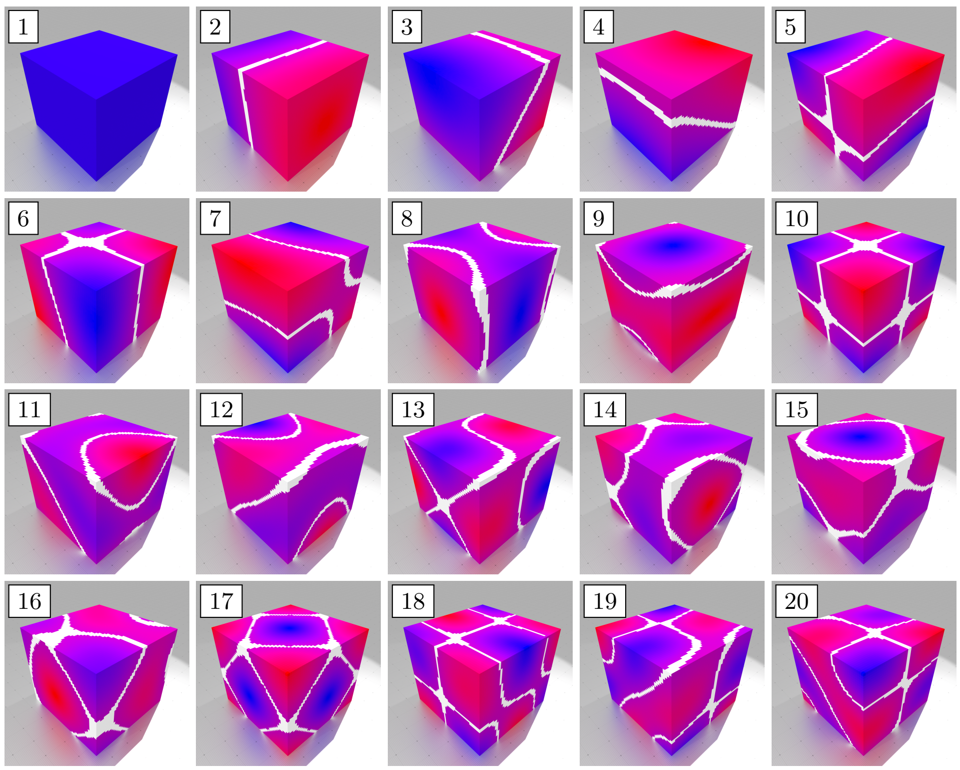

Fig. 3. Eigenfunctions display on a simple cube with faces aligned with $\mathbb{R}^3$ axis. Numbers on the top left of each figure represents the eigenvalue displayed in ascending order.

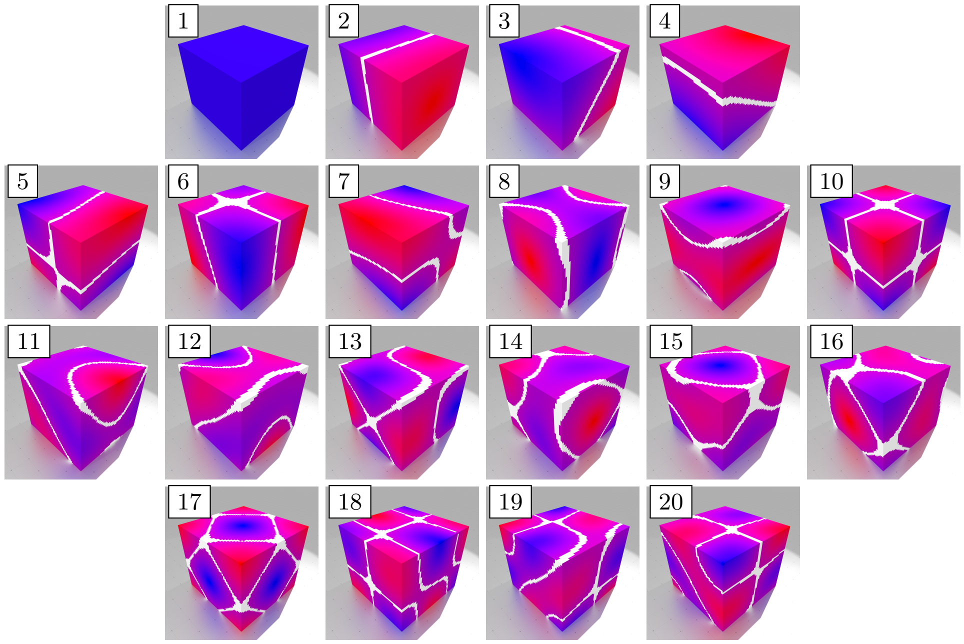

Fig. 4. Eigenfunctions display on a simple cube with faces aligned with $\mathbb{R}^3$ axis. Numbers on the top left of each figure represents the eigenvalue displayed in ascending order.

