Characterization of bijective discretized rotations by Gaussian integers

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Abstract

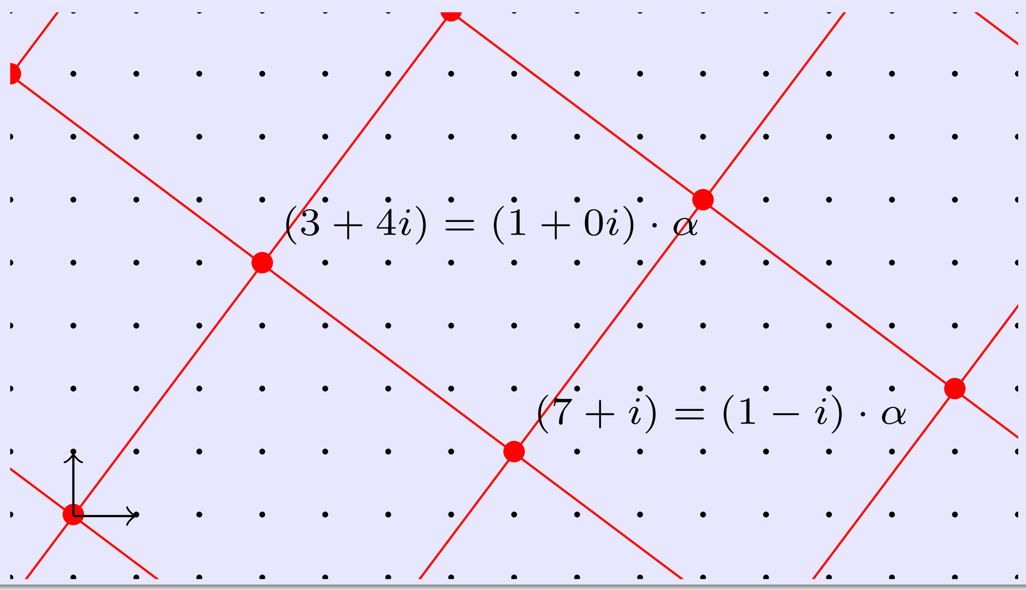
A discretized rotation is the composition of an Euclidean rotation with a rounding operation. It is well known that not all discretized rotations are bijective: e.g. two distinct points may have the same image by a given discretized rotation. Nevertheless, for a certain subset of rotation angles, the discretized rotations are bijective. In the regular square grid, the bijective discretized rotations have been fully characterized by [Nouvel and Rémila (IWCIA'2005)]. We provide a simple proof that uses the arithmetical properties of Gaussian integers.

Gaussian integers

Gaussian integers are the set $\mathbb{Z}[i] := \{u + vi \mid u, v \in \mathbb{Z}\}$, where $i^2 = -1$. Within the complex plane $\mathbb{C}$, they constitute the 2-dimensional integer lattice $\mathbb{Z}^2$.

- An *addition* by κ maps $\mathbb{Z}^2$ to $\mathbb{Z}^2 + (u, v)$ (*translation*).
- The *norm* of $\kappa = u + vi$ is defined by $N\kappa := \kappa\bar{\kappa} = u^2 + v^2$.
- A *multiplication* by κ maps $\mathbb{Z}^2$ to $\mathbb{Z}(u, v) + \mathbb{Z}(-v, u)$ (*rotation* by angle θ such that $\tan(\theta) = v/u$ and *scaling* by $\sqrt{N\kappa}$).

Gaussian integers $\equiv$ integers (Euclidean division, factorization into primes, gcd...).



Discrete rotations

Let $\alpha \in \mathbb{Z}[i]$ be equal to $a + bi$.

An Euclidean rotation is defined as follows:

$$r_\alpha : \mathbb{Z}[i] \to \mathbb{C}$$
$$\forall \kappa \in \mathbb{Z}[i],\ r_\alpha(\kappa) = \frac{\kappa\cdot\alpha}{\sqrt{N\alpha}}.$$

$\forall z \in \mathbb{C}$, $[z]$ is the unique Gaussian integer s.t. $z \in D([z])$, the discretization cell of $[z]$.
A discrete rotation is defined as follows:

$$[r_\alpha] : \mathbb{Z}[i] \to \mathbb{Z}[i]$$
$$\forall \kappa \in \mathbb{Z}[i],\ [r_\alpha](\kappa) = \left[\frac{\kappa\cdot\alpha}{\sqrt{N\alpha}}\right].$$

