# Subtraction Arc-Kayles

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Université de Lyon

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# Arc-Kayles

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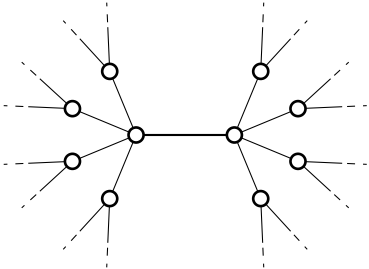
**Arc-Kayles : Schaefer, 1978**

- This game is played on a graph $G = (V, E)$.
- At each turn the current player chooses an edge.
- Its endpoints are deleted.
- The game ends when there is no more edges to play.



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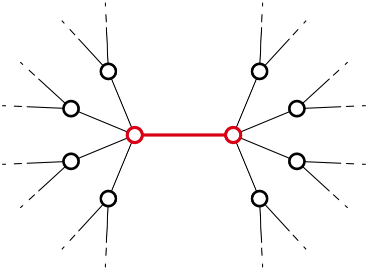

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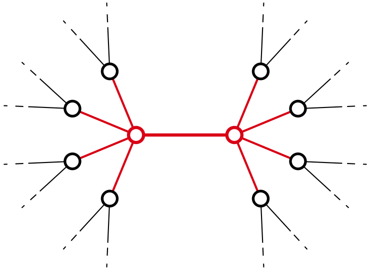

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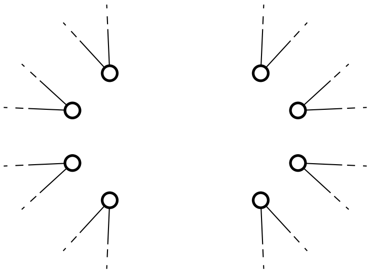

# Arc-Kayles

- Solved on paths (Guy and Smith, 1956)
- Solved on equimatchable graphs, cycles, wheels and generalized star graphs with three branches (Huggan and Stevens, 2016)
- FPT (Lampis and Mitsou, 2014)

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- This game is played on a graph $G = (V, E)$ with a weight function $\omega : V \to \mathbb{N}$.
- At each step the current player choose an edge.
- The weight of both endpoints is decreased by 1.
- Vertices with weight zero are removed.
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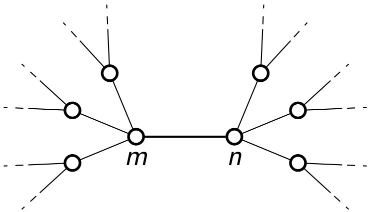

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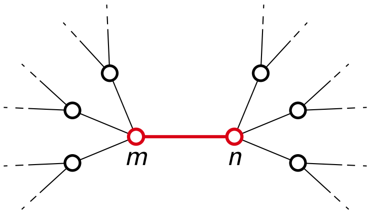

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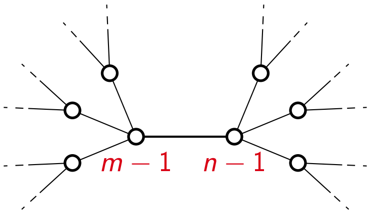
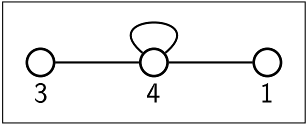


# Example



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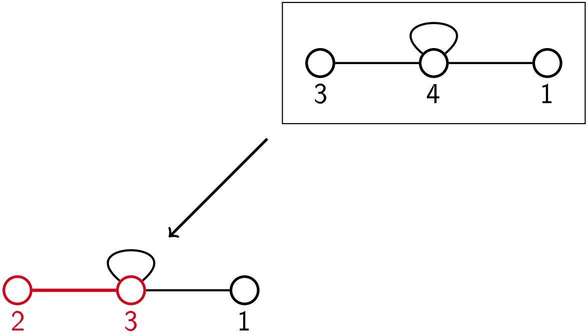

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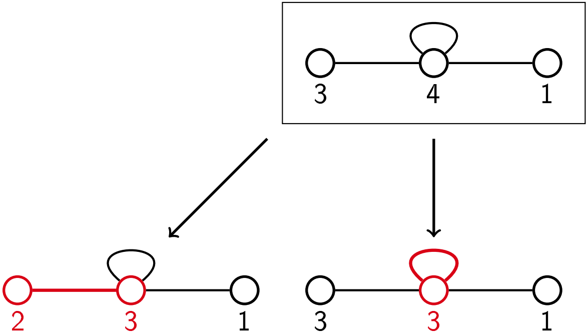

