Plane-probing algorithms for the analysis of digital surfaces

Tristan Roussillon

Université de Lyon, INSA Lyon, LIRIS, France

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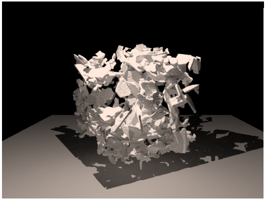
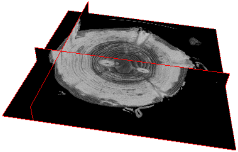
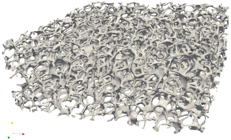
Data



(a)



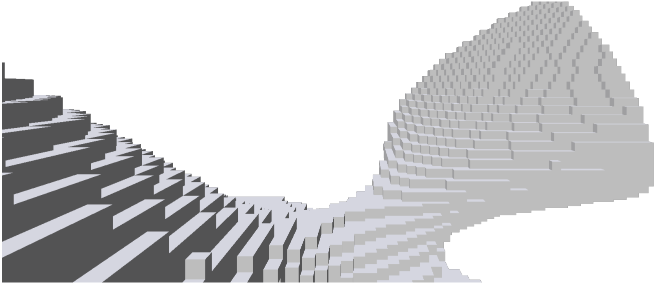
(b)



(c)

voxel sets in 3d digital images

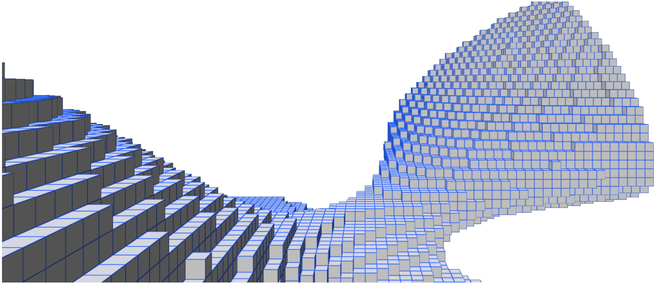
Digital surfaces



pros/cons

+ efficient spatial data structures
+ set operations (union, intersection, . . .)
+ integer-only, exact computations
+ . . .
− poor geometry

Digital surfaces



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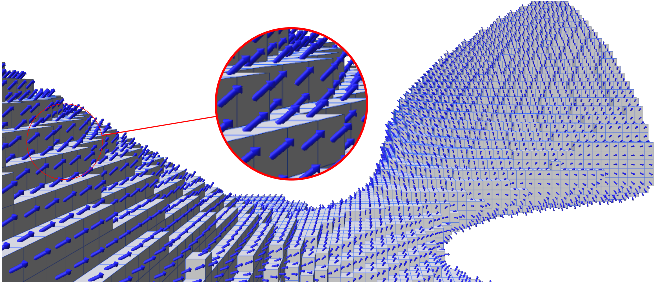
Analysis of digital surfaces

- enhance the geometry by estimating normal vectors

⇒ applications: measurements, deformation for simulation or tracking, surface fairing, rendering...



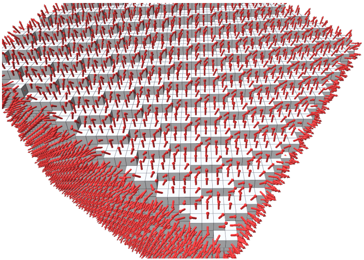
A lot of methods

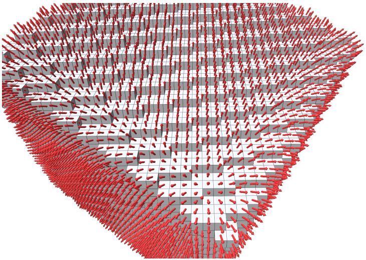
- fitting,
- Voronoi diagram,
- integral invariants,
- convolution,
- energy minimization,
- probabilistic approaches,
- ...

Flaw

Existing methods are not quite satisfactory

- parameter required ($\approx$ width of a neighborhood)
- that parameter is hard to pick
  - get decent estimates in flat/smooth parts
  - preserve sharp features





Challenge

Desiderata

- parameter-free method
- theoretical guarantees
  - exact on flat parts
  - converge on smooth parts as resolution increases

Key idea

- bound neighborhoods by their thickness instead of their width
- digitized planes have a thickness bounded by a small constant

Plane-probing algorithms

Definition

Given a digitized plane **P** and a starting point $\mathbf{p} \in \mathbf{P}$, a plane-probing algorithm computes the normal vector of **P** by sparsely probing it with the predicate “is $\mathbf{x} \in \mathbf{P}$?”.

H and R	[LPR2017] J-O. L., X. P., T. R. Two Plane-Probing Algorithms for the Computation of the Normal Vector to a Digital Plane. *J. Math. Imaging Vis.*, 59(1):23–39, 2017.
R^1	[LR2019] T. R., J-O. L., An efficient and quasi linear worst-case time algorithm for digital plane recognition, *DGCI'19*, LNCS, vol. 11414, p.380–393, 2019.
PH, PR, PR^1	[LMR2020] J-O. L., J. M., T. R. An Optimized Framework for Plane-Probing Algorithms, *J. Math. Imaging Vis.*, 62(5):718–736, 2020.

Implemented in DGtal (`dgtal.org`)

Outline

Context and motivation

Plane-probing algorithms

Generalized Euclidean algorithm

Delaunay triangulation

Generalization

Application to digital surfaces

One of the oldest algorithms

Euclidean algorithm

Given a couple of integers,

- subtract the smaller from the larger one, and repeat
- until both numbers are equal.

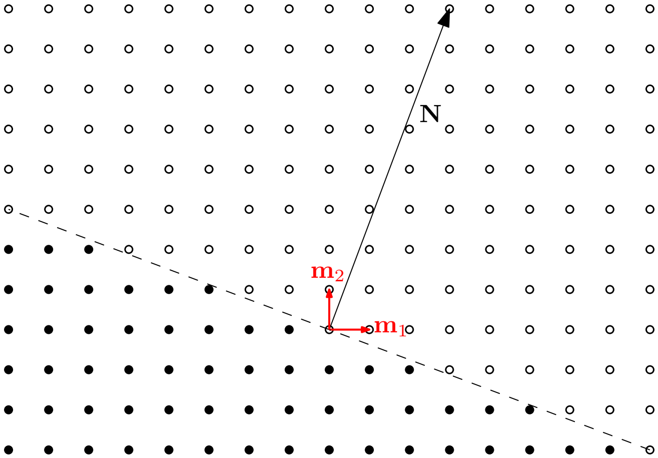
Example

step	0	1	2	3	4
a	3	3	3	1	1
b	8	5	2	2	1

we focus on the sequence of subtractions, assume $\gcd(a, b) = 1$

One geometrical interpretation of the Euclidean algorithm

$$\mathbf{m}_1 = (1,0), \quad \mathbf{m}_1 \cdot \mathbf{N} = a = 3$$
$$\mathbf{m}_2 = (0,1), \quad \mathbf{m}_2 \cdot \mathbf{N} = b = 8$$



