Decomposition of Rational Discrete Planes*

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Abstract. This paper is a contribution to the study of rational discrete planes, i.e., sets of points with integer coordinates lying between two parallel planes. Up to translation and symmetry, they are completely determined by a normal vector $\boldsymbol{a} \in \mathbb{N}^3$. Excepted for a few well-identified cases, it is shown that there are two approximations $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{N}^3$ of $\boldsymbol{a}$, satisfying $\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{c}$, such that the discrete plane of normal $\boldsymbol{a}$ can be partitioned into two sets having respectively the combinatorial structure of discrete planes of normal $\boldsymbol{b}$ and $\boldsymbol{c}$. Christoffel graphs are used to compactly encode the structure of discretes planes. This result may have practical interest in discrete geometry for the analysis of planar features.

Keywords: Discrete Plane · Christoffel Graph · Approximation.

1 Introduction

This paper is a contribution to the study of standard arithmetical rational discrete planes. They are defined from a non-zero normal vector $\boldsymbol{a} \in \mathbb{N}^3$ as follows:

$$\mathcal{P}_{\boldsymbol{a}} := \{\boldsymbol{x} \in \mathbb{Z}^3 \mid 0 \leq \boldsymbol{x} \cdot \boldsymbol{a} < \|\boldsymbol{a}\|_1\}.$$

Their combinatorial structure has been studied thirty years ago in [3]. The main definitions and results are recalled below. The adjacency graph associated to $\mathcal{P}_{\boldsymbol{a}}$ is a graph whose vertices are the points of $\mathcal{P}_{\boldsymbol{a}}$ and that has an edge between two distinct points $\boldsymbol{x}$ and $\boldsymbol{y}$ if and only if $\|\boldsymbol{x} - \boldsymbol{y}\|_1 = 1$. The elementary cycles, which are squares, are called faces. The adjacency graph is connected and, together with the set of faces, defines a two-dimensional combinatorial manifold without boundary [3] (Fig. 1).

In the adjacency graph associated to $\mathcal{P}_{\boldsymbol{a}}$, the set of edges incident to a given vertex $\boldsymbol{x}$ is determined by the quantity $\boldsymbol{x} \cdot \boldsymbol{a}$, called the height of $\boldsymbol{x}$ (Fig. 1). The arrangement of edges incident to equally high points is thus the same. In addition, there are only eight different arrangements of incident edges in all discrete planes and at most seven in a given one [3] (Fig. 2).

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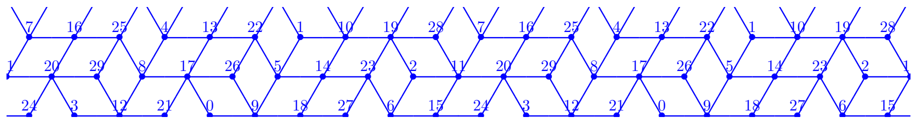
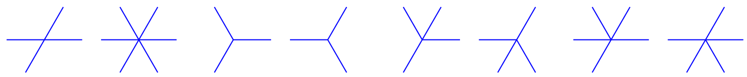


Fig. 1: $\mathcal{P}(4, 9, 17)$. The number displayed close to a point is its height.

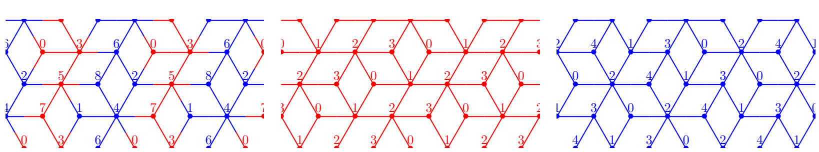


(a) (b) (c) (d) (e) (f) (g) (h)

Fig. 2: All arrangements of incident edges – (a) and (b) cannot be in the same discrete plane. This is close to a vertex-atlas in tiling theory [7, section 5.3].

Related Works. In 2d, it is well known that the smallest segment that periodically generates a rational discrete line can be uniquely partitioned into two segments, each of them being the minimal period of another rational discrete line. See, e.g., the splitting formula [8, pp. 153–157] or the standard factorization of Christoffel words [1, pp. 19–22]. This paper aims at extending such decomposition to 3d. However, there are infinitely many sets of faces that can periodically generate the same rational discrete plane and there is no canonical way of decomposing them. See, e.g., [2] for a practical method of generation and decomposition based on a geometrical extension of substitutions.

Contribution. In this paper, we propose another framework based on a symmetric version of Christoffel graphs, introduced in [5] as extensions of Christoffel words. In brief, they describe, for every height h, the arrangement of edges incident to the points of height h in the adjacency graph associated to $\mathcal{P}_{\boldsymbol{a}}$. They allow us to compare the arrangements of edges in the adjacency graphs of two different sets. We say that two sets have the same combinatorial structure if there is a bijection between their points and the arrangements of edges incident to them. We show that, excepted for a few cases, there are two approximations $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{N}^3$ of $\boldsymbol{a}$, such that $\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{c}$ and $\mathcal{P}_{\boldsymbol{a}}$ can be partitioned into two sets having respectively the combinatorial structure of $\mathcal{P}_{\boldsymbol{b}}$ and $\mathcal{P}_{\boldsymbol{c}}$. See Fig. 3.



(a) $\mathcal{P}(2, 3, 4)$ (b) $\mathcal{P}(1, 1, 2)$ (c) $\mathcal{P}(1, 2, 2)$

Fig. 3: Decomposition of $\mathcal{P}(2, 3, 4)$ in red and blue sets. The arrangements of edges in the red (resp. blue) set match those of $\mathcal{P}(1, 1, 2)$ (resp. $\mathcal{P}(1, 2, 2)$).

Outline. In section 2, we present several definitions based on Christoffel graphs. In section 3, we provide a condition for having a bijection between a subgraph and another graph. In section 4, we introduce approximations and gather several results about them. They are used in section 5 to prove that, excepted in a few cases, there is an approximation for which our condition for bijection is true.

2 Definitions

Given a directed graph $\mathcal{G} := (\mathcal{V}, \mathcal{A})$, let us introduce the function $arcs_{\mathcal{G}} : 2^{\mathcal{V}} \mapsto 2^{\mathcal{A}}$ that returns all the arcs emanating from the vertices of a given subset $\mathcal{V}' \subseteq \mathcal{V}$, i.e., such that $arcs_{\mathcal{G}}(\mathcal{V}') = \{(v_1, v_2) \in \mathcal{A} \mid v_1 \in \mathcal{V}'\}$.