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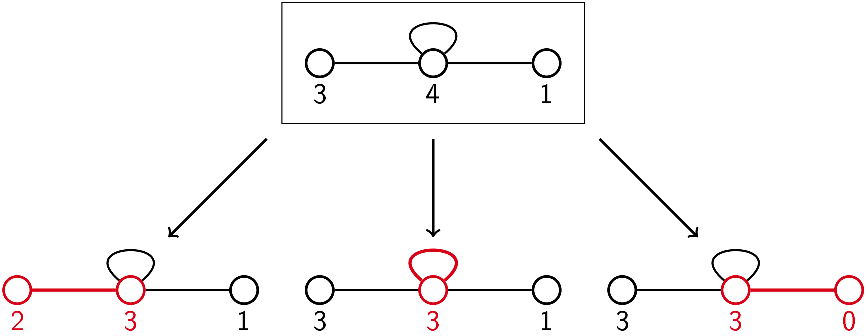

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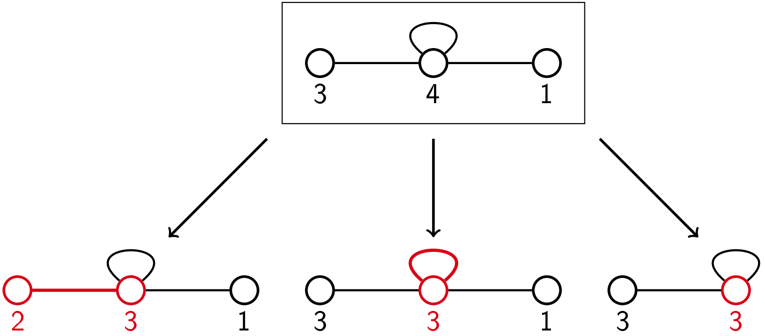
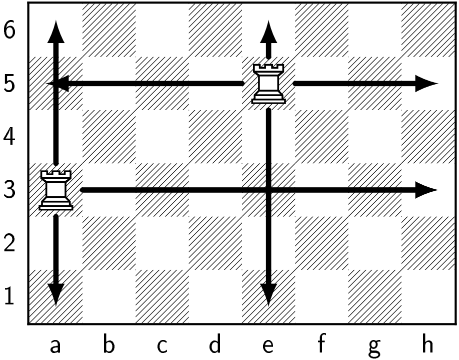


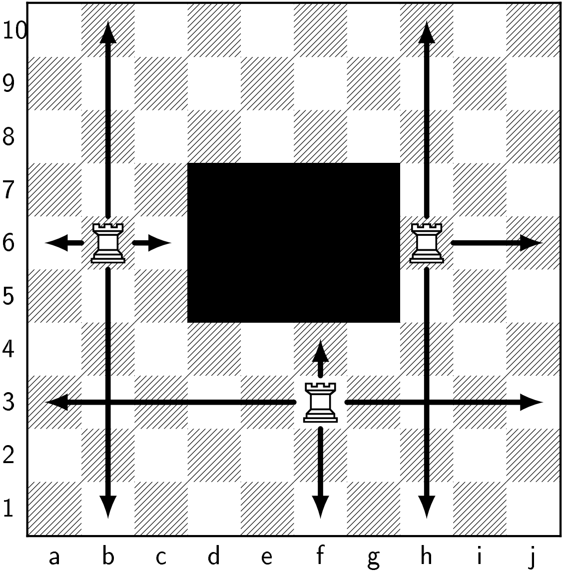
# Non-attacking rooks in a holed chessboard

# The game




Inspired by the non-attacking queens game(Noon and Van Brummelen, 2006)

# The game



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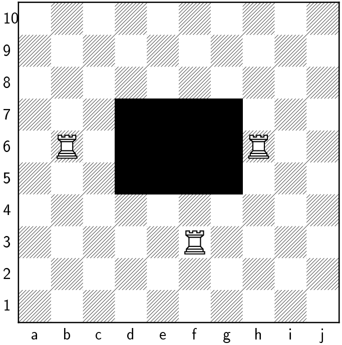