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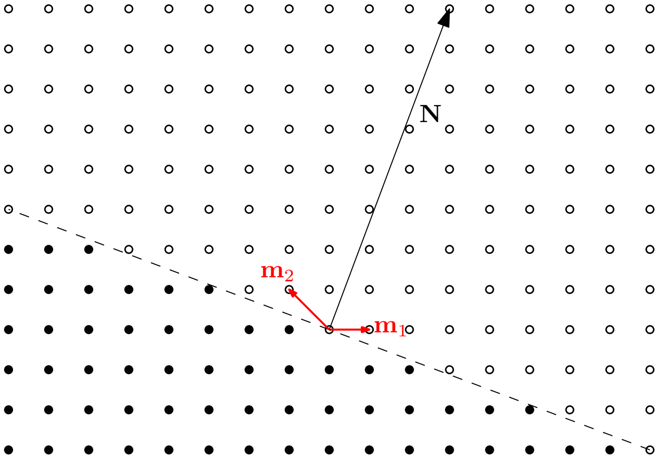
$$\mathbf{m}_2 = (-1,1), \quad \mathbf{m}_2 \cdot \mathbf{N} = b = 5$$



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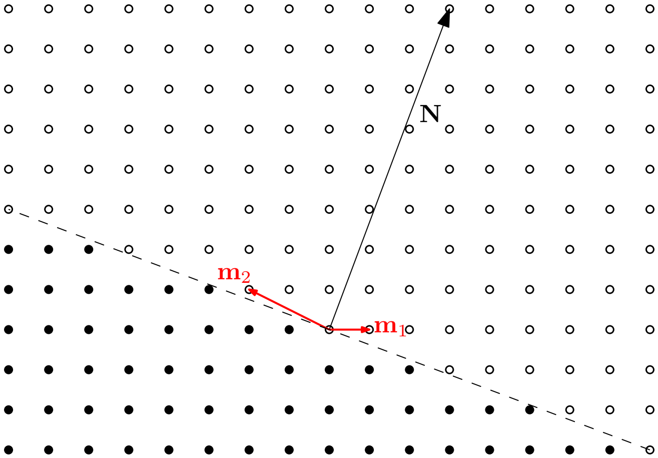
$$\mathbf{m}_1 = (1, 0), \qquad \mathbf{m}_1 \cdot \mathbf{N} = a = 3$$
$$\mathbf{m}_2 = (-2, 1), \quad \mathbf{m}_2 \cdot \mathbf{N} = b = 2$$



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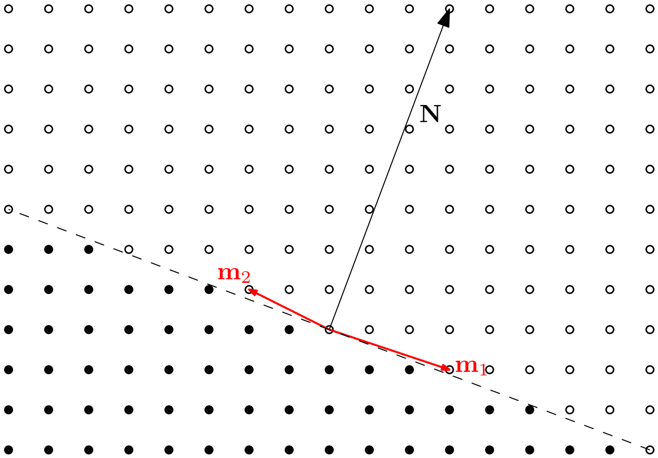

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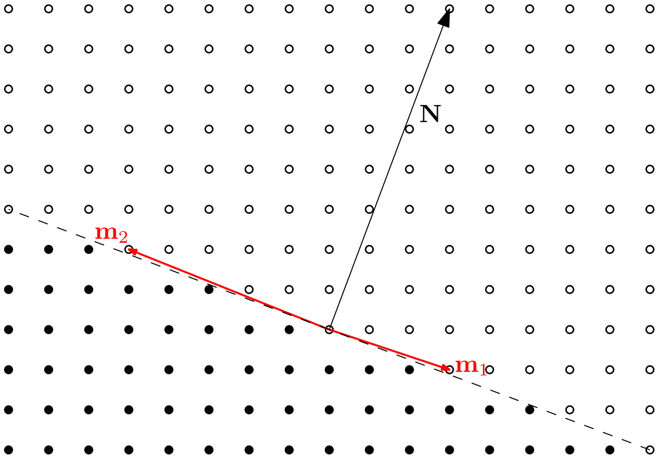
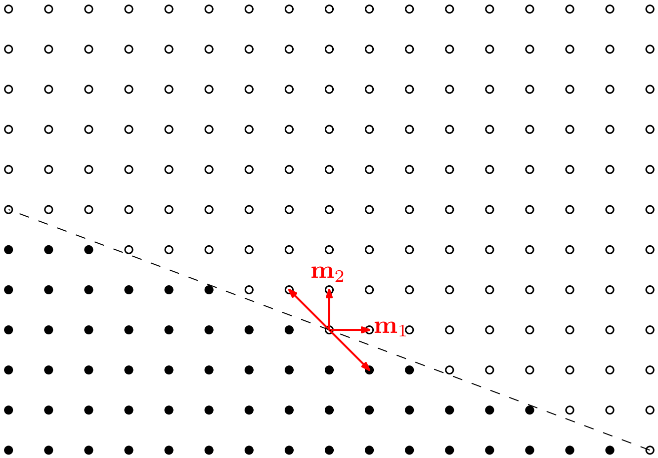
$$\mathbf{m}_1 = (3, -1), \quad \mathbf{m}_1 \cdot \mathbf{N} = a = 1$$
$$\mathbf{m}_2 = (-5, 2), \quad \mathbf{m}_2 \cdot \mathbf{N} = b = 1$$



An algorithm to compute **N**

N is unknown, but a predicate IsBlack is given

IsBlack($\mathbf{m}_1 - \mathbf{m}_2$)? IsBlack($\mathbf{m}_2 - \mathbf{m}_1$)?



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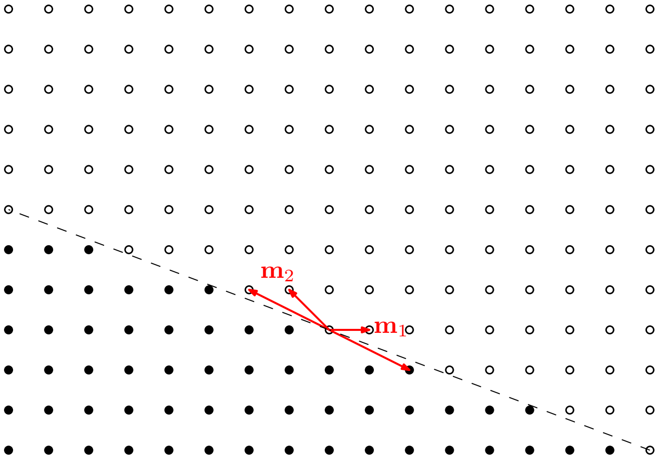