For a symmetric graph $\mathcal{G}$, i.e., such that $(v_1, v_2) \in \mathcal{A} \Leftrightarrow (v_2, v_1) \in \mathcal{A}$, a partition of $\mathcal{V}$ induces a partition of $\mathcal{A}$ such that the set of arcs emanating from a given vertex is included in the same subset of $\mathcal{A}$. Indeed, for two subsets $\mathcal{V}', \mathcal{V}''$ such that $\mathcal{V}' \cup \mathcal{V}'' = \mathcal{V}$ and $\mathcal{V}' \cap \mathcal{V}'' = \emptyset$, one has

$$arcs_{\mathcal{G}}(\mathcal{V}') \cup arcs_{\mathcal{G}}(\mathcal{V}'') = \mathcal{A}, \quad arcs_{\mathcal{G}}(\mathcal{V}') \cap arcs_{\mathcal{G}}(\mathcal{V}'') = \emptyset.$$

2.1 Christoffel Graph

Definition 1. *Given $A \in \mathbb{Z}_{>0}$, let us define the set $\mathcal{V}_A := \{0, \ldots, A-1\}$. The* symmetric Christoffel graph *of normal $\boldsymbol{a} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$, with $\gcd(\boldsymbol{a}) = 1$, is the pair $\mathcal{G}_{\boldsymbol{a}} := (\mathcal{V}_{\|\boldsymbol{a}\|_1}, \mathcal{A}_{\boldsymbol{a}})$, where*

$$\mathcal{A}_{\boldsymbol{a}} := \{(v_1, v_2) \mid v_1, v_2 \in \mathcal{V}_{\|\boldsymbol{a}\|_1}, 1 \leq i \leq 3, v_2 = v_1 \pm a_i\}.$$

We have symmetrized the original definition of Christoffel graph introduced in [5] in order to define partitions such that the set of arcs emanating from a given vertex is always included in the same subset of arcs. Since we only consider symmetric Christoffel graphs in the rest of the paper, we will now omit the term symmetric.

Two representations of the same Christoffel graph are shown in Fig. 4. The two arcs (v_1, v_2) and (v_2, v_1) are merged into one undirected edge in (b), while they are represented by two distinct segments in (c): one incident to v_1 for (v_1, v_2), the other incident to v_2 for (v_2, v_1). In the latter representation, the following convention is used for all vertices v: the segments at angle $4\pi/3$, 0, $2\pi/3$ respectively correspond to the arcs $(v, v + a_1)$, $(v, v + a_2)$, $(v, v + a_3)$ and symmetrically, the segments at angle $\pi/3$, π, $5\pi/3$ respectively correspond to the arcs $(v, v - a_1)$, $(v, v - a_2)$, $(v, v - a_3)$, where angles are measured counterclockwise with respect to the horizontal segment directed to the right.

2.2 Christoffel Subgraph

Given $A \in \mathbb{Z}_{>0}$, a bound $B \in \{0, \ldots, A-1\}$ and an offset $\delta \in \{-B+1, \ldots, A-B\}$ are used to define a subset of $\mathcal{V}_A$:

$$\mathcal{V}_A^{B,\delta} := \{k \in \mathcal{V}_A \mid (kB - \delta) \bmod A < B\}. \quad (1)$$

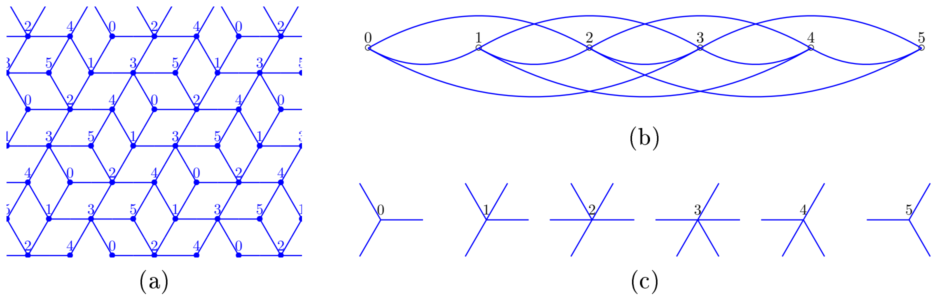
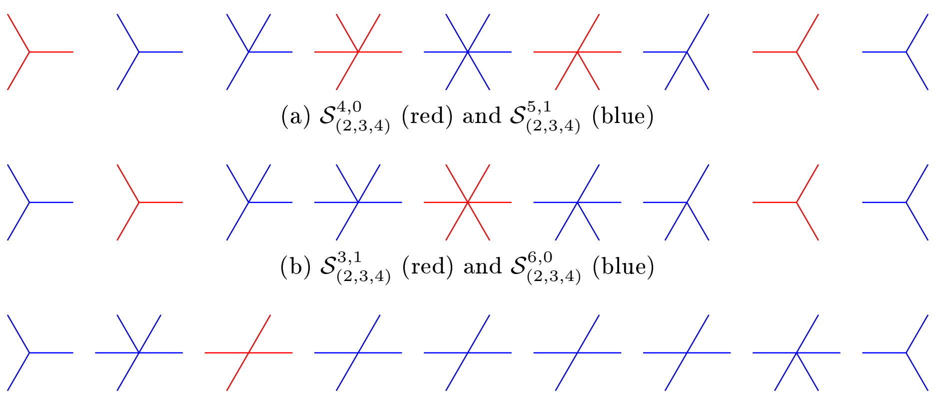


Fig. 4: $\mathcal{P}_{(1,2,3)}$ in (a). Two representations of $\mathcal{G}_{(1,2,3)}$ in (b) and (c). Observe how they encode the arrangements of edges incident to the points of $\mathcal{P}_{(1,2,3)}$.