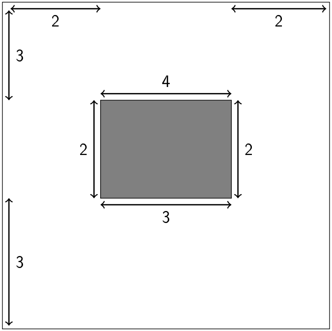

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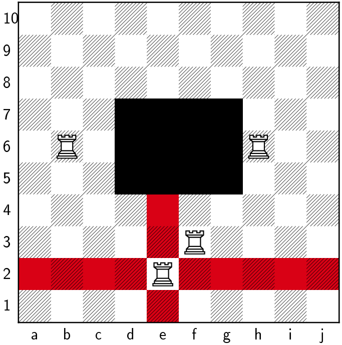
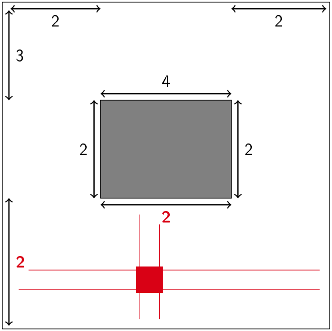
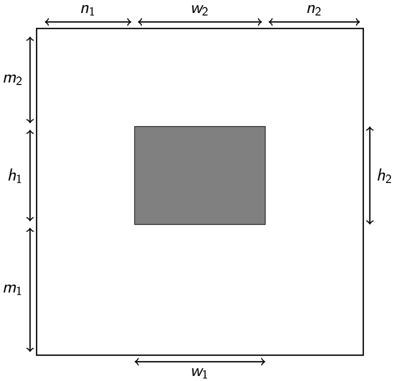


# Reduction to Subtraction Arc-Kayles



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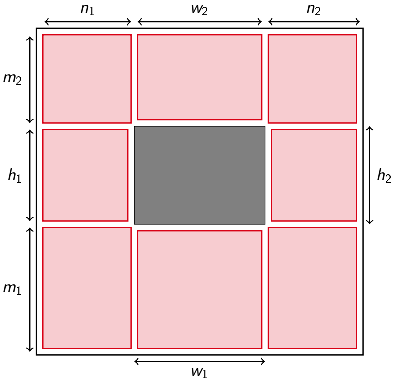

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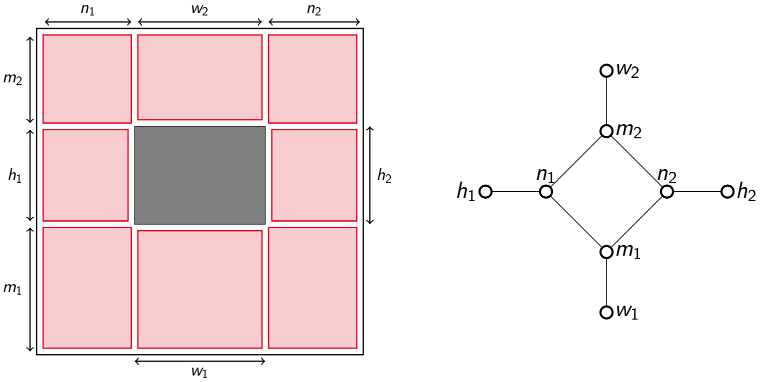

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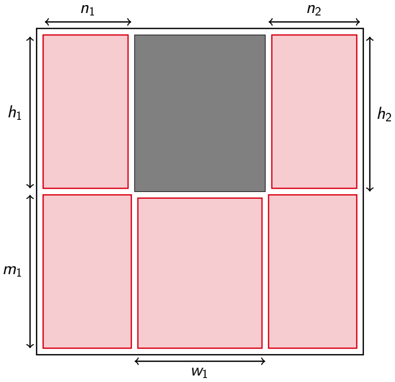
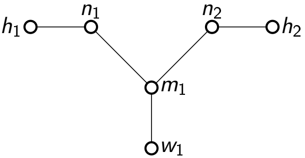


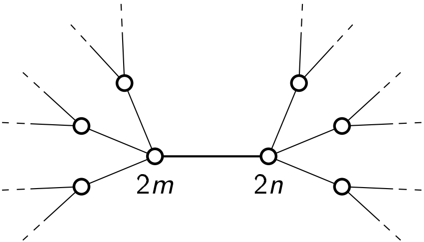



# First results

# All weights are even

**Lemma**

If all weights are even, then the outcome is $\mathcal{P}$



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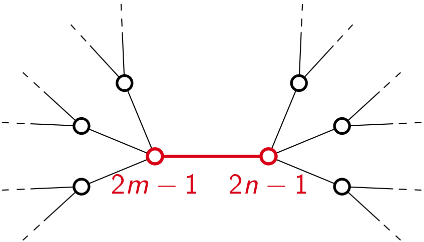