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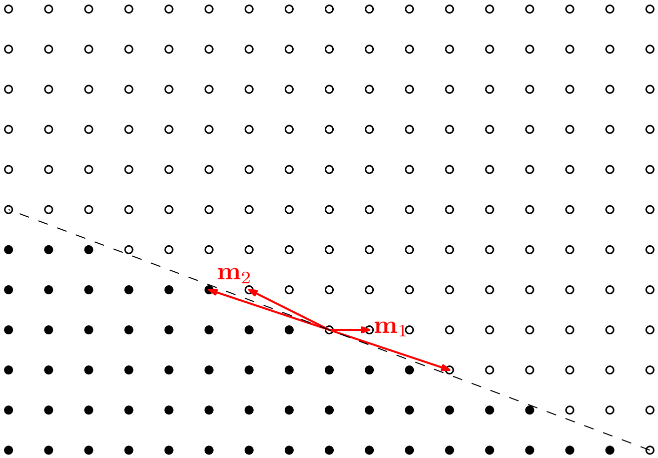

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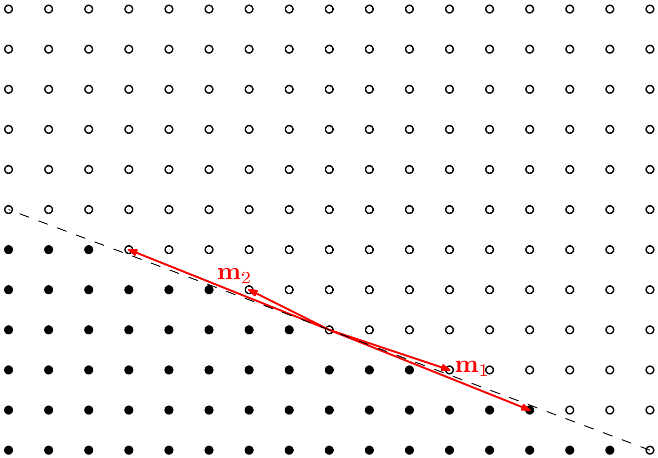

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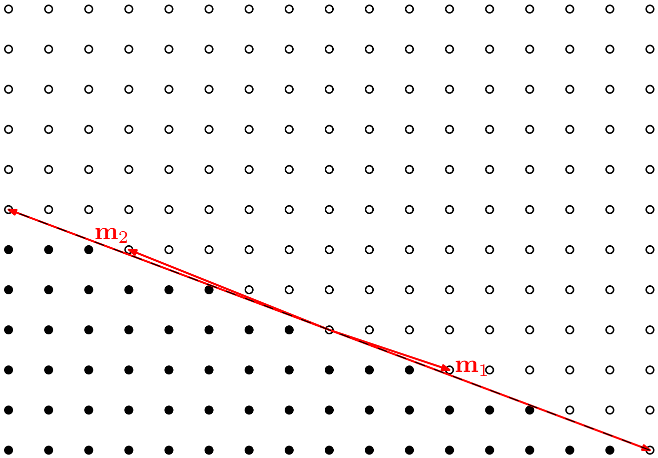

Extension to 3d

No unique extension to the Euclidean algorithm!

Assuming $0 \leq a \leq b \leq c$:

- *Brun*: $(a, b, c) \to (a, b, c - b)$;
- *Selmer*: $(a, b, c) \to (a, b, c - a)$;
- *Farey*: $(a, b, c) \to (a, b - a, c)$;
- *Fully-Subtractive*: $(a, b, c) \to (a, b - a, c - a)$;
- *Poincaré*: $(a, b, c) \to (a, b - a, c - b)$.
- ...

Note: the same operation is done at each step

A class of generalized Euclidean algorithms

Given three positive numbers (a, b, c), with $\gcd(a, b, c) = 1$,

- while they are not all equal to 1,
- subtract from a number $x \in \{a, b, c\}$ a strictly smaller number $y \in \{a, b, c\}$, $y < x$.

Example

$$\begin{aligned} \mathbf{m}_1 &= (1, 0, 0), \quad \mathbf{m}_1 \cdot \mathbf{N} = a = 1 \\ \mathbf{m}_2 &= (0, 1, 0), \quad \mathbf{m}_2 \cdot \mathbf{N} = b = 2 \\ \mathbf{m}_3 &= (0, 0, 1), \quad \mathbf{m}_3 \cdot \mathbf{N} = c = 3 \end{aligned}$$

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Example

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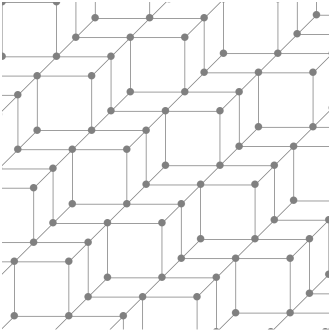
\end{aligned}$$

Digital plane

Let $\mathbf{N} \in \mathbb{Z}^3$ whose components (a, b, c) are coprime integers s.t. $0 < a \leq b \leq c$,

$$\mathbf{P_N} := \{\mathbf{x} \in \mathbb{Z}^3 \mid 0 \leq \mathbf{x} \cdot \mathbf{N} < \|\mathbf{N}\|_1\}$$



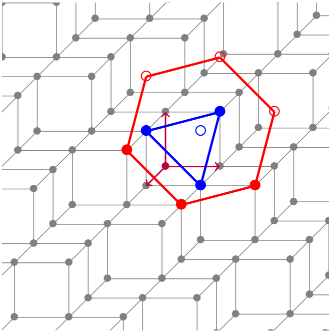
Interpretation of a generalized Euclidean algorithm

Internals

- $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3) := (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, $\mathbf{q} := (1, 1, 1) \notin \mathbf{P}_N$

$\Rightarrow$ triangle $(\mathbf{q} - \mathbf{m}_1, \mathbf{q} - \mathbf{m}_2, \mathbf{q} - \mathbf{m}_3)$

$\Rightarrow$ hexagon $\{\mathbf{q} + \mathbf{m}_i - \mathbf{m}_j \mid i, j \in \{1, 2, 3\}, i \neq j\}$



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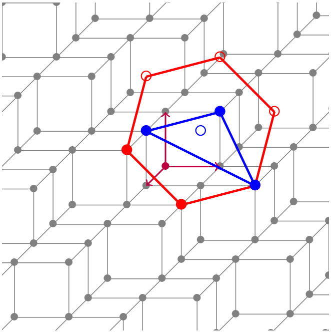

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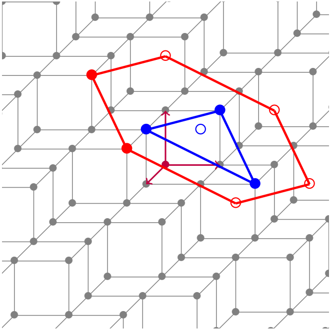

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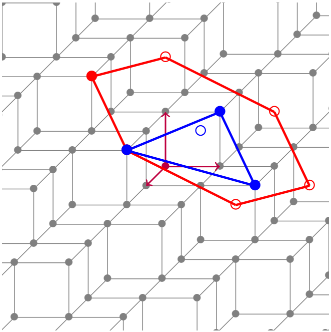

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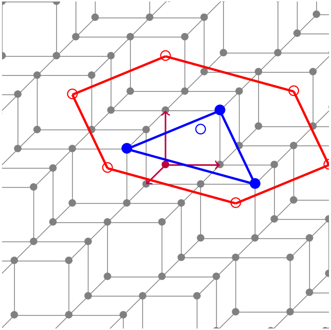


$\Rightarrow$ a plane-probing algorithm

$$\Pi := \{\mathbf{P_N} \mid \mathbf{N} \in \mathbb{Z}^3 \setminus \mathbf{0}\}$$

Input

- $\mathbf{P} \in \Pi$ described by the predicate InPlane: "is $\mathbf{x} \in \mathbf{P}$?"
- a starting point $\mathbf{p}$ s.t. InPlane($\mathbf{p}$), $\mathbf{q} := \mathbf{p} + (1, 1, 1)$

Main trick

- Assume $\mathbf{p} \cdot \mathbf{N} = 0$ ($\Rightarrow \mathbf{q} \cdot \mathbf{N} = \|\mathbf{N}\|_1$), where $\mathbf{N}$, the normal of $\mathbf{P}$
- InPlane($\mathbf{x}$) $\Leftrightarrow (\mathbf{x} - \mathbf{q}) \cdot \mathbf{N} < 0$.

