Fast winning strategies for the Maker-Breaker domination game

Valentin Gledel

LIRIS
Université Lyon 1
Lyon, France

Vesna Iršič

Faculty of Mathematics and Physics
University of Ljubljana
Ljubljana, Slovenia

Sandi Klavžar

Faculty of Mathematics and Physics
University of Ljubljana
Ljubljana, Slovenia

Abstract

The Maker-Breaker domination game is played on a graph G by Dominator and Staller. The players alternatively select a vertex of G that was not yet chosen in the course of the game. Dominator wins if at some point the vertices he has chosen form a dominating set. Staller wins if Dominator cannot form a dominating set. In this paper we introduce the Maker-Breaker domination number $\gamma_{\rm MB}(G)$ of G as the minimum number of moves of Dominator to win the game provided that he has a winning strategy and is the first to play. If Staller plays first, then the corresponding invariant is denoted $\gamma_{\rm MB}'(G)$. Comparing the two invariants it turns out that they behave much differently than the related game domination numbers. The invariant $\gamma_{\rm MB}(G)$ is also compared with the domination number. Using the Erdős-Selfridge Criterion a large class of graphs G is found for which $\gamma_{\rm MB}(G) > \gamma(G)$ holds. Residual graphs are introduced and used to bound/determine $\gamma_{\rm MB}(G)$ and $\gamma_{\rm MB}'(G)$. Using residual graphs, $\gamma_{\rm MB}(T)$ and $\gamma_{\rm MB}'(T)$ are determined for an arbitrary tree. The invariants are also obtained for cycles. A list of open problems and directions for further investigations is given.

Keywords: Maker-Breaker domination game, Maker-Breaker domination number, domination game, perfect matching, tree, cycle

Maker-Breaker games (as well as other *positional games*) have been introduced by Erdős and Selfridge in [13], and since then have been the subject of numerous studies, see [2,3,14,15]. Maker-Breaker games are played on hypergraphs by two players called Maker and Breaker. They take turns and at each turn the current player selects a new vertex. Maker wins if at some point of the game he has selected all vertices from one of the hyperedges, while Breaker wins if she can keep him from doing it. See [1] and [16] for general introductions on this field.

Very recently, the *Maker-Breaker domination game* was introduced in [12]. The game is played on a graph G with two players named Dominator and Staller. These names were selected to emphasize the domination nature of the game and to be consistent with the usual domination game where these two names are standard by now. (The domination game was introduced in [4] and further studied in dozens of papers, cf. [6,11,22,23,24].) The players alternatively select a vertex of G that was not yet chosen in the course of the game. Dominator wins if at some point, the vertices he has chosen form a dominating set. Staller wins if Dominator cannot form a dominating set. Note that the Maker-Breaker domination game is a Maker-Breaker game. Indeed, if for a graph G we build a hypergraph $\mathcal{F}$ with the same set of vertices as G, and in which the hyperedges are the

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dominating sets of G, then Dominator wins the Maker-Breaker domination game on G if and only if Maker wins the Maker-Breaker game on $\mathcal{F}$.

In several papers on Maker-Breaker games the authors were interested in the smallest number of moves needed for Maker to win, see [7,8,15]. Also, in [12] it was emphasized that when dealing with the Maker-Breaker games, there are two natural questions: (i) which player has a winning strategy and (ii) what is the minimum number of moves if Dominator has a winning strategy. In the seminal paper question (i) is investigated, while in this paper we study (ii). For this sake we say that if G is a graph, then the *Maker-Breaker domination number* $\gamma_{\rm MB}(G)$ of G is the minimum number of moves of Dominator to win the game provided that he has a winning strategy and is the first to play. Otherwise we set $\gamma_{\rm MB}(G) = \infty$. Similarly, $\gamma_{\rm MB}'(G)$ denotes is the minimum number of moves of Dominator in the game in which Staller plays first.