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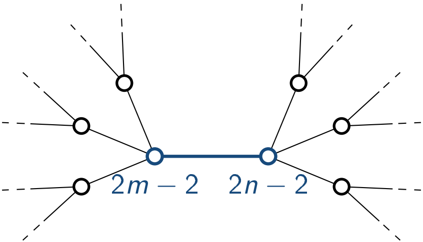
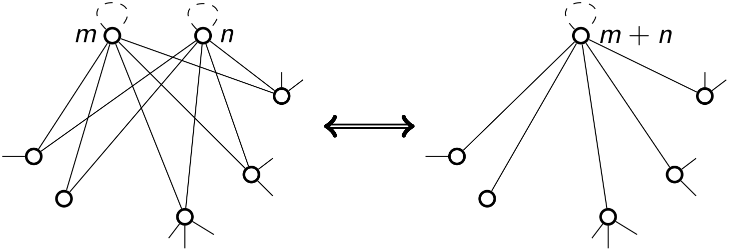


# Reduction lemmas

**Twin vertices lemma**

If two vertices have the **same neighbors** and either **have both loops or none of them have loops**, then the following reduction holds :



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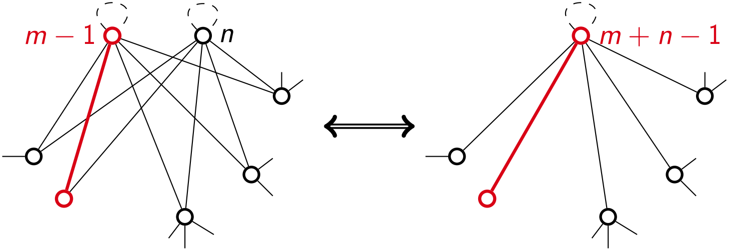

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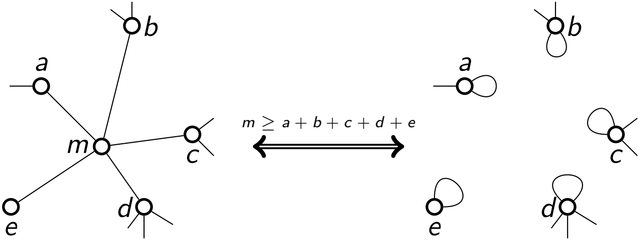
**Heavy vertex lemma**

If there is a vertex **without loop** and such that its weight is **greater than the sum of all of its neighbors' weight**, then the following reduction holds :



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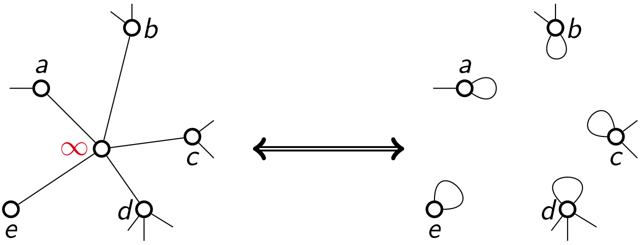

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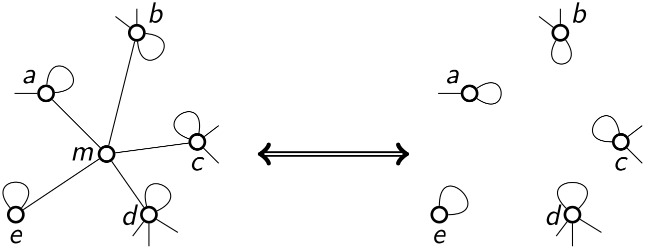
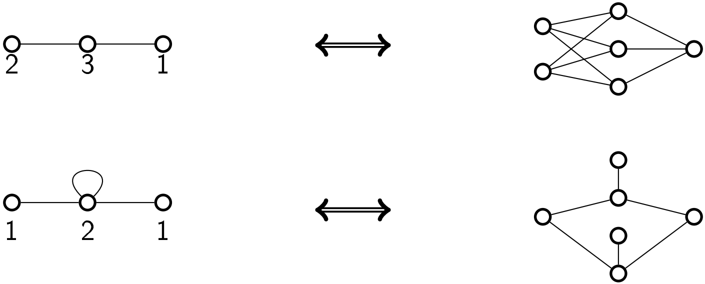
**Useless vertex lemma**

If there is a vertex without loop and such that **all its neighbors have loops** then the following reduction holds :



# Back to Arc-Kayles

Subtraction Arc-Kayles | Arc-Kayles

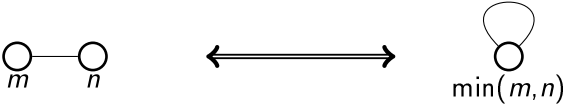


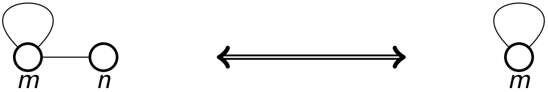
# Solving simple cases

# Simple cases


The game is already ended.