Properties of generalized Euclidean algorithms

At each step

P1 $\mathbf{p}$ and $\mathbf{q}$ both project into triangle $(\mathbf{q} - \mathbf{m}_1, \mathbf{q} - \mathbf{m}_2, \mathbf{q} - \mathbf{m}_3)$ along $(1, 1, 1)$

P2 matrix $\mathbf{M} := [\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3]$ is unimodular, i.e. $\det(\mathbf{M}) = 1$

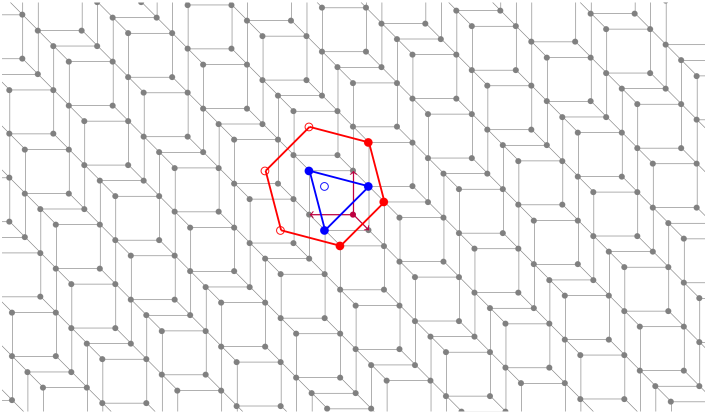
Termination

- number of steps $\leq \|\mathbf{N}\|_1 - 3$ (6 calls to InPlane per step)
- at the end, if $\mathbf{p} \cdot \mathbf{N} = 0$ ($\Rightarrow \mathbf{q} \cdot \mathbf{N} = \|\mathbf{N}\|_1$)
  $\forall k \in \{1, 2, 3\},\ \mathbf{m}_k \cdot \mathbf{N} = 1$
  $\Rightarrow$ the normal of triangle $(\mathbf{q} - \mathbf{m}_1, \mathbf{q} - \mathbf{m}_2, \mathbf{q} - \mathbf{m}_3)$ is $\mathbf{N}$

whichever the subtraction we choose

Example

Digital plane of normal $(5, 2, 3)$



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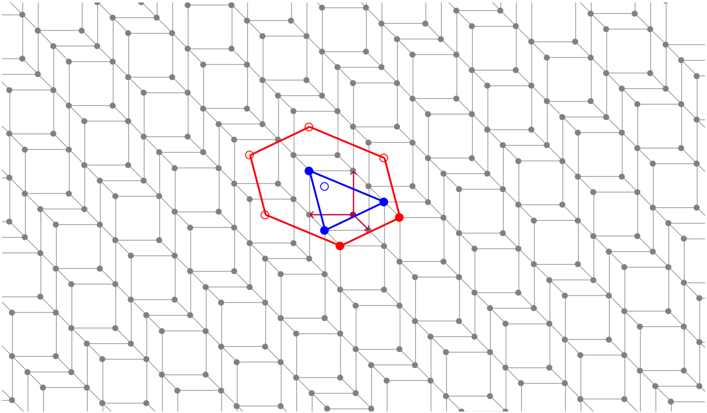

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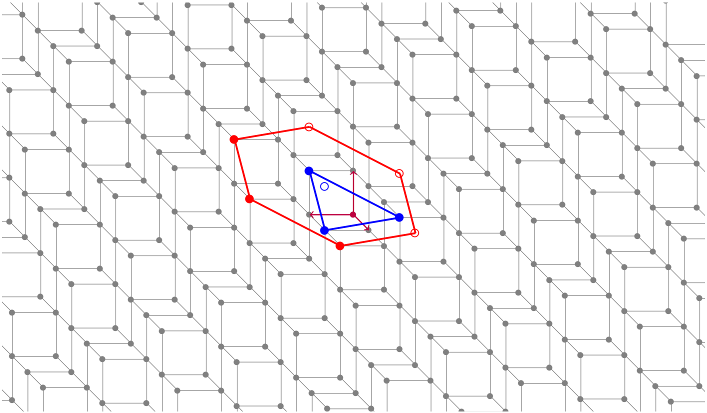

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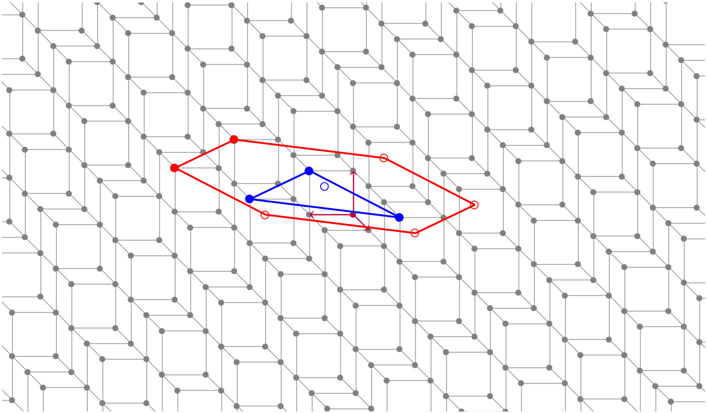

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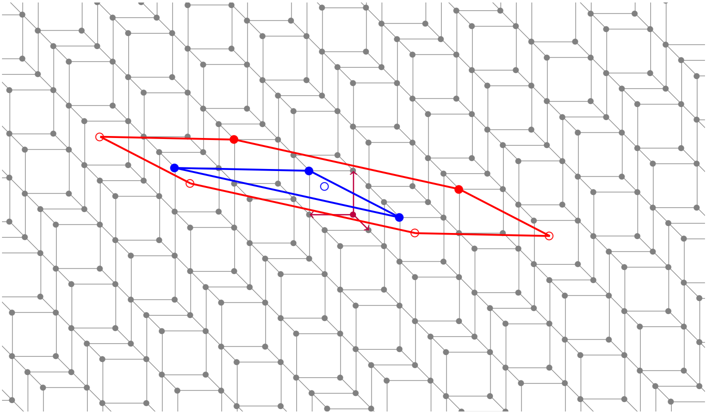