We proceed as follows. In the next section we list additional definitions and several known results needed in this paper, as well as prove some basic results on the Maker-Breaker domination number. In Section 2 we first compare $\gamma_{\rm MB}(G)$ with $\gamma_{\rm MB}'(G)$ and find out that they behave totally different than the related game domination invariants. We also compare $\gamma_{\rm MB}(G)$ with the domination number and using the Erdős-Selfridge Criterion prove that if the number of γ-sets of G is not too big, then $\gamma_{\rm MB}(G) > \gamma(G)$. In Section 3 we introduce residual graphs, determine (resp. bound) $\gamma_{\rm MB}'(G)$ (resp. $\gamma_{\rm MB}(G)$) in terms of the residual graph, and determine $\gamma_{\rm MB}(T)$ and $\gamma_{\rm MB}'(T)$ for an arbitrary tree. In the next section we obtain the invariants for cycles. We conclude with a list of open problems and directions for further investigations.

1 Preliminaries

Let $G = (V, E)$ be a graph. A vertex of G adjacent to a leaf is a *support vertex* of G. A *perfect matching* of G is a set of pairwise independent edges that cover $V(G)$. The order of G will be denoted with $n(G)$. If u is a vertex of G, then $N[u]$ denotes the *closed neighborhood* of u. If v is another vertex then we set $N[u, v] = N[u] \cap N[v]$. A set $D \subseteq V(G)$ is a *dominating set* of G if $\cup_{u \in D} N[u] = V(G)$. The *domination number* $\gamma(G)$ is the size of a smallest dominating set of G. A dominating set of size $\gamma(G)$ is called a *γ-set* of G.

The Maker-Breaker domination game is called a D-game (resp. S-game) if Dominator (resp. Staller) is the first to play a vertex. The sequence of vertices selected in a D-game will be denoted with $d_1, s_1, d_2, s_2, \ldots$, and the sequence of vertices selected in an S-game with $s_1', d_1', s_2', d_2', \ldots$ Suppose that Dominator wins a D-game. Then the last vertex played is by Dominator, let it be d_k. By the definition of the game, $\{d_1, \ldots, d_k\}$ is a dominating set of G. Similarly, if Dominator wins an S-game and the last vertex played by Dominator is d_ℓ', then $\{d_1', \ldots, d_\ell'\}$ is a dominating set of G. We say that a move of Staller is a *double threat* if it creates two possibilities for her to win in the next move and consequently Dominator cannot prevent Staller to win.

Let G be a graph, $k \geq 1$, and $u_1, \ldots, u_k, v_1, \ldots, v_k$ pairwise different vertices of G. Then we say that $X = \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$ is a *pairing dominating set* if

$$\bigcup_{i=1}^{k} N[u_i, v_i] = V(G)\,.$$

If $X = \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$ is a pairing dominating set such that $u_iv_i \in E(G)$ holds for $i \in [k]$, then we say that X is a *dominating matching*.

In the rest we will use this concept via the following interpretation proved in [12, Proposition 9].

Lemma 1.1 *Let $u_1, \ldots, u_k, v_1, \ldots, v_k$ be pairwise different vertices of a graph G, and let $X = \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$. Then X is a pairing dominating set if and only if every set $\{x_1, \ldots, x_k\}$, where $x_i \in \{u_i, v_i\}$, $i \in [k]$, is a dominating set of G.*

A direct application of this Lemma is the following fact.

Fact 1.2 [12, Proposition 10] *If G admits a pairing dominating set, then Dominator has a winning strategy on G in the D-game as well as in the S-game.*

The converse of Fact 1.2 does not hold in general. For instance, in [12, Figure 4] a chordal graph is presented on which Dominator has a winning strategy in both games but admits no pairing dominating set. On the other hand, the converse holds in the class of trees because if Dominator has a winning strategy on a tree T, then it was proved in [12] that T has a dominating matching. Moreover, the converse also holds for cographs.