The grundy value is $m$ modulo 2.



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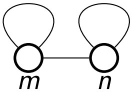


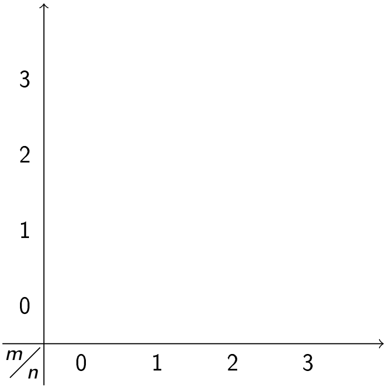

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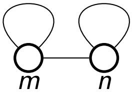
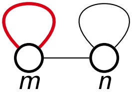
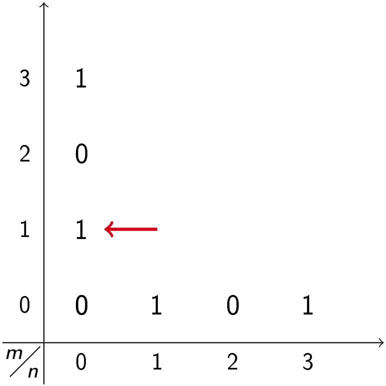



| $m$ \ $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 3 | 1 | | | |
| 2 | 0 | | | |
| 1 | 1 | | | |
| 0 | 0 | 1 | 0 | 1 |

# Simple cases



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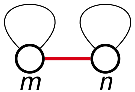


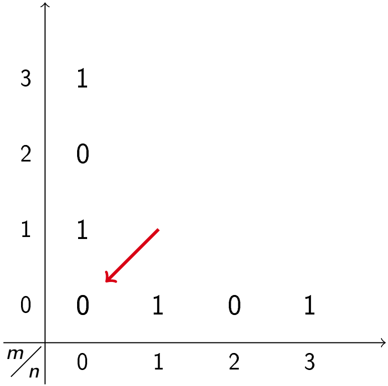

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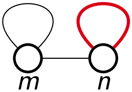


| | | | | |
|---|---|---|---|---|
| 3 | 1 | | | |
| 2 | 0 | | | |
| 1 | 1 | | | |
| 0 | 0 | 1 | 0 | 1 |
| m/n | 0 | 1 | 2 | 3 |

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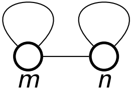


| m / n | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 3 | 1 | 2 | | |
| 2 | 0 | 3 | | |
| 1 | 1 | 2 | 3 | 2 |
| 0 | 0 | 1 | 0 | 1 |

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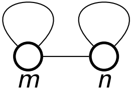


| | | | | |
|---|---|---|---|---|
| 3 | 1 | 2 | 1 | |
| 2 | 0 | 3 | 0 | 1 |
| 1 | 1 | 2 | 3 | 2 |
| 0 | 0 | 1 | 0 | 1 |
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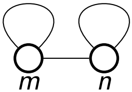
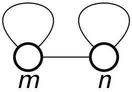
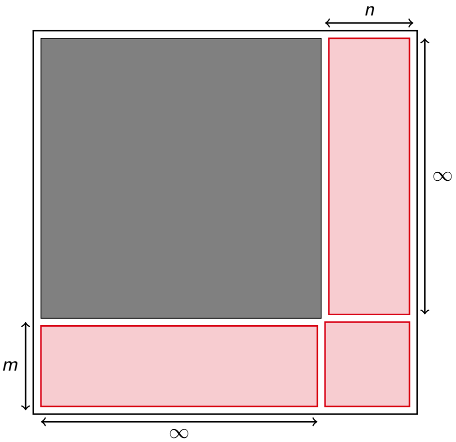



| | | | | |
|---|---|---|---|---|
| 3 | 1 | 2 | 1 | 2 |
| 2 | 0 | 3 | 0 | 1 |
| 1 | 1 | 2 | 3 | 2 |
| 0 | 0 | 1 | 0 | 1 |
| $m/n$ | 0 | 1 | 2 | 3 |

The game is $\mathcal{P}$ when both $m$ and $n$ are even.