The following table shows how to compute two subsets of the vertices $\{0, \dots, 8\}$ (in blue and red) thanks to (1), where (B, δ) are equal to $(4, 0)$ (resp. $(5, 1)$) in the second (resp. third) line.

k	0	1	2	3	4	5	6	7	8
$4k$ mod 9	0	4	8	3	7	2	6	1	5
$(5k - 1)$ mod 9	8	4	0	5	1	6	2	7	3

Then, let us denote by $\mathcal{S}_{\boldsymbol{a}}^{B,\delta} := \left(\mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}, arcs_{\mathcal{G}_{\boldsymbol{a}}}\left(\mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}\right)\right)$ the subgraph of $\mathcal{G}_{\boldsymbol{a}}$ of bound B and offset δ. To say it simply, it contains all the arcs of $\mathcal{G}_{\boldsymbol{a}}$ emanating from at least one vertex of $\mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}$. Several subgraphs of $\mathcal{G}_{(2,3,4)}$ and $\mathcal{G}_{(1,1,7)}$ are illustrated in Fig. 5.



(a) $\mathcal{S}_{(2,3,4)}^{4,0}$ (red) and $\mathcal{S}_{(2,3,4)}^{5,1}$ (blue)

(b) $\mathcal{S}_{(2,3,4)}^{3,1}$ (red) and $\mathcal{S}_{(2,3,4)}^{6,0}$ (blue)

(c) $\mathcal{S}_{(1,1,7)}^{1,2}$ (red) and $\mathcal{S}_{(1,1,7)}^{(8,-1)}$ (blue)

Fig. 5: Subgraphs of $\mathcal{G}_{(2,3,4)}$ (a-b) and $\mathcal{G}_{(1,1,7)}$ (c). Since the vertices $\{0, 1, \dots, 8\}$ are placed from left to right in increasing order, the numbers are omitted.

The rationale for the offset is two-fold. On the one hand, it allows us to easily describe the relative complement of a subgraph with respect to a graph. See Fig. 5 and the next subsection. On the other hand, it allows us to always find a subgraph such that it corresponds to a Christoffel graph, excepted for a few values of $\boldsymbol{a}$, as it will become clear in section 5.

2.3 Relative Complement

The relative complement of $\mathcal{S}_{\boldsymbol{a}}^{B,\delta}$ with respect $\mathcal{G}_{\boldsymbol{a}}$ is defined as

$$\overline{\mathcal{S}_{\boldsymbol{a}}^{B,\delta}} := \left(\mathcal{V}_{\|\boldsymbol{a}\|_1} \setminus \mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}, \mathcal{A}_{\boldsymbol{a}} \setminus arcs_{\mathcal{G}_{\boldsymbol{a}}}\left(\mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}\right)\right).$$

Note that $\mathcal{S}_{\boldsymbol{a}}^{B,\delta} \cup \overline{\mathcal{S}_{\boldsymbol{a}}^{B,\delta}} = \mathcal{G}_{\boldsymbol{a}}$ by definition, where the union is done independently on the vertices and arcs.

The relative complement of a subgraph with respect to a graph is also a subgraph in itself with appropriate bound and offset.

Lemma 1. *One has* $\overline{\mathcal{S}_{\boldsymbol{a}}^{B,\delta}} = \mathcal{S}_{\boldsymbol{a}}^{C,\gamma}$*, where* $C := \|\boldsymbol{a}\|_1 - B$ *and* $\gamma := -\delta + 1$.

Note that $\gamma \in \{-C+1, \dots, \|\boldsymbol{a}\|_1 - C\}$, see subsection 2.2.

Proof. Let us first focus on the vertices. By definition of $\mathcal{V}_{\|\boldsymbol{a}\|_1}$ and $\mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}$, the set $\mathcal{V}_{\|\boldsymbol{a}\|_1} \setminus \mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}$ is equal to $\{k \in \mathcal{V}_{\|\boldsymbol{a}\|_1} \mid B \leq (kB - \delta) \bmod \|\boldsymbol{a}\|_1\}$.

Denoting by q_k and r_k, respectively the quotient and remainder of the Euclidean division of $kB - \delta$ by $\|\boldsymbol{a}\|_1$, one has $kB - \delta = q_k\|\boldsymbol{a}\|_1 + r_k$ and $B \leq r_k < \|\boldsymbol{a}\|_1$ for all $k \in \mathcal{V}_{\|\boldsymbol{a}\|_1} \setminus \mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}$. However, $kB - \delta = q_k\|\boldsymbol{a}\|_1 + r_k$ is equivalent to

$$\begin{aligned}
&(-kB+\delta) + (k\|\boldsymbol{a}\|_1 - 1) &&= (-q_k\|\boldsymbol{a}\|_1 - r_k) + (k\|\boldsymbol{a}\|_1 - 1) \\
\iff &k(\|\boldsymbol{a}\|_1 - B) + \delta - 1 &&= (k - 1 - q_k)\|\boldsymbol{a}\|_1 + (\|\boldsymbol{a}\|_1 - r_k - 1) \\
\iff &kC - \gamma &&= (k - 1 - q_k)\|\boldsymbol{a}\|_1 + (\|\boldsymbol{a}\|_1 - r_k - 1).
\end{aligned}$$

Furthermore, $B \leq r_k < \|\boldsymbol{a}\|_1$ is equivalent to

$$\begin{aligned}
&-1 < (\|\boldsymbol{a}\|_1 - r_k - 1) \leq \|\boldsymbol{a}\|_1 - B - 1 \\
\iff \quad &0 \leq (\|\boldsymbol{a}\|_1 - r_k - 1) < \|\boldsymbol{a}\|_1 - B.
\end{aligned}$$

The two above results imply that $\|\boldsymbol{a}\|_1 - r_k - 1$ is the remainder of the Euclidean division of $kC - \gamma$ by $\|\boldsymbol{a}\|_1$ and is strictly less than $\|\boldsymbol{a}\|_1 - B = C$ for all $k \in \mathcal{V}_{\|\boldsymbol{a}\|_1} \setminus \mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}$.

As a consequence,

$$\begin{aligned}
\mathcal{V}_{\|\boldsymbol{a}\|_1} \setminus \mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta} &= \{k \in \mathcal{V}_{\|\boldsymbol{a}\|_1} \mid B \leq (kB - \delta) \bmod \|\boldsymbol{a}\|_1\} \\
&= \{k \in \mathcal{V}_{\|\boldsymbol{a}\|_1} \mid (kC - \gamma) \bmod \|\boldsymbol{a}\|_1 < C\} = \mathcal{V}_{\|\boldsymbol{a}\|_1}^{C,\gamma}.
\end{aligned}$$

Finally, by the definitions of $\mathcal{G}_{\boldsymbol{a}}$ and $arcs_{\mathcal{G}_{\boldsymbol{a}}}$, $\mathcal{V}_{\|\boldsymbol{a}\|_1} \setminus \mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta} = \mathcal{V}_{\|\boldsymbol{a}\|_1}^{C,\gamma}$ implies $\mathcal{A}_{\boldsymbol{a}} \setminus arcs_{\mathcal{G}_{\boldsymbol{a}}}\left(\mathcal{V}_{\|\boldsymbol{a}\|_1}^{B,\delta}\right) = arcs_{\mathcal{G}_{\boldsymbol{a}}}\left(\mathcal{V}_{\|\boldsymbol{a}\|_1}^{C,\gamma}\right)$, which concludes. □

2.4 Arc-preserving Bijection

The goal of the paper is to compare a subgraph with another graph.