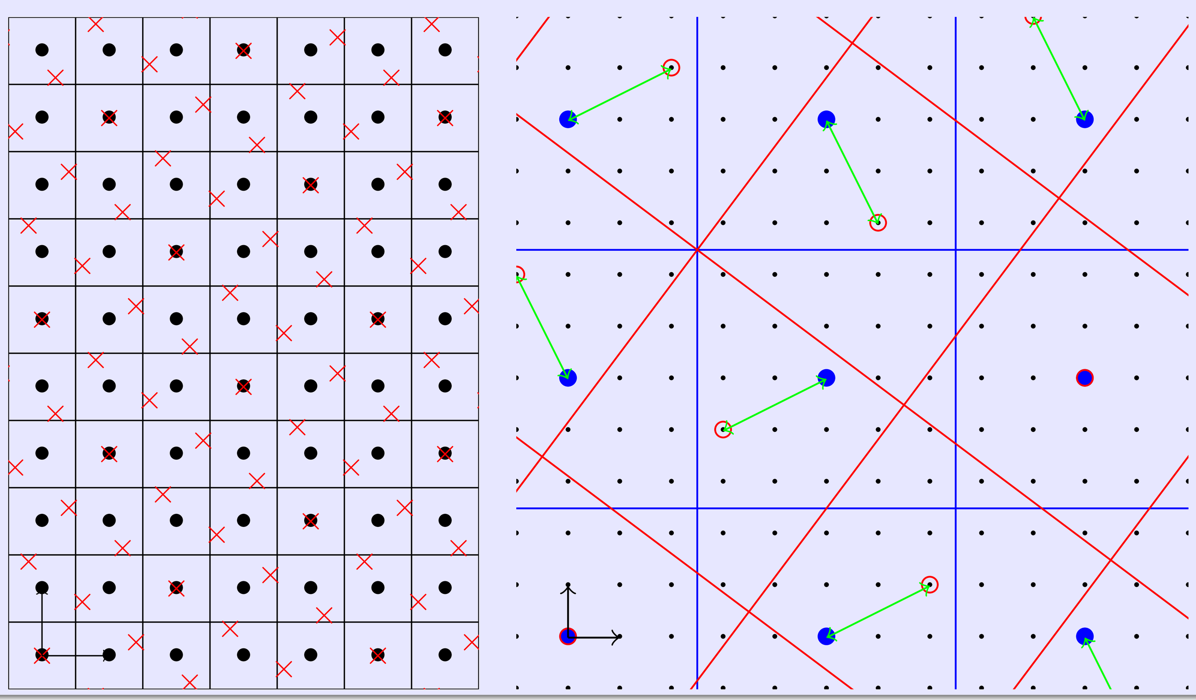
We focus on Pythagorean rotation angles, i.e. angles such that $\sqrt{N\alpha} = c \in \mathbb{Z}$.
For any primitive solution of $N\alpha = c^2$, there exists a unique $\gamma := p + qi$ such that $0 < q < p$, $\gcd(p, q) = 1$, $p - q$ odd and