Lemma 1.3 (No-Skip Lemma) *In an optimal strategy of Dominator to achieve $\gamma_{\rm MB}(G)$ or $\gamma_{\rm MB}'(G)$ it is never an advantage for him to skip a move. Moreover, if Staller skips a move it can only be an advantage for Dominator.*

If G is a graph and $S \subseteq V(G)$, then let $G|S$ denote that graph G in which the vertices from S are declared to be already dominated, that is, Dominator is not obliged to dominate them in the rest of the game. Then we have the following Continuation Principle, a proof of which is much simpler that the corresponding principle for the domination game [21].

Remark 1.4 (Continuation Principle) Let G be a graph with $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_{\rm MB}(G|A) \le \gamma_{\rm MB}(G|B)$ and $\gamma'_{\rm MB}(G|A) \le \gamma'_{\rm MB}(G|B)$.

Indeed, the remark follows from the fact that Dominator can apply the same strategy in $G|A$ as in $G|B$.

Suppose that $\gamma_{\rm MB}(G) < \infty$. Then in any winning strategy of Dominator, he will play at most half of the vertices (because Staller will play the other half) which in turn implies that

$$1 \le \gamma_{\rm MB}(G) \le \left\lceil \frac{n(G)}{2} \right\rceil . \tag{1}$$

The bound is sharp, consider for instance the disjoint union of K_1 and several copies of K_2. It is also easy to see that all the possible values from (1) can be realized by considering the disjoint union of a complete graph and an appropriate number of K_2s. Similarly, for the S-game, assuming that $\gamma'_{\rm MB}(G) < \infty$, we have

$$1 \le \gamma'_{\rm MB}(G) \le \left\lfloor \frac{n(G)}{2} \right\rfloor , \tag{2}$$

where again all the values can be realized.

Later we will apply the celebrated Erdős-Selfridge Criterion for Maker-Breaker games that reads as follows.

Theorem 1.5 (Erdős-Selfridge Criterion [13]) *If $\mathcal{F}$ is a hypergraph, then*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2} \ \Rightarrow \ \mathcal{F} \textit{ is a Breaker's win}.$$

This theorem together with its proof can also be found in the book [16, Theorem 2.3.3].

2 Maker-Breaker domination numbers

In this section we first compare $\gamma_{\rm MB}(G)$ with $\gamma'_{\rm MB}(G)$ and construct graphs for all possible values of the invariants. In the second part we compare $\gamma_{\rm MB}(G)$ with the domination number and using the Erdős-Selfridge Criterion find a large class of graphs G for which $\gamma_{\rm MB}(G) > \gamma(G)$ holds.

2.1 Realizations of Maker-Breaker domination numbers

One of the fundamental theorems on the domination game proved in [4,21] asserts that the difference between the value of the game domination number when Dominator starts and when Staller starts is bounded by 1, that is to say $|\gamma_g(G) - \gamma'_g(G)| \le 1$ holds for every graph G. The next result reveals that the situation with the Maker-Breaker domination number is dramatically different.

Theorem 2.1 *If G is a graph, then $\gamma(G) \le \gamma_{\rm MB}(G) \le \gamma'_{\rm MB}(G)$. Moreover, for any integers r, s, t, where $2 \le r \le s \le t$, there exists a graph G such that $\gamma(G) = r$, $\gamma_{\rm MB}(G) = s$, and $\gamma'_{\rm MB}(G) = t$.*

Note that if $\gamma(G) = 1$, then also $\gamma_{\rm MB}(G) = 1$. Hence Theorem 2.1 does not extend to the case $r = 1$. On the other hand, if G_t, $t \ge 1$, is the graph obtained from t disjoint triangles by identifying a vertex from each of the triangles (so that this new vertex is of degree $2t$), then $\gamma(G_t) = 1$, $\gamma_{\rm MB}(G_t) = 1$, and $\gamma'_{\rm MB}(G_t) = t$.

Theorem 2.1 extends also to highly connected graphs. To see this, consider the graphs $H_{k,r,s,t}$, $2 \le r \le s \le t$, $k \ge 1$, that are schematically drawn in Fig. 1. Here, each vertex of a K_k clique is adjacent to each vertex of the clique K_{k+r}. One can see that $\gamma(H_{k,r,s,t}) = r$, $\gamma_{\rm MB}(H_{k,r,s,t}) = s$, and $\gamma'_{\rm MB}(H_{k,r,s,t}) = t$. Moreover, $H_{k,r,s,t}$ is $(k+1)$-connected.