Definition 2. *Let* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^3 \backslash \{\boldsymbol{0}\}$*, with* $\gcd(\boldsymbol{a}) = \gcd(\boldsymbol{b}) = 1$*, be such that* $\boldsymbol{a} - \boldsymbol{b} \in \mathbb{N}^3 \backslash \{\boldsymbol{0}\}$ *and* δ *be in* $\{-\|\boldsymbol{b}\|_1 + 1, \ldots, \|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1\}$*.* $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}$ *is said to agree with* $\mathcal{G}_{\boldsymbol{b}}$*, which is denoted by* $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$*, if there exists a bijection* $f : \mathcal{V}_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta} \mapsto \mathcal{V}_{\|\boldsymbol{b}\|_1}$ *such that for all* $i \in \{1, 2, 3\}$*,* $(v, v + a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \Leftrightarrow (f(v), f(v) + b_i) \in \mathcal{G}_{\boldsymbol{b}}$ *and* $(v, v - a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \Leftrightarrow (f(v), f(v) - b_i) \in \mathcal{G}_{\boldsymbol{b}}$*.*

In other words, $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$ means that there exists a bijection f, such that, for every vertex v in $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}$, v and $f(v) \in \mathcal{G}_{\boldsymbol{b}}$ are surrounded by the same kinds of arcs. You can observe that the blue subgraph in Fig. 5 (b) can be mapped to the graph drawn in Fig. 4 (c), i.e., $\mathcal{S}_{(2,3,4)}^{6,0} \simeq \mathcal{G}_{(1,2,3)}$.

Note that we do not rely directly on a subgraph isomorphism definition, since it involves a *vertex-induced* subgraph, i.e., such that the endpoints of the edges are all in the vertex subset, which is not the case here.

3 A Criterion for Bijection

In order to compare $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}$ with $\mathcal{G}_{\boldsymbol{b}}$, let us consider the following:

Definition 3. *The bijection function* $f_A^{B,\delta} : \mathcal{V}_A^{B,\delta} \mapsto \mathcal{V}_B$ *is defined such that*

$$f_A^{B,\delta}(k) := \left\lfloor \frac{kB - \delta}{A} \right\rfloor. \tag{2}$$

Corollary 1 provides a simple criterion for $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$. It depends on $\boldsymbol{b}$ and δ and comes from the two following lemmae:

Lemma 2. *If* $k \in \mathcal{V}_A^{B,\delta}$*, then* $f_A^{B,\delta}(k) \in \mathcal{V}_B$*.*

Proof. Using $k \in \mathcal{V}_A^{B,\delta} \subseteq \mathcal{V}_A$ and $\delta \in \{-B + 1, \ldots, A - B\}$ (see subsections 2.1 and 2.2), one can show that

$$0 \leq \left\lfloor \frac{kB - \delta}{A} \right\rfloor \leq \frac{kB - \delta}{A} < B.$$

Indeed, the upper bound is implied by $k \leq A - 1$ and $-B < \delta$. The lower bound is trivial if $\delta \leq 0$. Otherwise, it is enough to notice that $k \in \mathcal{V}_A^{B,\delta}$ implies $\frac{\delta}{B} \leq k$. It follows that $0 \leq kB - \delta$, which concludes. □

Lemma 3. *For all* $i \in \{1, 2, 3\}$*, for all* $k \in \mathcal{V}_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta}$*,*

$(C1)$ *if* $b_i\|\boldsymbol{a}\|_1 - a_i\|\boldsymbol{b}\|_1 \leq \mu$, *then* $(k, k + a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \Rightarrow (l, l + b_i) \in \mathcal{G}_{\boldsymbol{b}}$,

$(C2)$ *if* $-(b_i\|\boldsymbol{a}\|_1 - a_i\|\boldsymbol{b}\|_1) \leq \mu$, *then* $(k, k - a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \Leftarrow (l, l - b_i) \in \mathcal{G}_{\boldsymbol{b}}$,

$(C3)$ *if* $\nu \leq b_i\|\boldsymbol{a}\|_1 - a_i\|\boldsymbol{b}\|_1$, *then* $(k, k + a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \Leftarrow (l, l + b_i) \in \mathcal{G}_{\boldsymbol{b}}$,

$(C4)$ *if* $\nu \leq -(b_i\|\boldsymbol{a}\|_1 - a_i\|\boldsymbol{b}\|_1)$, *then* $(k, k - a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \Rightarrow (l, l - b_i) \in \mathcal{G}_{\boldsymbol{b}}$,

where $l = f_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta}(k)$*,* $\mu = \|\boldsymbol{b}\|_1 - 1 + \delta$ *and* $\nu = -\|\boldsymbol{a}\|_1 + \|\boldsymbol{b}\|_1 + \delta$*.*

Proof. First, note that the numbers l and $k\|\boldsymbol{b}\|_1 - l\|\boldsymbol{a}\|_1 - \delta$ are respectively the quotient and remainder of the Euclidean division of $k\|\boldsymbol{b}\|_1 - \delta$ by $\|\boldsymbol{a}\|_1$.

Let us denote by P_0 the proposition $\delta \leq k\|\boldsymbol{b}\|_1 - l\|\boldsymbol{a}\|_1 < \delta + \|\boldsymbol{b}\|_1$. Since $k\|\boldsymbol{b}\|_1 - l\|\boldsymbol{a}\|_1 - \delta = (k\|\boldsymbol{b}\|_1 - \delta) \bmod \|\boldsymbol{a}\|_1$, $k \in \mathcal{V}_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta}$ implies P_0.