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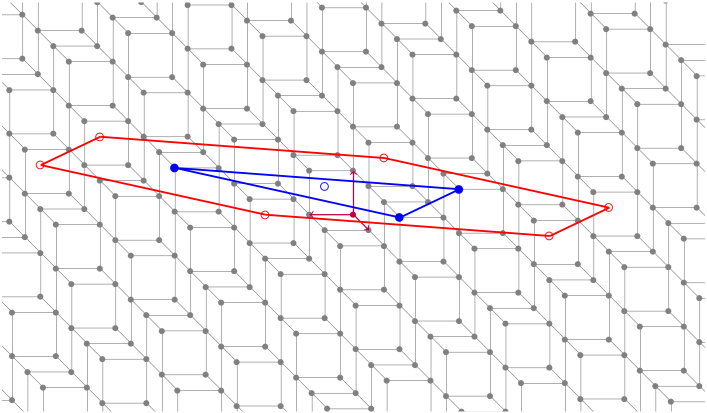
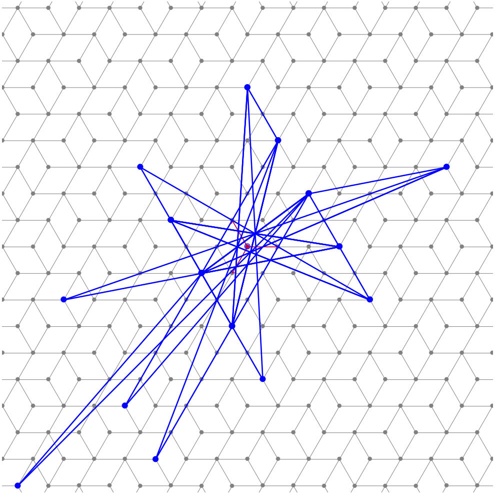


All possible final triangles

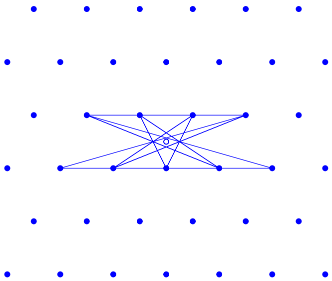
Digital plane of normal $(2, 3, 5)$



About final triangles

- vertices $\in \Lambda := \{\mathbf{x} \in \mathbb{Z}^3 \mid \mathbf{x} \cdot \mathbf{N} = \|\mathbf{N}\|_1 - 1\}$
- do not contain any other point of Λ (P2)
- projection of $\mathbf{p}$ along $(1,1,1)$ (P1)

Digital plane of normal $(2,2,5)$



Towards a selection criterion

- The Delaunay triangulation of Λ gives acute triangles
- **p** projects into one of them (if no co-circularity)

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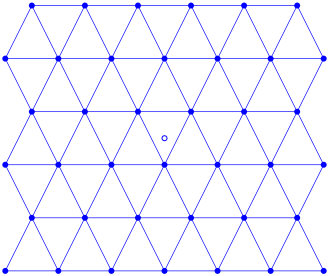

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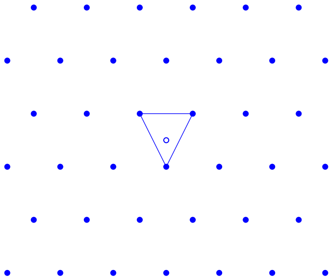
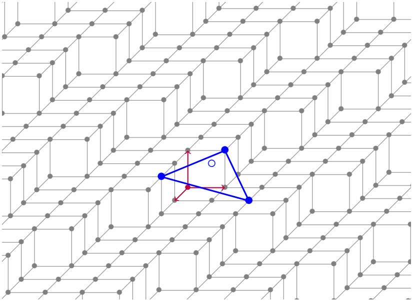


Algorithm H (candidates in an hexagon)

At each step:

- consider a candidate set S
- filter S through InPlane
- pick a *closest* point $s^\star$: the circumsphere of $T \cup s^\star$ doesn't contain any other
- update T with this point



The last triangle is very often acute, but not always

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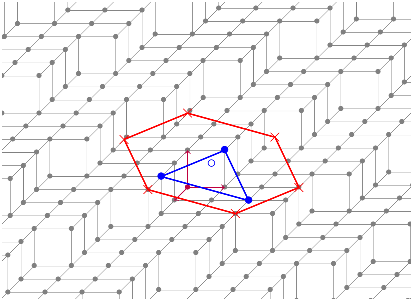

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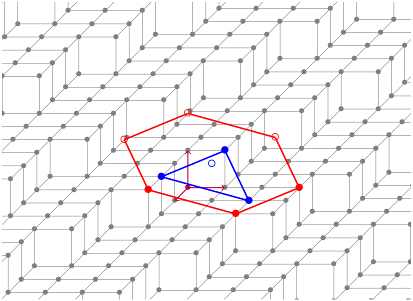

The last triangle is very often acute, but not always

Algorithm H (candidates in an hexagon)

At each step:

- consider a candidate set S
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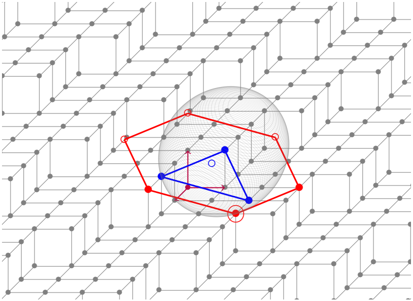

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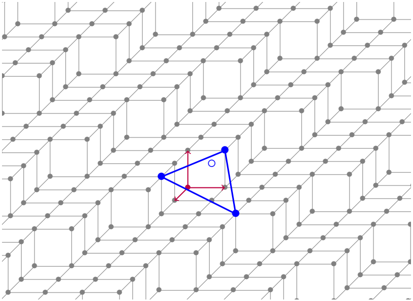

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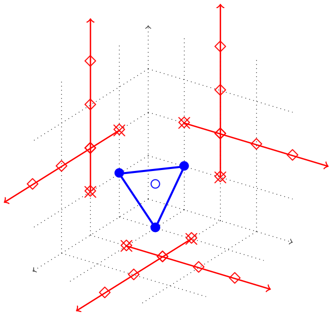
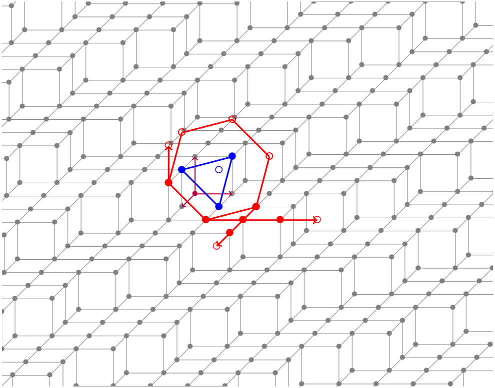
Algorithm R (candidates along rays)



- same algorithm as before, only S differs
- S is infinite but the filtering by InPlane gives a finite point set
- $O(\|\mathbf{N}\|_1)$ steps, $O(\log(\|\mathbf{N}\|_1))$ calls to InPlane per step
- the last triangle is always acute