2.2 Relation with the domination number

As already observed above, $\gamma_{\rm MB}(G) = 1$ if and only if $\gamma(G) = 1$. In general it would be interesting to characterize the graphs G such that $\gamma_{\rm MB}(G) = \gamma(G) = k$, where $k \ge 2$ is a fixed integer. For $k = 2$ the answer is simple:

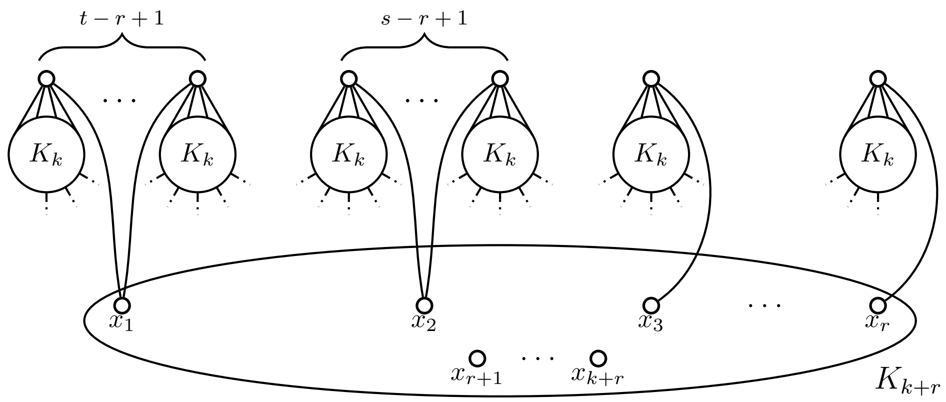

Fig. 1. Graph $H_{k,r,s,t}$

Proposition 2.2 *Let G be a graph with $\gamma(G) = 2$. Then $\gamma_{\rm MB}(G) = \gamma(G) = 2$ if and only if G has a vertex that lies in at least two γ-sets of G.*

Proposition 2.2 can be rephrased to hold for larger k also, but this would be more or less just rephrasing the definitions. It would be more interesting to find a structural characterization of the corresponding graphs. This task, however, seems difficult. On the other hand, the Erdős-Selfridge Criterion gives a sufficient condition for $\gamma_{\rm MB}(G) > \gamma(G)$. Let $X_\gamma(G)$ be the number of γ-sets of a graph G, cf. [9]. Then:

Proposition 2.3 *If G is a graph and $X_\gamma(G) < 2^{\gamma(G)-1}$, then $\gamma_{\rm MB}(G) > \gamma(G)$.*

Consider the cycles C_{3k-1}, $k \geq 1$. It is known and easy to see that $\gamma(C_{3k-1}) = k$. We now determine the number of γ-sets of C_{3k-1}. Each vertex from a γ-set dominates itself and its two neighbors. As there are k such triplets and $3k-1$ vertices in the graph, there is only one vertex that is dominated by two vertices from the γ-set, all others are dominated exactly once. Thus if the vertex that is dominated twice is fixed, then the γ-set of the cycle is uniquely determined. As there are $3k-1$ choices for this vertex, we have $X_\gamma(C_{3k-1}) = 3k-1$. If $k \geq 5$, then $X_\gamma(C_{3k-1}) = 3k-1 < 2^{k-1} = 2^{\gamma(C_{3k-1})-1}$, and by Proposition 2.3, we conclude that $\gamma_{\rm MB}(C_{3k-1}) > k = \gamma(C_{3k-1})$. Actually, $\gamma_{\rm MB}(C_{3k-1})$ is much bigger than $\gamma_g(C_{3k-1})$ as we will see in Section 4.