# Simple cases



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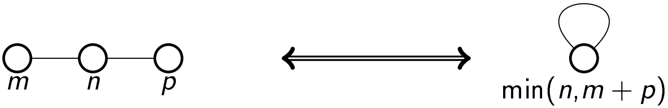


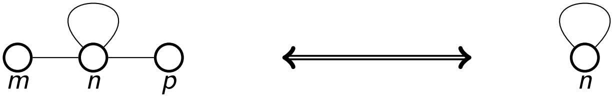

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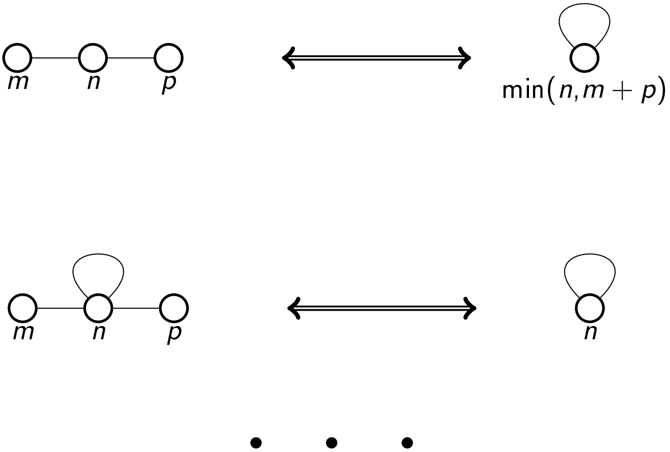
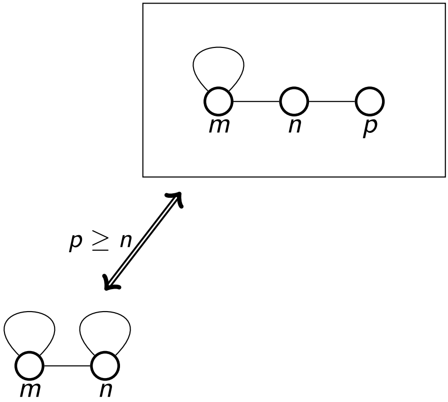


# First difficulty



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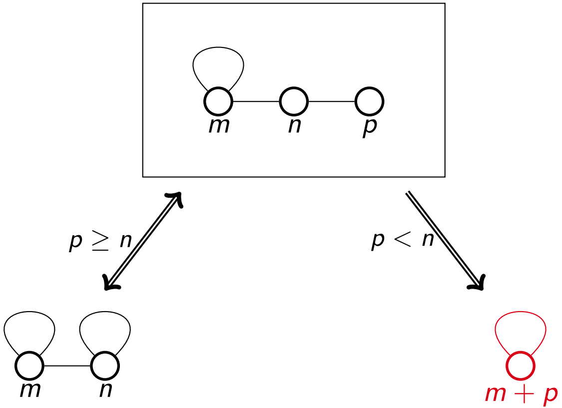
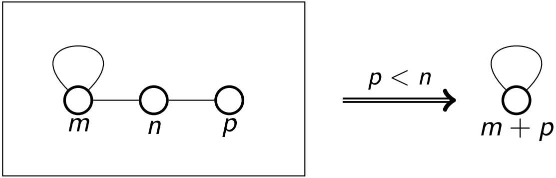


# General method



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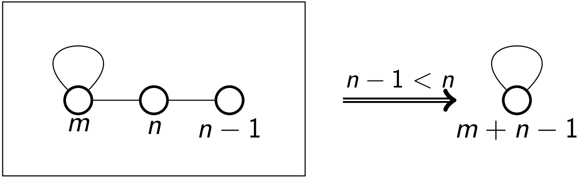