$$\alpha = \gamma\cdot\gamma\,,$$
$$c = \gamma\cdot\bar{\gamma}\,.$$

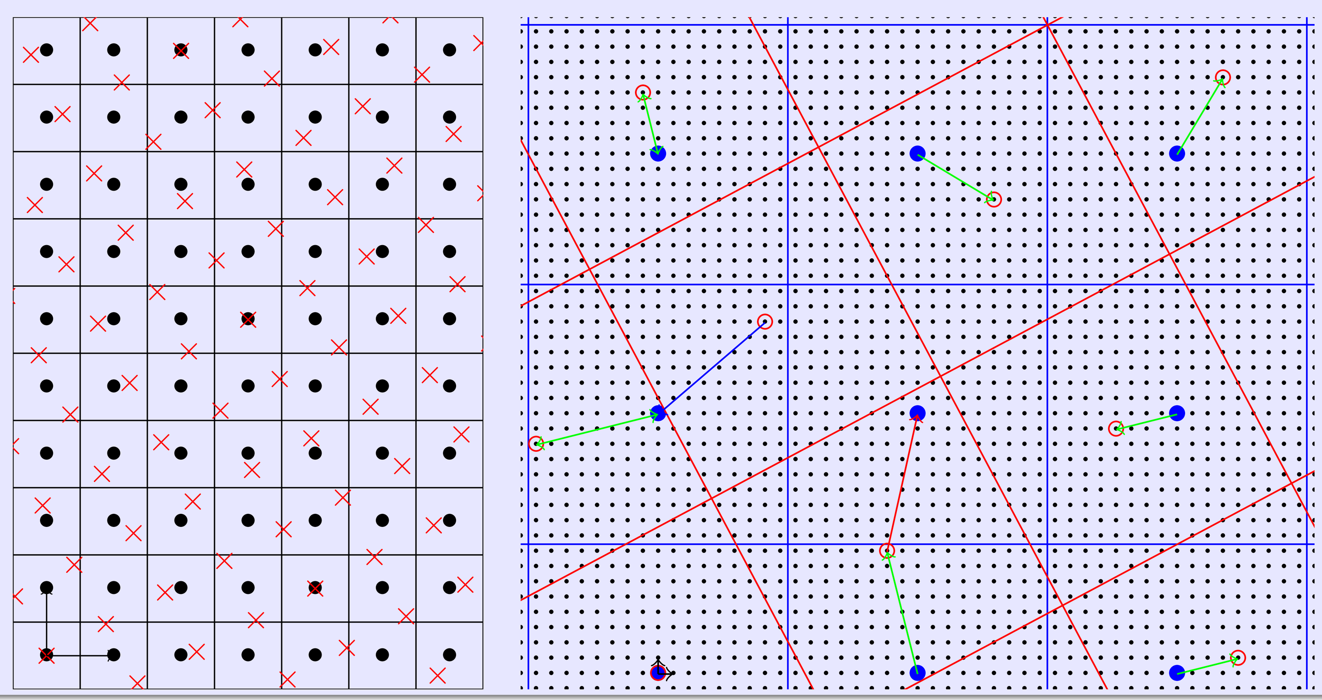
Characterization of bijective rotations [Nouvel and Rémila (IWCIA'2005)]

The discretized rotation $[r_\alpha]$ is bijective iff $\gamma = (k+1) + ki$, $k \in \mathbb{Z}^+$.

Bijective case: $\gamma = 2 + i$, $\alpha 3 + 4i$, $c = 5$.



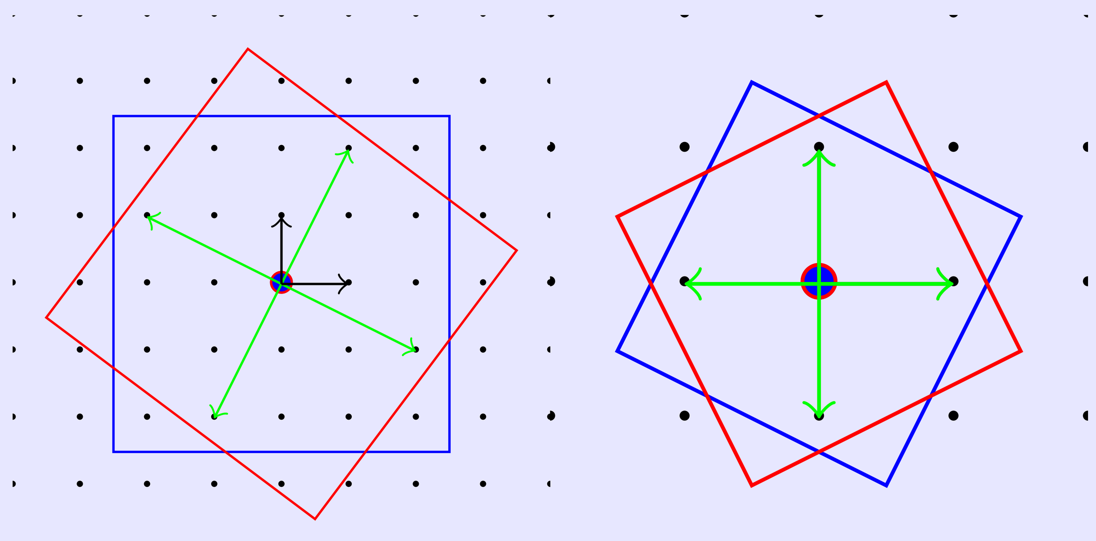
Not bijective case: $\gamma = 4 + i$, $\alpha = 15 + 8i$, $c = 17$.