The converse of Proposition 2.3 does not hold as the following example shows. If $k \in \{3,4\}$, then $X_\gamma(C_{3k-1}) = 3k-1 > 2^{k-1} = 2^{\gamma(C_{3k-1})-1}$, but as we will see in Section 4, $\gamma(C_{3k-1}) = k < k+1 = \lfloor \frac{3k-1}{2} \rfloor = \gamma_{\rm MB}(C_{3k-1})$.

3 Residual graphs

In this section we study the Maker-Breaker domination number on a construction that might be of independent interest and that will be later used to determine the invariant for trees.

If G is a graph, then we say that the *residual graph* $R(G)$ of G is the graph obtained from G by iteratively removing pendant paths P_2 until no such path is present. By a *pendant* P_2 we mean P_2 attached to G with an edge. Hence, when such a pendant P_2 is removed, exactly two vertices and two edges are removed. When $G = P_2$, we can also remove it and obtain the empty graph.

Note that $H = R(G)$ for some graph G if and only if H is the empty graph, $H = K_1$, or each support vertex of H has degree at least 3. This is in particular true if H has no support vertices. We further observe:

Lemma 3.1 *If G is a graph, then $R(G)$ is unique (up to isomorphism).*

Note that also that if $R(G) \neq K_1$, then $G \setminus V(R(G))$ is unique. To see that it is not unique in general, consider a path P_{2k+1}, $k \geq 2$, and different sequences of removing pendant P_2s.

Lemma 3.2 *Let G be a graph and $R(G)$ a residual graph of G. Then*

(i) $G \setminus V(R(G))$ is a forest that has a unique perfect matching, and

(ii) G has a perfect matching if and only if $R(G)$ has a perfect matching.

For the proof of the main result of this section, we also need the following.

Lemma 3.3 *If T is a tree that admits a perfect matching and $v \in V(T)$, then Staller has a strategy for the S-game such that Dominator has to select at least $\frac{n(T)}{2}$ vertices to dominate T and v is played by Staller in her last move.*

Proof. We prove the claim by induction on $n(T)$. If $T = P_2$ and $v \in V(P_2)$, then Staller can play on v and Dominator has to reply on the other vertex.

Let now $n(T) \geq 4$ and consider T as a BFS-tree rooted at an arbitrary vertex r. Let x be a leaf of this BFS-tree at the largest distance from r and let y be the neighbor of x. Then $\deg(y) = 2$ because T has a perfect matching. Let z be the other neighbor of y. Set $T' = T \setminus \{x, y\}$. As T has a perfect matching, xy belongs to it, hence T' also has a perfect matching. If $v \in V(T')$, then Staller starts on y, Dominator has to reply on x (otherwise Staller would win) and then Staller applies her strategy on T' (by the induction hypothesis). If $v \in \{x, y\}$, then she applies her strategy on T' with her last move on z, and then plays v in her last move. Note that if Dominator plays on v while Staller is playing on T', then Staller wins the game as she can prevent Dominator from playing on one pair of vertices from the matching in T'.

From the above strategy of Staller we conclude that the total number of Dominator's moves was $\frac{n(T')}{2} + 1 = \frac{n(T)}{2}$. □

Note that by the strategy from the proof of lemma 3.3, unless Staller wants to play on a leaf, she plays on the support vertex, forcing Dominator to reply on its neighboring leaf and separating this P_2 from the rest of the graph.

Theorem 3.4 *Let $R(G)$ be a residual graph of G and let $H = G \setminus V(R(G))$. Then*

(i) $\gamma'_{\rm MB}(G) = \frac{n(H)}{2} + \gamma'_{\rm MB}(R(G))$,

(ii) $\frac{n(H)}{2} + \gamma_{\rm MB}(R(G)) - 1 \leq \gamma_{\rm MB}(G) \leq \frac{n(H)}{2} + \gamma_{\rm MB}(R(G))$.