In the four cases, the same proof by contradiction is used. It can be coarsely described as follows: we consider an hypothesis H, and an implication $P \Rightarrow Q$; we show that, assuming H, the negation of the implication, i.e., $(P \wedge \neg Q)$ contradicts P_0, which means that $P \Rightarrow Q$ must be true.

The hypotheses are given after the ***if*** in the claim of the lemma and involve bounds on the quantity $b_i\|\boldsymbol{a}\|_1 - a_i\|\boldsymbol{b}\|_1$.

The implications and the converse propositions are given in Table 1. We indeed can derive arithmetic constraints from the fact that both v_1 and v_2 must be in $\mathcal{V}_{\|\boldsymbol{a}\|_1}$ for an arc (v_1, v_2) being part of $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}$. For instance, knowing that $k \in \mathcal{V}_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta}$, $(k, k + a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \subseteq \mathcal{G}_{\boldsymbol{a}}$ is equivalent to $k < \|\boldsymbol{a}\|_1 - a_i$, while $(k, k - a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \subseteq \mathcal{G}_{\boldsymbol{a}}$ is equivalent to $k \geq a_i$. Obviously, the same applies for $\mathcal{G}_{\boldsymbol{b}}$. Indeed, since $k \in \mathcal{V}_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta}$, one has $l \in \mathcal{V}_{\|\boldsymbol{b}\|_1}$ by Lemma 2. Knowing that $l \in \mathcal{V}_{\|\boldsymbol{b}\|_1}$, $(l, l + b_i) \in \mathcal{G}_{\boldsymbol{b}}$ is equivalent to $l < \|\boldsymbol{b}\|_1 - b_i$, while $(l, l - b_i) \in \mathcal{G}_{\boldsymbol{b}}$ is equivalent to $l \geq b_i$.

Case	$P \Rightarrow Q$	$P \wedge \neg Q$
(C1)	$(k < \|\boldsymbol{a}\|_1 - a_i) \Rightarrow (l < \|\boldsymbol{b}\|_1 - b_i)$	$(k < \|\boldsymbol{a}\|_1 - a_i) \wedge (l \geq \|\boldsymbol{b}\|_1 - b_i)$
(C2)	$(l \geq b_i) \Rightarrow (k \geq a_i)$	$(l \geq b_i) \wedge (k < a_i)$
(C3)	$(l < \|\boldsymbol{b}\|_1 - b_i) \Rightarrow (k < \|\boldsymbol{a}\|_1 - a_i)$	$(l < \|\boldsymbol{b}\|_1 - b_i) \wedge (k \geq \|\boldsymbol{a}\|_1 - a_i)$
(C4)	$(k \geq a_i) \Rightarrow (l \geq b_i)$	$(k \geq a_i) \wedge (l < b_i)$

Table 1: The implications to show are on the left. The converse propositions, which contradicts P_0, are on the right.

From the constraints given by $P \wedge \neg Q$ (see Table 1), we introduce two integral slack variables to have expressions for k and l. Then, we compute a bound for $k\|\boldsymbol{b}\|_1 - l\|\boldsymbol{a}\|_1$ that contradicts P_0.

In the first two cases, we have an upper bound for k and a lower one for l:

- $\epsilon_A \geq 1$ is such that $k = (\|\boldsymbol{a}\|_1 - a_i) - \epsilon_A$ in (C1) and $k = a_i - \epsilon_A$ in (C2),
- $\epsilon_B \geq 0$ is such that $l = (\|\boldsymbol{b}\|_1 - b_i) + \epsilon_B$ in (C1) and $l = b_i + \epsilon_B$ in (C2).

Now, substituting k and l by their values in $k\|\boldsymbol{b}\|_1 - l\|\boldsymbol{a}\|_1$, we get for (C1):

$$\begin{aligned} k\|\boldsymbol{b}\|_1 - l\|\boldsymbol{a}\|_1 = (&\|\boldsymbol{a}\|_1\|\boldsymbol{b}\|_1 - \|\boldsymbol{b}\|_1\|\boldsymbol{a}\|_1) \\ &\underbrace{+(b_i\|\boldsymbol{a}\|_1 - a_i\|\boldsymbol{b}\|_1)}_{\leq \mu = \|\boldsymbol{b}\|_1 - 1 + \delta} \underbrace{-\|\boldsymbol{b}\|_1\epsilon_A - \|\boldsymbol{a}\|_1\epsilon_B}_{\leq -\|\boldsymbol{b}\|_1} \leq \delta - 1, \end{aligned}$$

which raises a contradiction, because $k\|\boldsymbol{b}\|_1 - l\|\boldsymbol{a}\|_1 \geq \delta$. We thus conclude that $(k < \|\boldsymbol{a}\|_1 - a_i) \Rightarrow (l < \|\boldsymbol{b}\|_1 - b_i)$ and $(k, k + a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \Rightarrow (l, l + b_i) \in \mathcal{G}_{\boldsymbol{b}}$.

Likewise, for (C2),

$$k\|\boldsymbol{b}\|_1 - l\|\boldsymbol{a}\|_1 = \underbrace{-(b_i\|\boldsymbol{a}\|_1 - a_i\|\boldsymbol{b}\|_1)}_{\leq \mu = \|\boldsymbol{b}\|_1 - 1 + \delta} \underbrace{-\|\boldsymbol{b}\|_1\epsilon_A - \|\boldsymbol{a}\|_1\epsilon_B}_{\leq -\|\boldsymbol{b}\|_1} \leq \delta - 1,$$

which means that $(k, k - a_i) \in \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \Leftarrow (l, l - b_i) \in \mathcal{G}_{\boldsymbol{b}}$.

For the last two cases, (C3) and (C4), we can similarly conclude using ν. □

The following Corollary sums up the previous results:

Corollary 1. *If* $\big\|\|\boldsymbol{b}\|_1\boldsymbol{a} - \|\boldsymbol{a}\|_1\boldsymbol{b}\big\|_\infty \leq \min\left(|\|\boldsymbol{b}\|_1 - 1 + \delta|, |\|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1 - \delta|\right)$ *then* $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$.

In the next section, we introduce the concept of approximation, which will be linked later with Corollary 1.