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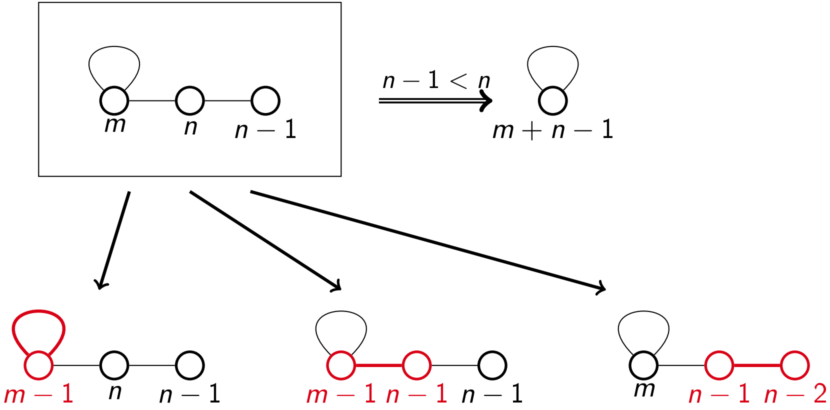

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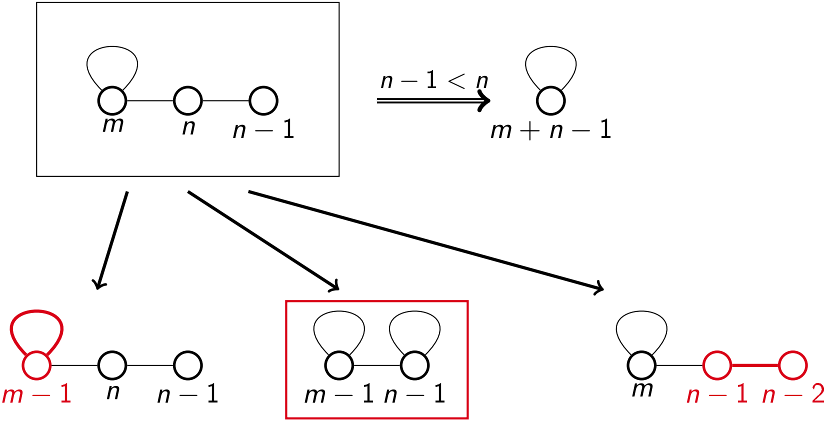
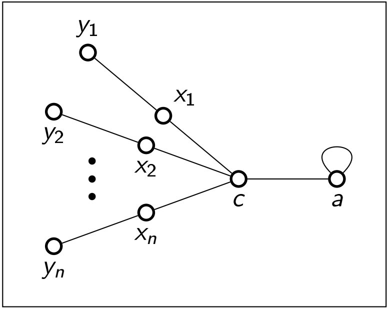


# Main results



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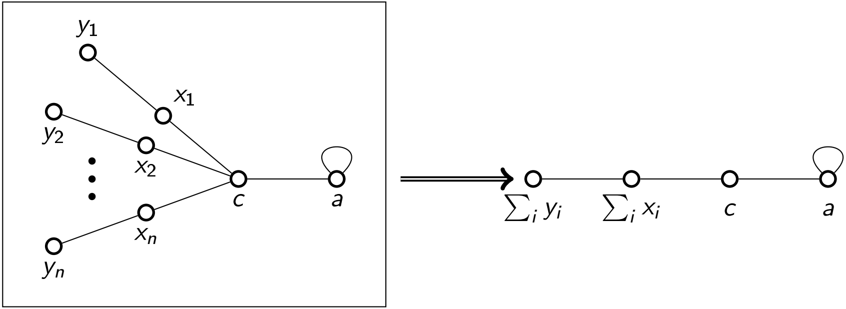

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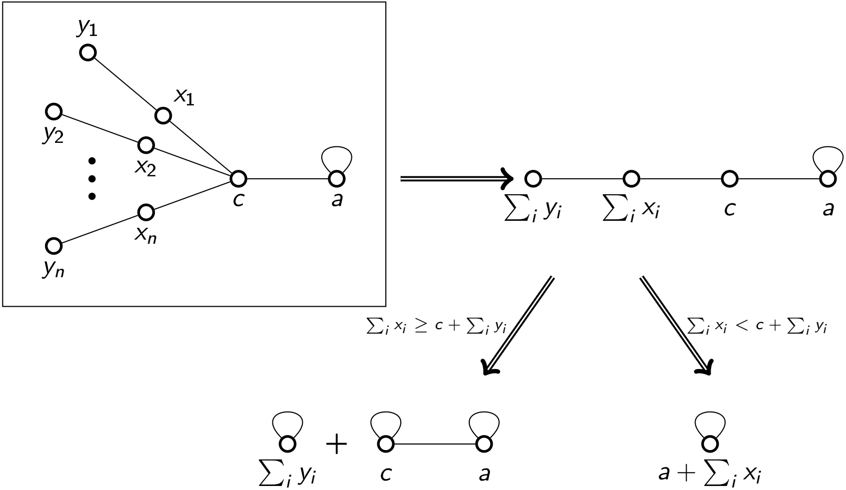