Proof. (i) H has a perfect matching and is a forest by Lemma 3.2(i). Let S-game be played on G and consider the following strategy of Staller. By Lemma 3.3 she can play on each tree of H and play last on the vertex of this tree adjacent to $R(G)$. Dominator has to reply on the matching (otherwise Staller wins the game). Thus, Dominator makes (at least) $\frac{n(H)}{2}$ moves on H. Moreover, Staller plays on vertices adjacent to $R(G)$, hence no vertex in $R(G)$ will be dominated by the time Staller makes her first move in $R(G)$. Next, Staller is the player to make the first move on $R(G)$ and she follows her optimal strategy there to ensure at least $\gamma'_{\rm MB}(R(G))$ moves of Dominator.

On the other hand, Dominator's strategy is to then reply wherever Staller plays, H or $R(G)$, with its strategy on this graph. As H has a perfect matching, Dominator makes no more than $\frac{n(H)}{2}$ moves on H. Moreover, he makes at most $\gamma'_{\rm MB}(R(G))$ moves on $R(G)$. Hence, we have $\gamma'_{\rm MB}(G) = \frac{n(H)}{2} + \gamma'_{\rm MB}(R(G))$.

(ii) Suppose now that the D-game is played on G. To prove the upper bound, Dominator's strategy is to start on $R(G)$ and then reply on $R(G)$ or H if Staller plays there. As H has a perfect matching, Dominator makes no more than $\frac{n(H)}{2}$ moves on H. Moreover, he makes at most $\gamma_{\rm MB}(R(G))$ moves on $R(G)$. Hence we get the upper bound $\gamma_{\rm MB}(G) \leq \frac{n(H)}{2} + \gamma_{\rm MB}(R(G))$.

To prove the lower bound, consider the following strategy of Staller depending on the first move of Dominator. We will distinguish two cases, the second with two subcases, which are schematically depicted in Fig. 2.

1 The first move of Dominator is on $R(G)$.
Staller first applies her strategy from Lemma 3.3 on each tree of H, playing the vertex adjacent to $R(G)$ as her last move on each of the trees. With this, she forces Dominator to play (at least) $\frac{n(H)}{2}$ moves on H. After that we have an ordinary D-game played on $R(G)$, so at least $\gamma_{\rm MB}(R(G))$ moves are made on it by Dominator if Staller follows her strategy there.

2 The first move of Dominator is on H.
Let d_1 be the vertex Dominator plays in his first move, let T be the connected component of H containing d_1 (recall that T is a tree), let P be the shortest path between d_1 and $R(G)$ in T, and let M be the unique perfect matching of T (cf. Lemma 3.2(i)).
In this case, Staller first applies her strategy from Lemma 3.3 on all the other trees of H, playing the vertex adjacent to $R(G)$ as her last move on each tree. Next, Staller applies her strategy from Lemma 3.3 on the edges from M, which are not incident with P. Additionally, she plays last on the vertices closest to P. After that, only $R(G)$, P, and maybe some vertices adjacent to P, remain undominated.
2.1 At least one vertex adjacent to P is still undominated (see Fig. 2).