4 Diophantine Approximation

Definition 4. *Let* $\boldsymbol{a} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ *such that* $\gcd(\boldsymbol{a}) = 1$. *A vector* $\boldsymbol{b} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ *is an* approximation *of vector* $\boldsymbol{a}$ *if and only if* $\boldsymbol{a} - \boldsymbol{b} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ *and*

$$\left\| \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|_1} - \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|_1} \right\|_\infty < \frac{1}{2\|\boldsymbol{b}\|_1}. \tag{3}$$

In addition, an approximation $\boldsymbol{b}$ *is* reduced *if and only if* $\gcd(\boldsymbol{b}) = 1$.

That definition is closely related to the simultaneous approximation of fractions. When the denominator is denoted by q, $\frac{1}{2q}$ is usually considered as a trivial bound, see [4, Section 5.2]. What is relatively uncommon here is that we consider only rationals and that the denominator are sums of numerators: $\|\boldsymbol{b}\|_1 = \sum_i b_i$.

Vectors for which the left-hand side of (3) is strictly less than the trivial bound are called "approximations", while the others do not deserve to be called "approximations". The rationale for the strict inequality sign is technical: it allows us to have an "if and only if" in Lemma 7 of the next section.

Multiplying both sides of (3) by the product $\|\boldsymbol{b}\|_1\|\boldsymbol{a}\|_1$, one obtains:

$$\big\|\|\boldsymbol{b}\|_1\boldsymbol{a} - \|\boldsymbol{a}\|_1\boldsymbol{b}\big\|_\infty < \frac{\|\boldsymbol{a}\|_1}{2}. \tag{4}$$

Note that the condition of Corollary 1 also involves the left-hand side of (4). An approximation $\boldsymbol{b}$ will turn out to be a good candidate to have $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$. However, it must be proved first that an approximation exists.

4.1 Existence of an Approximation

Lemma 4. *Let* $\boldsymbol{a} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ *be such that* $\gcd(\boldsymbol{a}) = 1$. *There exists at least one approximation of* $\boldsymbol{a}$ *in* $\mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ *if and only if* $\boldsymbol{a}$ *is not a permutation of* $(0, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$ *or* $(1, 1, 2)$.

Proof. ($\Longrightarrow$) One can check by enumeration that permutations of $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ and $(1,1,2)$ admit no approximation.

($\Longleftarrow$) Suppose that $\|\boldsymbol{a}\|_1 \leq 4$. A convenient permutation of $(0,0,1)$ is an approximation of any permutation of $(0,1,2)$ and $(0,1,3)$. Other cases are permutations of $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ and $(1,1,2)$, which are excluded.

Assume now that $\|\boldsymbol{a}\|_1 > 4$. Let us consider the open ball $\mathcal{B} := \{\boldsymbol{x} \in \mathbb{R}^3 \mid \|\boldsymbol{x}\|_\infty < 1/2\}$ and the images $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}$ under the orthogonal projection onto $(1,1,1)$ and onto the orthogonal complementary subspace respectively. Since both images have the same volume [6] and that the first one is trivially equal to $\sqrt{3}$, one has $vol(\mathcal{B}_2) = \sqrt{3}$.

Now, let us consider the open straight segment $\mathcal{S} := \{\lambda \boldsymbol{a} \mid \lambda \in (-1,1)\}$ and the image $\mathcal{S}_1$ of $\mathcal{S}$ under the orthogonal projection onto $(1,1,1)$. It is easy to see that $vol(\mathcal{S}_1) = \frac{2}{\sqrt{3}}\|\boldsymbol{a}\|_1$.

Finally, let us consider the dilation of $\mathcal{S}$ by $\mathcal{B}_2$, i.e., $\mathcal{D} := \mathcal{S} \oplus \mathcal{B}_2$. It is a symmetric and convex region. Note that the volume of $\mathcal{D} = \mathcal{S} \oplus \mathcal{B}_2$ is equal to the volume of $\mathcal{S}_1 \oplus \mathcal{B}_2$, because one can transform one to the other by a volume-invariant shearing. Thus,

$$vol(\mathcal{D}) = vol(\mathcal{B}_2)vol(\mathcal{S}_1) = \sqrt{3}\frac{2}{\sqrt{3}}\|\boldsymbol{a}\|_1 = 2\|\boldsymbol{a}\|_1 > 2 \cdot 4 = 8.$$

By Minkowski's theorem, since the volume of $\mathcal{D}$ is strictly greater than 2^3, $\mathcal{D}$ contains at least a non-zero integer point $\boldsymbol{b}$.

By definition of $\mathcal{D}$, we have $-\|\boldsymbol{a}\|_1 < \boldsymbol{b} \cdot (1,1,1) < \|\boldsymbol{a}\|_1$ and, due to the symmetry, one can assume without loss of generality that $0 < \boldsymbol{b} \cdot (1,1,1)$. Furthermore, $\frac{\|\boldsymbol{b}\|_1}{\|\boldsymbol{a}\|_1}\boldsymbol{a}$ is the projection of $\boldsymbol{b}$ onto $\mathcal{S}$ along projecting lines orthogonal to $(1,1,1)$. Since both $\boldsymbol{b}$ and its projection are in $\mathcal{D}$ by definition, we have

$$\left\| \frac{\|\boldsymbol{b}\|_1}{\|\boldsymbol{a}\|_1}\boldsymbol{a} - \boldsymbol{b} \right\|_\infty < \frac{1}{2} \quad \text{which implies} \quad \big\| \|\boldsymbol{b}\|_1 \boldsymbol{a} - \|\boldsymbol{a}\|_1 \boldsymbol{b} \big\|_\infty < \frac{\|\boldsymbol{a}\|_1}{2}. \qquad (5)$$

Consequently, $\boldsymbol{b}$ is an approximation of $\boldsymbol{a}$. □

4.2 Reduced Approximations

If an approximation exists, then a reduced one exists. That claim is rather obvious but crucial to link approximations with Christoffel subgraphs, because the latter are defined for vectors with coprime coordinates.

Lemma 5. *If* $\boldsymbol{b}$ *is an approximation of* $\boldsymbol{a} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ *such that* $\gcd(\boldsymbol{a}) = 1$*, then there exists a* reduced *approximation* $\boldsymbol{b}^\star$ *of* $\boldsymbol{a}$*.*

Proof. Let us define $\boldsymbol{b}^\star := \boldsymbol{b}/\gcd(\boldsymbol{b})$. It is clear that $\boldsymbol{b}^\star, \boldsymbol{a} - \boldsymbol{b}^\star \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ and

$$\max_i (|\|\boldsymbol{b}^\star\|_1 \boldsymbol{a}_i - \|\boldsymbol{a}\|_1 \boldsymbol{b}^\star_i|) \leq \max_i (|\|\boldsymbol{b}\|_1 \boldsymbol{a}_i - \|\boldsymbol{a}\|_1 \boldsymbol{b}_i|) < \frac{\|\boldsymbol{a}\|_1}{2},$$

which means that $\boldsymbol{b}^\star$ is a reduced approximation of $\boldsymbol{a}$. □

Furthermore, in order to be able to deal with the relative complement of a subgraph with respect to a Christoffel graph, the following result is useful.