Example

Digital plane of normal $(2, 3, 9)$



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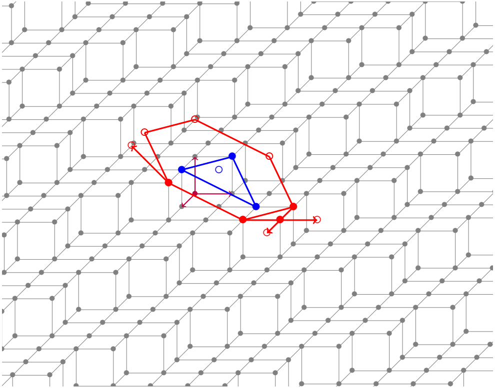

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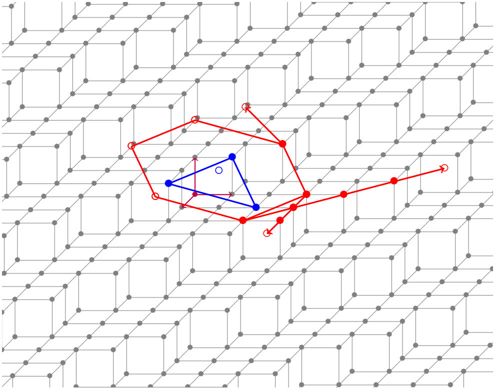

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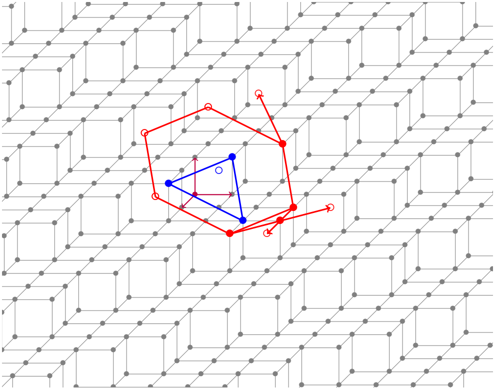

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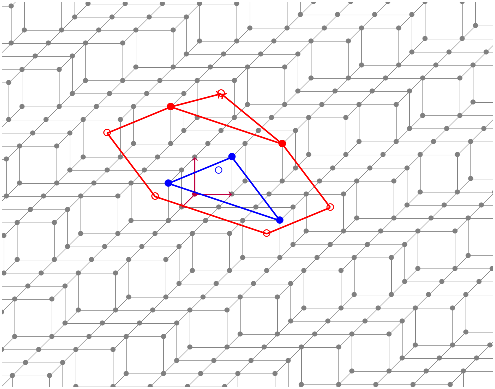

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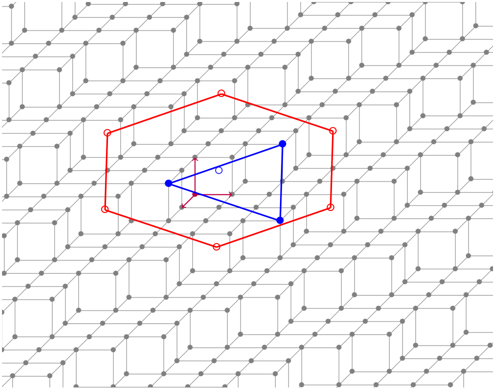


Algorithm R^1

Features

- has the same output as R
- but $O(\|\mathbf{N}\|_1)$ calls to InPlane instead of $O(\|\mathbf{N}\|_1 \log \|\mathbf{N}\|_1)$

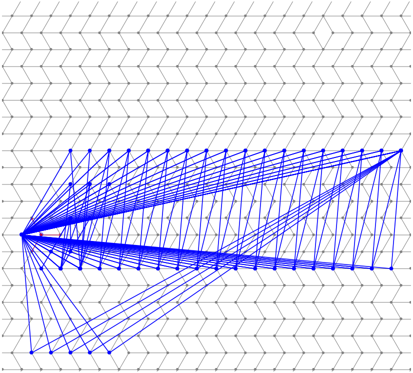
How?

1. local probing: 6 rays $\rightarrow$ at most 2 rays and 1 point
2. geometrical study: 2 rays $\rightarrow$ 1 ray and 1 point
3. efficient algorithm: 1 ray and 1 point $\rightarrow$ a *closest* point

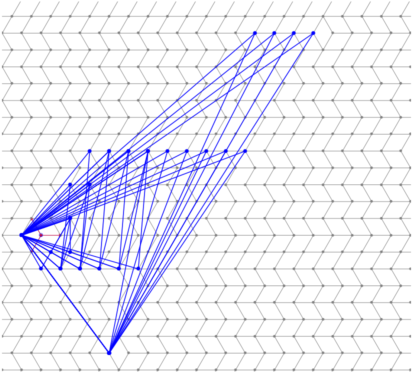
Example

Digital plane of normal $(67, 1, 91)$

H

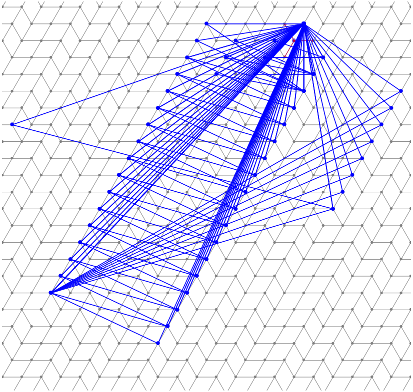


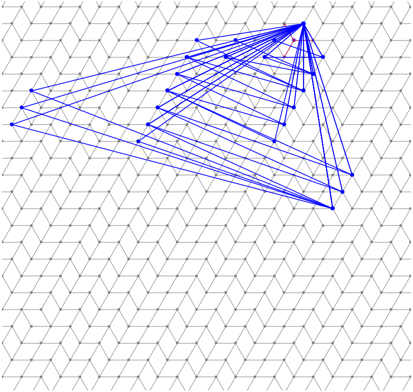
R/R^1



Example

Digital plane of normal $(1, 73, 100)$





Recap

Main features

- $\mathbf{N}$ from a point $\mathbf{p}$ s.t. $\mathbf{p} \cdot \mathbf{N} = 0$
- by sparse and local computations:
  - $\mathbf{p}$ projects into all triangles
  - with R and R^1, the current triangle is acute every two steps, always acute at the end
- $O(\|\mathbf{N}\|_1)$ calls to InPlane with H and R^1, $O(\|\mathbf{N}\|_1 \log(\|\mathbf{N}\|_1))$ with R

Drawbacks

1. do not retrieve **N** from any point
2. do not retrieve all triangles of the lattice Λ

Problem #1: starting from any point

Input

- $\mathbf{P}$ of normal $\mathbf{N}$
- InPlane: "is $\mathbf{x} \in \mathbf{P}$?"