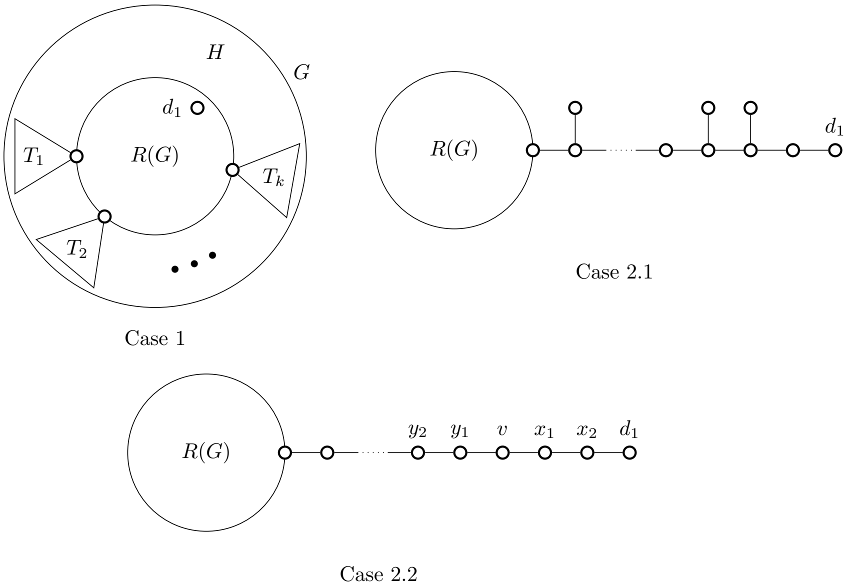

Fig. 2. Representations of the cases from the proof of Theorem 3.4

Let u be an undominated vertex adjacent to P. Staller plays on its neighbor on P, forcing Dominator to reply on u. Staller does so on each such vertex. After that, the only undominated vertices lie on P, moreover, up to now, at least one move of Dominator was played on each already completely dominated edge from M.

As long as there are some more undominated edges from M on P, at least one of them, say $e \in M$, is incident to a vertex s of P already played by Staller. Her strategy is to play on the vertex of e which is at distance 2 from s. Then Dominator has to reply on the other vertex of e, otherwise Staller wins by playing it. Hence, Staller can force Dominator to reply on all remaining edges.

2.2 The only undominated vertices in H lie on P.

Staller's strategy is to play on the vertex v of P at distance 3 from d_1. Dominator has to reply on a neighbor of v, otherwise one of the neighbors of v is not dominated and Staller can win by playing that vertex and creating a double threat. Indeed, in this case, two undominated adjacent vertices are played by Staller, and no matter where Dominator answers, she can play another consecutive vertex and win the game.

Let x_i be a vertex at distance i from v on P in the direction of d_1, and y_i be a vertex at distance i from v on P in the direction of $R(G)$ for all possible $i \geq 1$, see Fig. 2 again.

If Dominator replies on x_1, then Staller's next move is y_2. Now, Dominator has to reply on y_1, otherwise Staller wins. Then Staller repeats this strategy until P is dominated, i.e., she plays on the vertices y_{2k} in the increasing order, and Dominator is forced to reply on y_{2k-1}.

If Dominator replies on y_1, then Staller replies on x_2. After that, Dominator has to play x_1. Next, Staller applies the same strategy as before, taking y_1 as the new d_1.

In both cases, Dominator is forced to play at least one move on each edge of the matching M, hence at least $\frac{n(T)}{2}$ moves are made on T. On $H - T$, at least $\frac{n(H-T)}{2}$ moves are made by Lemma 3.3.

After T is completely dominated, Staller follows her optimal strategy on $R(G)$, but it might happen that one vertex u in $R(G)$ is already dominated (by a move of Dominator in H close to $R(G)$). As Staller's strategy on H forces Dominator to answer on H, Staller will be the first player to play on $R(G)$. But as she can imagine that Dominator's move was u, we have

$$\gamma'_{\rm MB}(R(G)|u) \geq \gamma'_{\rm MB}(R(G)|N[u]) \geq \gamma_{\rm MB}(R(G)) - 1\,,$$

hence the total number of moves on $R(G)$ is at least $\gamma_{\rm MB}(R(G)) - 1$.

In either case, Dominator played at least $\frac{n(H)}{2} + \gamma_{\rm MB}(R(G)) - 1$ moves, which proves the lower bound. □

Note that in the inequality $\gamma'_{\rm MB}(G|u) \geq \gamma_{\rm MB}(G) - 1$ from the above proof, the equality can be attained. For example, consider the graph G on Fig. 3. Clearly, $\gamma_{\rm MB}(G) = 2$ and $\gamma'_{\rm MB}(G|u) = 1$.

Appending to G some trees with perfect matchings, where at least one of them is attached to u, we get

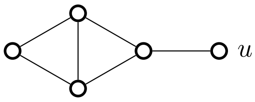

Fig. 3. The graph G with the property $\gamma'_{\rm MB}(G|u) = \gamma_{\rm MB}(G) - 1$.

graphs that attain the lower bound from Theorem 3.4(ii).