Lemma 6. *Let $\boldsymbol{b}$ be a reduced approximation of $\boldsymbol{a} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ such that* $\gcd(\boldsymbol{a}) = 1$*. Then $\boldsymbol{a} - \boldsymbol{b}$ is also a reduced approximation of $\boldsymbol{a}$.*

Proof. Let $\boldsymbol{c} := \boldsymbol{a} - \boldsymbol{b}$. From the hypothesis, we have that $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{a} - \boldsymbol{c} = \boldsymbol{b} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$. On the other hand, one has for all $i \in \{1, 2, 3\}$,

$$\begin{aligned} \big|\|\boldsymbol{c}\|_1 \boldsymbol{a}_i - \|\boldsymbol{a}\|_1 \boldsymbol{c}_i\big| &= \big|(\|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1)\boldsymbol{a}_i - \|\boldsymbol{a}\|_1(\boldsymbol{a}_i - \boldsymbol{b}_i)\big| \\ &= \big|\|\boldsymbol{a}\|_1 \boldsymbol{a}_i - \|\boldsymbol{a}\|_1 \boldsymbol{a}_i - \|\boldsymbol{b}\|_1 \boldsymbol{a}_i + \|\boldsymbol{a}\|_1 \boldsymbol{b}_i\big| \\ &= \big| - (\|\boldsymbol{b}\|_1 \boldsymbol{a}_i - \|\boldsymbol{a}\|_1 \boldsymbol{b}_i)\big| \leq \big\|\|\boldsymbol{b}\|_1 \boldsymbol{a} - \|\boldsymbol{a}\|_1 \boldsymbol{b}\big\|_\infty < \tfrac{1}{2}\|\boldsymbol{a}\|_1, \end{aligned}$$

where we used (4). This means that $\boldsymbol{c}$ is a reduced approximations of $\boldsymbol{a}$. □

5 Existence of a Partition

In this section, we show that there always exist $\boldsymbol{b}$ and δ, such that $\mathcal{G}_{\boldsymbol{a}} = \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \cup \overline{\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}}$, $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$ and $\overline{\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}} \simeq \mathcal{G}_{(\boldsymbol{a}-\boldsymbol{b})}$. This result is based on the following:

Lemma 7. *Let $\boldsymbol{a} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ be such that* $\gcd(\boldsymbol{a}) = 1$*. Then $\boldsymbol{b}$ is an approximation of $\boldsymbol{a}$ if and only if there exists $\delta \in \{-\|\boldsymbol{b}\|_1 + 1, \ldots, \|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1\}$ such that*

$$\big\|\|\boldsymbol{b}\|_1 \boldsymbol{a} - \|\boldsymbol{a}\|_1 \boldsymbol{b}\big\|_\infty \leq \min\left(|\|\boldsymbol{b}\|_1 - 1 + \delta|, |\|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1 - \delta|\right).$$

Proof. For sake of shortness, let us set $q := \big\|\|\boldsymbol{b}\|_1 \boldsymbol{a} - \|\boldsymbol{a}\|_1 \boldsymbol{b}\big\|_\infty$. The goal is to search for δ such that $q \leq \min\left(|\|\boldsymbol{b}\|_1 - 1 + \delta|, |\|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1 - \delta|\right)$. However, $\delta \in \{-\|\boldsymbol{b}\|_1 + 1, \ldots, \|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1\}$ implies $0 \leq \|\boldsymbol{b}\|_1 - 1 + \delta$ and $0 \leq \|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1 - \delta$. Therefore, one can equivalently search for δ such that both of these conditions are true:

$$\begin{aligned} q &\leq \|\boldsymbol{b}\|_1 + \delta - 1 &&\Leftrightarrow q - \|\boldsymbol{b}\|_1 + 1 \leq \delta, \\ q &\leq \|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1 - \delta &&\Leftrightarrow \delta \leq \|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1 - q. \end{aligned}$$

We conclude that such δ exists if and only if the lower bound is less than the upper bound, i.e., if and only if

$$q - \|\boldsymbol{b}\|_1 + 1 \leq \|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1 - q \Leftrightarrow q \leq \frac{\|\boldsymbol{a}\|_1 - 1}{2},$$

which is true if and only if $\boldsymbol{b}$ is an approximation of $\boldsymbol{a}$. □

Based on all previous results, the main result of the paper follows.

Theorem 1. *Let $\boldsymbol{a} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ be such that* $\gcd(\boldsymbol{a}) = 1$*. If $\boldsymbol{a}$ is not a permutation of one of the vectors $(0, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$ and $(1, 1, 2)$, then there exist a reduced approximation $\boldsymbol{b} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ of $\boldsymbol{a}$ and there exists an offset δ such that*

$$\mathcal{G}_{\boldsymbol{a}} = \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \cup \overline{\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}}, \quad \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}} \quad \text{and} \quad \overline{\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}} \simeq \mathcal{G}_{(\boldsymbol{a}-\boldsymbol{b})}.$$

Proof. From Lemma 4 and Lemma 5, there exists a reduced approximation $\boldsymbol{b} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ of $\boldsymbol{a}$. There are two consequences. On the one hand, $\gcd(\boldsymbol{b}) = 1$, which means that one can define $\mathcal{G}_{\boldsymbol{b}}$. On the other hand, Lemma 7 and Corollary 1 together prove that there exists an offset δ such that $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$ (with bijection function $f_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta}$; see Definition 3).

Then, note that $\overline{\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}} = \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{c}\|_1,\gamma}$ by Lemma 1, where $\boldsymbol{c} := \boldsymbol{a} - \boldsymbol{b}$, $\gamma := -\delta + 1$. In addition, $\boldsymbol{c}$ is also a reduced approximation of $\boldsymbol{a}$ by Lemma 6. Since, $\gcd(\boldsymbol{c}) = 1$, one can define $\mathcal{G}_{\boldsymbol{c}}$.