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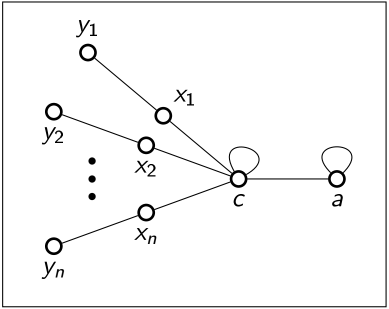

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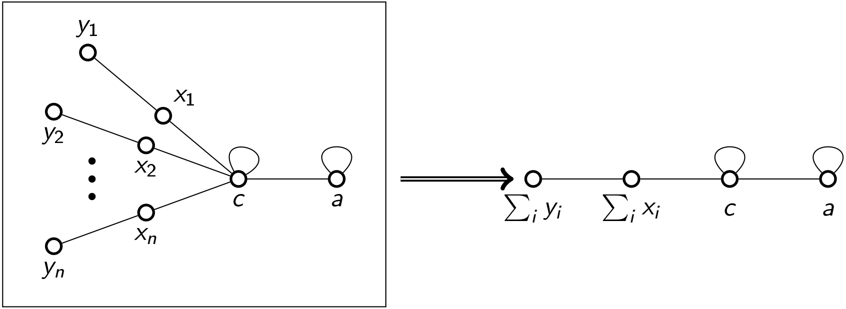
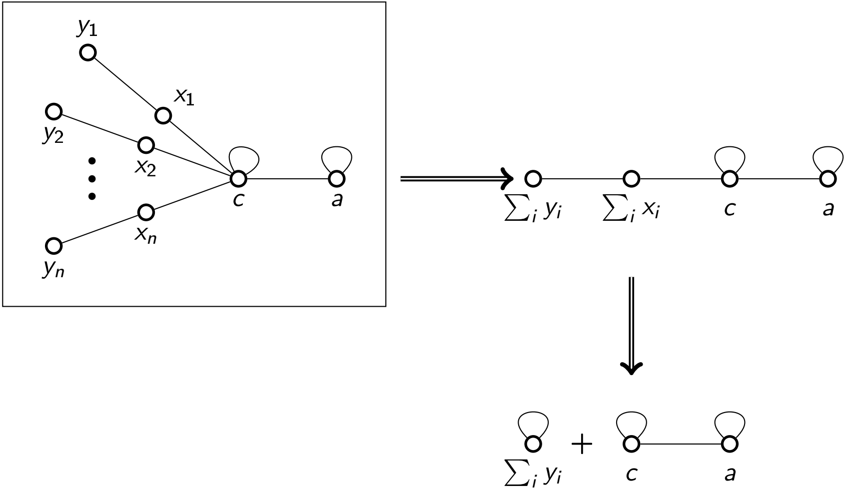


# Main results



# Periodicity

**Periodicity theorem**

The function $x \to outcome(G(x, \omega_2, \dots \omega_n))$ is ultimately 2-periodic with preperiod at most $2 \sum_{i \geq 2} \omega_i$.

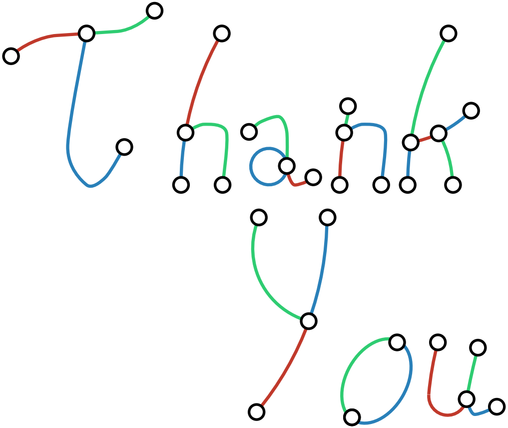
# Conclusion

**Results**

- Introduction of a generalisation of Arc-Kayles
- Application to non-attacking rooks on a holed chessboard
- Complete characterisation of trees of depth 2
- Periodicity for one vertex

**Perspectives**

- Solving more complex graphs
- Solving the general case of rooks on a holed chessboard
- Studying the complexity of the game


Thank
you