To conclude the section we apply the residual construction to determine the Maker-Breaker domination number of trees. This contrasts the domination game where no such result is known, cf. [5,17,18].

Theorem 3.5 *If T a tree, then*

$$\gamma_{\rm MB}(T) = \begin{cases} \frac{n(T)}{2}; & T \text{ has a perfect matching}, \\ \frac{n(T)-1}{2}; & R(T) \cong K_1, \\ \frac{n(T)-k+1}{2}; & R(T) \cong K_{1,k} \text{ for } k \geq 3, \\ \infty; & \text{otherwise}, \end{cases}$$

and

$$\gamma'_{\rm MB}(T) = \begin{cases} \frac{n(T)}{2}; & T \text{ has a perfect matching}, \\ \infty; & \text{otherwise}. \end{cases}$$

Note that by Theorem 3.5, $\gamma_{\rm MB}$ and $\gamma'_{\rm MB}$ of trees are polynomial.

4 Cycles

The D-game domination number and the S-game domination number of cycles are given with the following formulas:

$$\gamma_g(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \mod 4, \\ \left\lceil \frac{n}{2} \right\rceil; & \text{otherwise}, \end{cases} \qquad \gamma'_g(C_n) = \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil - 1; & n \equiv 2 \mod 4, \\ \left\lceil \frac{n-1}{2} \right\rceil; & \text{otherwise}. \end{cases}$$

This fundamental result was first obtained in an unpublished manuscript [20]. The result appeared for the first time in press in the paper [19], where an alternative proof is given. For the total domination game, parallel results were obtain in [10]. The latter paper investigates the total domination game on paths and cycles only. So the (total) game domination number of cycles is far from being straightforward. Here we determine the Maker-Breaker domination number of cycles, a task that turned out to be less involved.

Using Lemma 1.1 and Fact 1.2 it is easy to see that Dominator has a winning strategy on even cycles. By observing that removing the neighborhood of any vertex of an odd cycles leaves this graph with a perfect matching we can also see that Dominator has a winning strategy on odd cycles. However it is interesting to remark that Dominator cannot do better than these strategies.

Theorem 4.1 *If $n \geq 3$, then*

$$\gamma_{\rm MB}(C_n) = \gamma'_{\rm MB}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor .$$

5 Concluding remarks

To conclude the paper we list several problems and directions for further investigation of the Maker-Breaker domination number.

(i) For the upper bound in (1) we have provided examples of graphs that attain the equality. These examples are not connected and it is not difficult to achieve the equality with connected graphs of even order. However, we do not know of any connected graph of odd order (different from K_1) for which the equality in (1) is achieved. More generally, we ask for a characterization of the extremal graphs with respect to (1) and (2).

(ii) As we already mentioned, it would be interesting to find a structural characterization of the graphs G for which $\gamma_{\rm MB}(G) = \gamma(G) = k$ holds, where $k \geq 2$ is a fixed integer.

(iii) It would also be interesting to investigate $\gamma_{\rm MB}(G \,\square\, H)$ and $\gamma'_{\rm MB}(G \,\square\, H)$, where G and H are arbitrary graphs and $G \,\square\, H$ is the Cartesian product of G and H. In particular, it would be interesting to determine $\gamma_{\rm MB}(P_n \,\square\, P_m)$ (and $\gamma'_{\rm MB}(P_n \,\square\, P_m)$), as well as $\gamma_{\rm MB}(G \,\square\, K_2)$ (and $\gamma'_{\rm MB}(G \,\square\, K_2)$) for an arbitrary graph G.

(iv) If G is a cograph, then it is not difficult to determine whether Dominator or Staller wins the Maker-Breaker domination game [12]. On the other hand, it does not seem straightforward to determine the Maker-Breaker domination numbers of cographs.

(v) In this paper we have considered the Maker-Breaker domination number which is an optimization problem from Dominator's point of view. It would likewise be of interest to consider the Staller's point of view, that is, assuming that Staller wins on a graph G, what is the minimum number of moves with which she can achieve the goal?

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