Equivalence used so far

- assume $\mathbf{q} \cdot \mathbf{N} = \|\mathbf{N}\|_1$
- $\text{InPlane}(\mathbf{x}) \Leftrightarrow (\mathbf{x} - \mathbf{q}) \cdot \mathbf{N} < 0$

Generalized equivalence

- assume $\mathbf{q} \cdot \mathbf{N} \geq \|\mathbf{N}\|_1$
- $\exists l \in \mathbb{N}$ s.t. $\text{InPlane}(\mathbf{q} + l(\mathbf{x} - \mathbf{q})) \Leftrightarrow (\mathbf{x} - \mathbf{q}) \cdot \mathbf{N} < 0.$

Predicate NotAbove

Data: InPlane, $\mathbf{q}$ and an integer $L \geq 2\|\mathbf{N}\|_1$

Input: A point $\mathbf{x} \in \mathbb{Z}^3$ s.t. $\mathbf{q} \cdot \mathbf{N} - \|\mathbf{N}\|_1 \leq \mathbf{x} \cdot \mathbf{N}$

Output: True iff $(\mathbf{x} - \mathbf{q}) \cdot \mathbf{N} < 0$ in $O(\log(L))$ calls to InPlane

$\mathbf{u} \leftarrow \mathbf{x} - \mathbf{q}$; `// direction`

$l \leftarrow 1$;

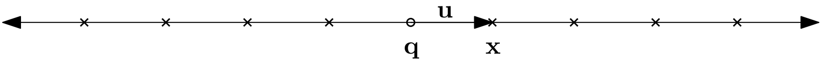
while $l < L$ **do**

  if InPlane($\mathbf{q} + l\mathbf{u}$) **then** **return** *True* ;

  if InPlane($\mathbf{q} - l\mathbf{u}$) **then** **return** *False* ;

  $l \leftarrow 2l$;

return *False*;



it is enough to use NotAbove instead of InPlane

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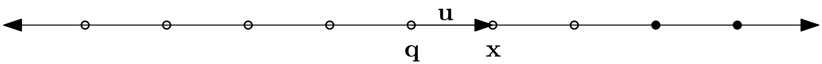
```
u ← x − q ; // direction
l ← 1;
while l < L do
    if InPlane(q + lu) then return True ;
    if InPlane(q − lu) then return False ;
    l ← 2l;
return False;
```



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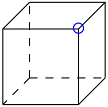
Problem #2: retrieving all triangles

Triangle → Parallelepiped

- top point $\mathbf{q}$
- upper triangle $(\mathbf{q} - \mathbf{m}_1, \mathbf{q} - \mathbf{m}_2, \mathbf{q} - \mathbf{m}_3)$
- lower triangle $(\mathbf{q} - \mathbf{m}_2 - \mathbf{m}_3, \mathbf{q} - \mathbf{m}_3 - \mathbf{m}_1, \mathbf{q} - \mathbf{m}_1 - \mathbf{m}_2)$
- bottom point $\mathbf{q} - \sum_k \mathbf{m}_k$



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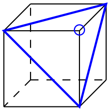

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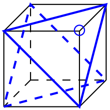

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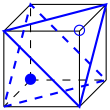

Staying close to the digital plane

Update rule

- when the parallelepiped has less than 4 vertices in **P**,
  - $\Rightarrow$ the lower triangle is updated (top moves, not bottom)
- otherwise
  - $\Rightarrow$ the upper triangle is updated (bottom moves, not top)
- invariant: at least one point in **P** (bottom), one not (top)

Generalized versions of H, R and R^1

For each $X \in \{H, R, R^1\}$, PX uses a parallelepiped and the above update rule with NotAbove instead of InPlane.

Recap

Main features

- $\mathbf{N}$ from any point $\mathbf{p}$ such that InPlane($\mathbf{p}$),
- all triangles of the lattice $\Lambda = \{\mathbf{x} \in \mathbb{Z}^3 \mid \mathbf{x} \cdot \mathbf{N} = \|\mathbf{N}\|_1 - 1\}$
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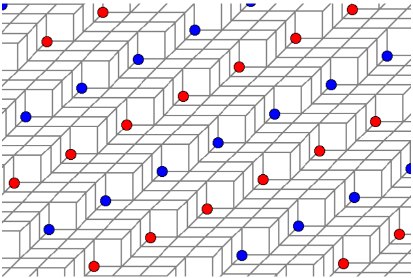

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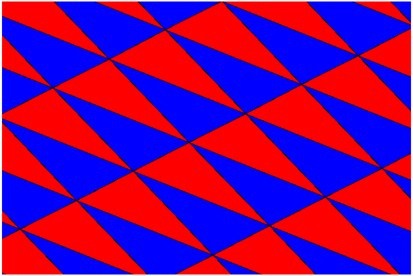


Outline

Context and motivation

Plane-probing algorithms

- Generalized Euclidean algorithm
- Delaunay triangulation
- Generalization

Application to digital surfaces

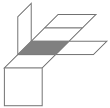
A similar algorithm for a digital surface S

Input

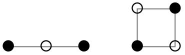
- a predicate InSurface : $\mathbf{x} \in S$?
- a starting square face s in S

Additional constraints

- find an origin and a basis from s



- stop if non-planar configurations (parallelepiped/hexagon/rays)



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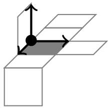

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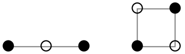
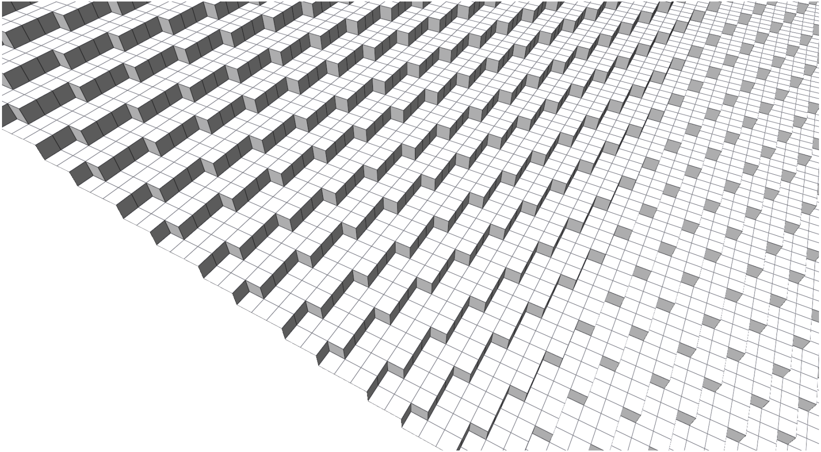


Example: flat parts and sharp edges



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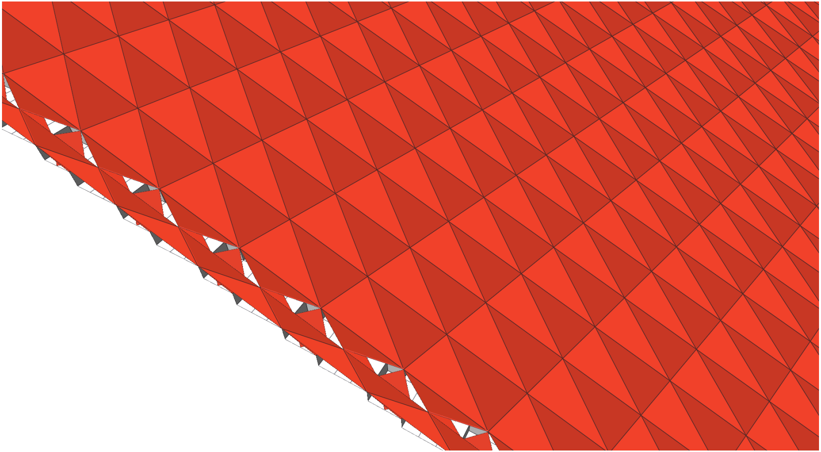