For that subgraph, the condition of Corollary 1 writes

$$\left\| \|\boldsymbol{c}\|_1 \boldsymbol{a} - \|\boldsymbol{a}\|_1 \boldsymbol{c} \right\|_\infty \leq \min\left(|\|\boldsymbol{c}\|_1 - 1 + \gamma|, |\|\boldsymbol{a}\|_1 - \|\boldsymbol{c}\|_1 - \gamma|\right). \tag{6}$$

Remind that $\left\| \|\boldsymbol{c}\|_1 \boldsymbol{a} - \|\boldsymbol{a}\|_1 \boldsymbol{c} \right\|_\infty = \left\| \|\boldsymbol{b}\|_1 \boldsymbol{a} - \|\boldsymbol{a}\|_1 \boldsymbol{b} \right\|_\infty$ (see the proof of Lemma 6). In addition, note that

(i) $\|\boldsymbol{c}\|_1 - 1 + \gamma = (\|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1) - 1 + (-\delta + 1) = \|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1 - \delta$,
(ii) $\|\boldsymbol{a}\|_1 - \|\boldsymbol{c}\|_1 - \gamma = \|\boldsymbol{a}\|_1 - (\|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1) - (-\delta + 1) = \|\boldsymbol{b}\|_1 - 1 + \delta$.

As a consequence, (6) is equivalent to

$$\left\| \|\boldsymbol{b}\|_1 \boldsymbol{a} - \|\boldsymbol{a}\|_1 \boldsymbol{b} \right\|_\infty \leq \min\left(|\|\boldsymbol{b}\|_1 - 1 + \delta|, |\|\boldsymbol{a}\|_1 - \|\boldsymbol{b}\|_1 - \delta|\right),$$

which is true by Lemma 7 and one can conclude as above that $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{c}\|_1,\gamma} \simeq \mathcal{G}_{\boldsymbol{c}}$ □

Remark 1. We have not tried to identify a set Σ such that $f_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta}$ makes $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$ for all $\boldsymbol{b} \in \Sigma$ and *only them*. Indeed, in order to have an "only if" part in Theorem 1, it would be necessary to show that not only $f_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta}$, but all bijection functions fail to make $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$ if $\boldsymbol{b} \notin \Sigma$. That would require other results than those provided above, which are mainly based on $f_{\|\boldsymbol{a}\|_1}^{\|\boldsymbol{b}\|_1,\delta}$.

Remark 2. A few vectors are excluded from Theorem 1 because they do not have an approximation. They are, up to a permutation of the coordinates, $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ and $(1,1,2)$. It is easy to see that the Christoffel graphs defined from those vectors all contain specific local configurations (see Fig. 6). Thus, they cannot contain subgraphs corresponding to a Christoffel graph defined from a shorter vector. That is why it is not possible to have a result similar to Theorem 1 for them.

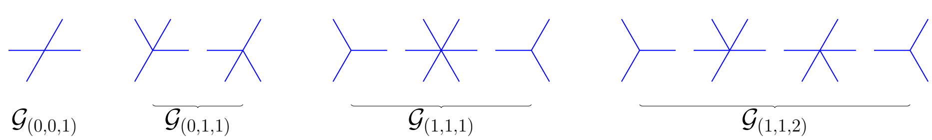


Fig. 6: Undecomposable Christoffel graphs.

Remark 3. As shown in Fig. 5 (a-b), there may be several possible $\boldsymbol{b}$ and δ.

6 Conclusion and Perspectives

We have shown in Theorem 1 that for all $\boldsymbol{a} \in \mathbb{N}^3 \setminus \{\boldsymbol{0}\}$ such that $\gcd(\boldsymbol{a}) = 1$ and which is not a permutation of $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ or $(1,1,2)$, there are an approximation $\boldsymbol{b}$ and an offset δ such that $\mathcal{G}_{\boldsymbol{a}} = \mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \cup \overline{\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}}$, $\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta} \simeq \mathcal{G}_{\boldsymbol{b}}$ and $\overline{\mathcal{S}_{\boldsymbol{a}}^{\|\boldsymbol{b}\|_1,\delta}} \simeq \mathcal{G}_{(\boldsymbol{a}-\boldsymbol{b})}$. In other words, $\mathcal{P}_{\boldsymbol{a}}$ can be partitioned into two parts having respectively the combinatorial structure of $\mathcal{P}_{\boldsymbol{b}}$ and $\mathcal{P}_{(\boldsymbol{a}-\boldsymbol{b})}$. Compare Fig. 7 with Fig. 3.

A short-term perspective is to focus on the geometrical aspects of the partition. It seems that a rational discrete plane is decomposed into parallel strips. See Fig. 3 (a). Are the subsets in each strip connected? What are the thickness and direction of the strips? Another perspective is to efficiently compute one or all approximations for which a decomposition is possible. Finally, we are also interested in the practical application of these studies in discrete geometry.

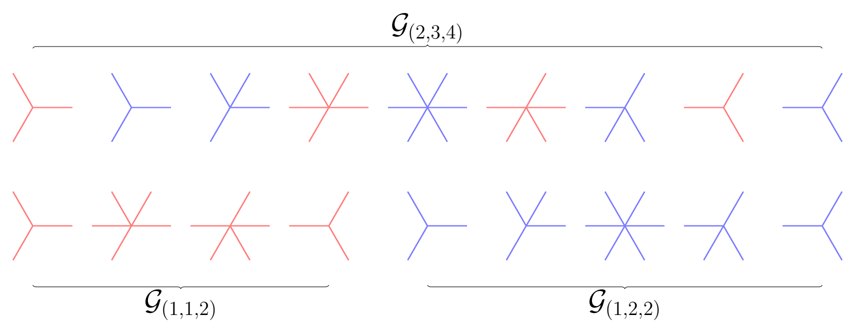


Fig. 7: $\mathcal{G}_{(2,3,4)} = \mathcal{S}_{(2,3,4)}^{4,0} \cup \mathcal{S}_{(2,3,4)}^{5,1}$, $\mathcal{S}_{(2,3,4)}^{4,0} \simeq \mathcal{G}_{(1,1,2)}$, $\mathcal{S}_{(2,3,4)}^{5,1} \simeq \mathcal{G}_{(1,2,2)}$.

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