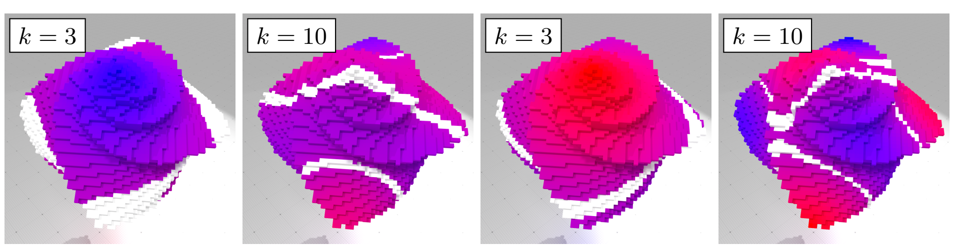

Fig. 5. Eigenfunctions display on the octa-flower form. (*First row*) using L_{DEC}, (*second row*) with $L_h^\star$TODODODODODODODO

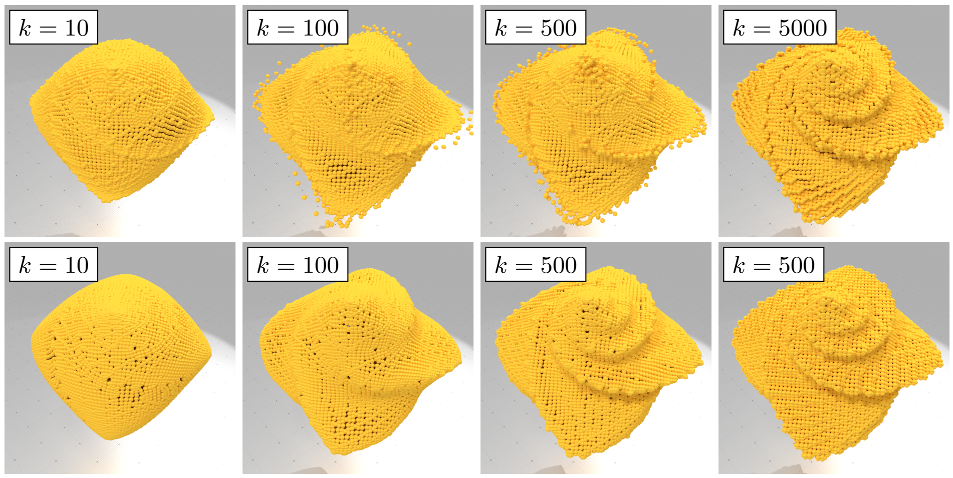

Fig. 6. Images of the reconstruction using an increasing number of eigenvectors k. (*First row*) using L_{DEC}, (*second row*) with $L_h^\star$

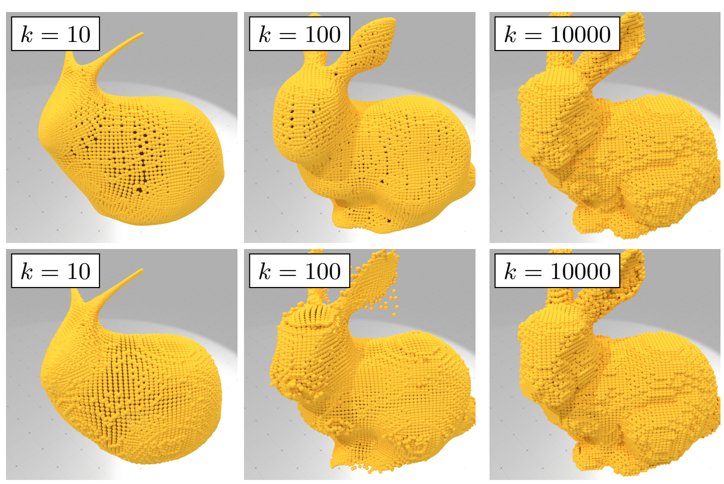

Fig. 7. Images of the reconstruction using an increasing number of eigenvectors k. (*First row*) using L_{DEC}, (*second row*) with $L_h^\star$

3.4 Distance maps

We implemented here the work described in [3].

Ressortir figures pour $L_h^\star$

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