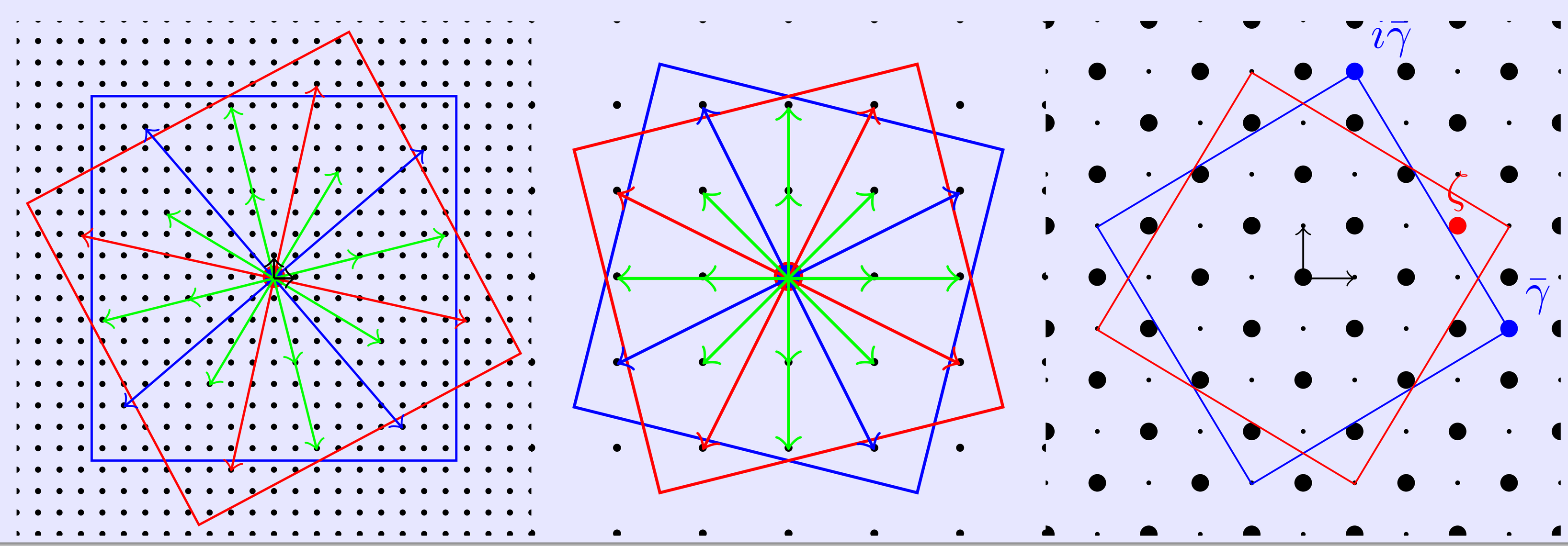
Idea of the proof that uses arithmetical properties of Gaussian integers

- We do not divide $\kappa\cdot\alpha$ by c, but we consider the result with respect to the discretization cells of the scaled lattice $c\mathbb{Z}^2$
- $\forall \kappa, \lambda \in \mathbb{Z}[i]$, we focus on the difference $s_{\alpha,c}(\kappa,\lambda) := \kappa\cdot\alpha - \lambda\cdot c$, called *remainder*, such that $\kappa\cdot\alpha \in cD(\lambda)$ and $\lambda\cdot c \in \alpha D(\kappa)$, ie. such that $s_{\alpha,c}(\kappa,\lambda) \in cD(0)$ and $s_{\alpha,c}(\kappa,\lambda) \in \alpha D(0)$.
- $\forall \kappa, \lambda \in \mathbb{Z}[i]$, $s_{\alpha,c}(\kappa,\lambda) = \gamma s_{\gamma,\bar{\gamma}}(\kappa,\lambda)$ because $\gcd(\alpha, c) = \gamma$.
- Let $S_{\bar{\gamma}}$ (resp. S_γ) be equal to $\{\rho \in \mathbb{Z}[i] \mid \rho \in \bar{\gamma}D(0)\}$ (resp. $S\{\rho \in \mathbb{Z}[i] \mid \rho \in \gamma D(0)\}$). The discretized rotation $[r_\alpha]$ is bijective iff $S_{\bar{\gamma}} = S_\gamma$.

Bijective case ($\gamma = 2 + i$)



Not bijective case ($\gamma = 4 + i$)



Future works

Other Algebraic integers could be of interest for Digital Geometry purposes.
E.g. Eisenstein integers and discretized rotations on the *hexagonal lattice*.