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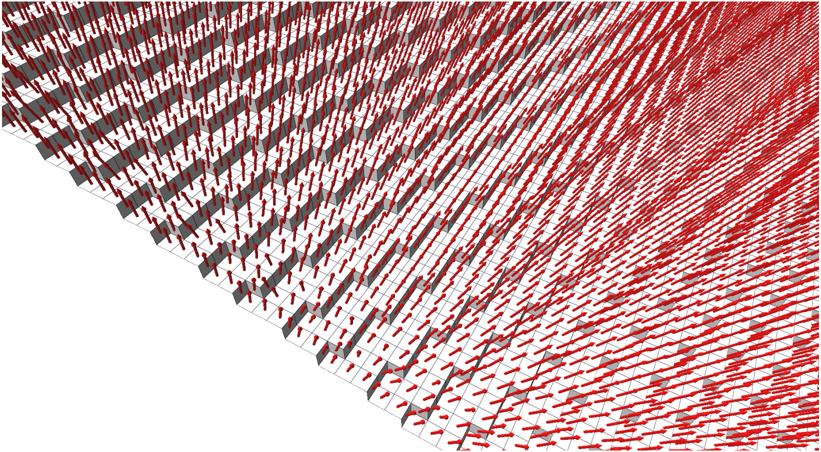

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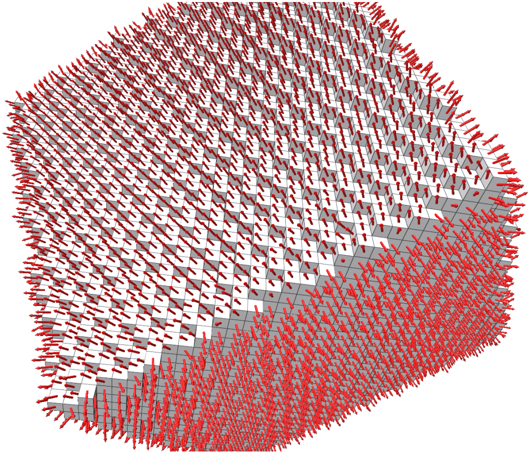
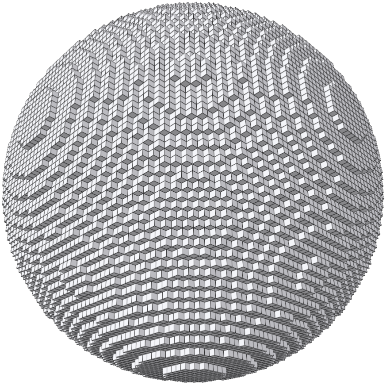


Example: convex shapes



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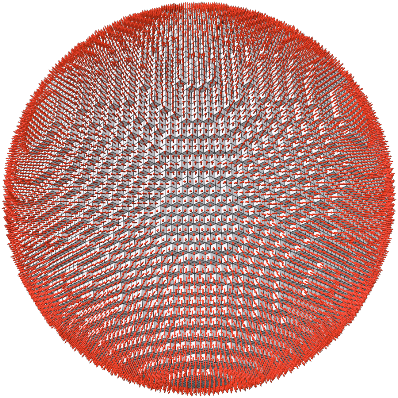

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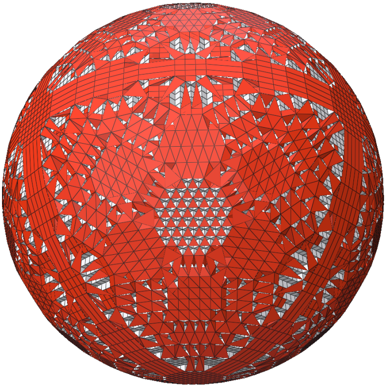
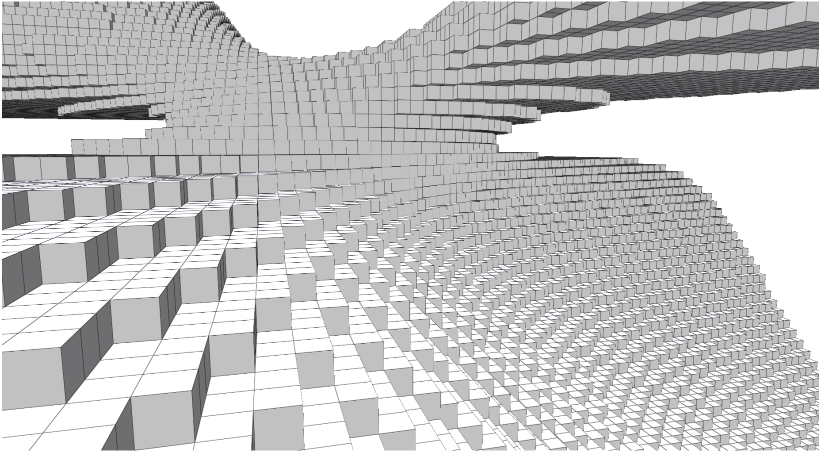


Example: not convex shapes



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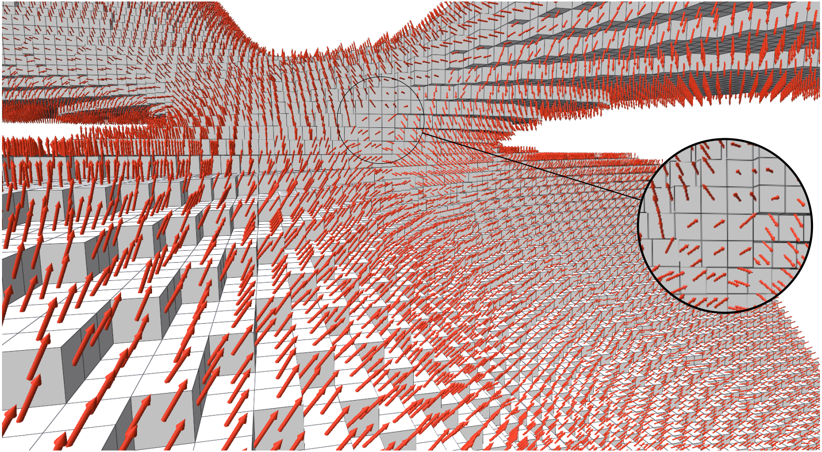


Perspectives

Digital planes

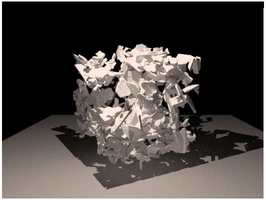
- What piece of digital plane is enough to find **N**?

Digital surfaces

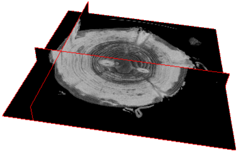
- try all candidates, obtuse triangles may be interesting
- perform a dense probing to process non-convex parts
- estimator: multigrid convergence, experimental comparison
- reconstruction: find of way of gluing triangles together

The end

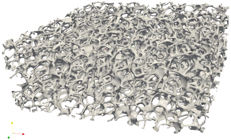
My first answer:



(a) snow



(b) wood



(c) foam