Recent Progress in Improvement of Extreme Discrepancy and Star Discrepancy of One-dimensional Sequences.

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In this communication, we report on recent progress in improvement of extreme discrepancy and star discrepancy of one-dimensional sequences. Namely, we present a permutation of "Babylonian" sequences in base 60, which improves the best known results for star discrepancy obtained by Henri Faure in 1981 [1], and a permutation of sequences in base 84, which improves the best known results for extreme discrepancy obtained by Henri Faure in 1992 [2]. Our best result for star discrepancy in base 60 is $2119/(2360 \log 60) \approx 0.219298$ (Faure's best result in base 12 is $1919/(3454 * Log(12)) \approx 0.223585$); our best result for extreme discrepancy in base 84 is $130/(83 \log 84) \approx 0.353494$ (Faure's best result in base 36 is $23/(35 * Log(6)) \approx 0.366758$);

1 Introduction

First, let us recall some definitions commonly used in the specialized literature [1, 2, 3].

Let $X = (x_n)_{n \geq 1}$ be a sequence defined on one-dimensional interval $[0, 1]$, and $A(\alpha, N, X)$ the number of $n \leq N$ such that $0 \leq x_n < \alpha$. The *remainder* E is defined as $E(\alpha, N, X) = A(\alpha, N, X) - \alpha N$; $E([\alpha, \beta[; N, X) = E(\beta, N, X) - E(\alpha, N, X)$, where $0 \leq x_n < \alpha < \beta \leq 1$.

The *extreme discrepancy* is defined as $D(N, X) = \sup_{\alpha,\beta} |E([\alpha, \beta[; N, X)|$, and the *star discrepancy* is defined as $D^*(N, X) = \sup_{\alpha,\beta} |E(\alpha, N, X)|$.

The superior limits of extreme and star discrepancy are defined as

$$s(X) = \overline{\lim_{N}}(D(N)/Log(N)$$

and

$$s^*(X) = \overline{\lim_{N}}(D^*(N)/Log(N),$$

Given an integer $n \geq 1$ in b-adic representation $\sum_{j=0}^{\infty} a_j(n)n^j$ and the permutations $(\sigma_j)_{j\geq 0}$ of the set $\{0, 1, \ldots, b-1\}$, the generalized van der Corput sequence $S_{b,\sigma}$ in fixed base b is defined by

$$S_{b,\sigma} = \sum_{j=0}^{\infty} \sigma(a_j(n))n^{-j-1}.$$

Theorem 1 (Faure 1981) *The terms of extreme and star discrepancy of $S_{b,\sigma}$ can be expressed, for any $N \geq 1$, as follows*

$$D^+(S_{b,\sigma}, N) = \sum_{j=1}^{\infty} \Psi^+_{b,\sigma_{j-1}}\left(\frac{N}{b^j}\right)$$

$$D^-(S_{b,\sigma}, N) = \sum_{j=1}^{\infty} \Psi^-_{b,\sigma_{j-1}}\left(\frac{N}{b^j}\right)$$

$$D(S_{b,\sigma}, N) = \sum_{j=1}^{\infty} \Psi_{b,\sigma_{j-1}}\left(\frac{N}{b^j}\right)$$

$$D^*(S_{b,\sigma}, N) = \max(D^+(S_{b,\sigma}, N), D^-(S_{b,\sigma}, N)).$$

Theorem 2 (Faure 1981) *The asymptotic behavior of the extreme discrepancy of $S_{b,\sigma}$ can be expressed in terms of the constant $\alpha_{b,\sigma}$, defined as*

$$\alpha_{b,\sigma} = \inf_{n\geq 1} \sup_{x\in R} \left(\frac{1}{n}\sum_{j=1}^{\infty} \Psi_{b,\sigma_j}\left(\frac{x}{b^j}\right)\right)$$

Then,

$$s(S_{b,\sigma}) = \overline{\lim}_{N\to\infty} \frac{D(S_{b,\sigma}, N)}{\log N} = \frac{\alpha_{b,\sigma}}{\log b}.$$

Theorem 3 (Faure 1981) *Let $A \subset N$ defined as $A = \bigcup_{H=1}^{\infty} A_H$ and $A_H = \{H(H-1)+1, \ldots, H^2\}$. Let τ be a permutation defined as $\tau(k) = b-1-k$, where $0 \leq k \leq b-1$. Then, the permutation $\Sigma_A = (\sigma_j)_{j\geq 1}$*

is defined as $\sigma_j = \sigma$ if $j \in A$ and $\sigma_j = \tau \circ \sigma$ if $j \notin A$. The asymptotic behavior of the star discrepancy of S_{b,Σ_A} can be expressed in terms of $\alpha^+_{b,\sigma}$ and $\alpha^-_{b,\sigma}$ as follows:

$$s^*(S_{b,\Sigma_A}) = \overline{\lim}_{N\to\infty} \frac{D^*(S_{b,\sigma},N)}{\log N} = \frac{\alpha^+_{b,\sigma} + \alpha^-_{b,\sigma}}{2\log b}$$

where

$$\alpha^+_{b,\sigma} = \inf_{n\geq 1} \sup_{x\in R} \left(\frac{1}{n} \sum_{j=1}^{\infty} \Psi^+_{b,\sigma_j}\left(\frac{x}{b^j}\right)\right)$$

and

$$\alpha^-_{b,\sigma} = \inf_{n\geq 1} \sup_{x\in R} \left(\frac{1}{n} \sum_{j=1}^{\infty} \Psi^-_{b,\sigma_j}\left(\frac{x}{b^j}\right)\right).$$

Following [1, 2], we define $\Psi^-_{b,\sigma}$, $\Psi^+_{b,\sigma}$ and $\Psi_{b,\sigma}$ on intervals $I^n_h = [h/b^n, (h+1)/b^n]$. The interval I^n_h is called *dominated* if there exists a set J of integers with $h \notin J$ such that $F_n(x) \leq max_{j\in J} F_n(x + (j-h)/b^n$ for all $x \in I^n_h$. Otherwise, the interval is *dominant*.

2 Main results

Theorem 4 *Let base $b = 84$ and the permutation (σ_j), defined as*

$(\sigma_j) = (0, 22, 64, 32, 50, 76, 10, 38, 56, 18, 72, 45, 6, 28, 59, 79, 41, 13, 67, 25, 54, 2, 36, 70, 16, 48, 81, 30, 61, 8, 43, 74, 20, 52, 4, 34, 66, 15, 46, 77, 26, 11, 62, 39, 82, 57, 23, 69, 33, 3, 51, 19, 73, 42, 7, 60, 29, 80, 47, 14, 65, 35, 1, 53, 24, 68, 12, 40, 78, 58, 27, 5, 44, 71, 17, 55, 37, 83, 21, 49, 75, 9, 31, 63)$.

The extreme discrepancy of the sequence $S_{84,\sigma}$ is

$$s(S_{84,\sigma}) = 130/(83 \log 84) \approx 0.353494.$$

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Theorem 5 *Let base $b = 60$ and the permutation (σ_j), defined as*

$(\sigma_j) = (0, 15, 30, 40, 2, 48, 20, 35, 8, 52, 23, 43, 12, 26, 55, 4, 32, 45, 17, 37, 6, 50, 28, 10, 57, 21, 41, 13, 33, 54, 1, 25, 46, 18, 38, 5, 49, 29, 9, 58, 22, 42, 14, 34, 53, 3, 27, 47, 16, 36, 7, 51, 19, 44, 31, 11, 56, 24, 39, 59)$.

The star discrepancy of the sequence S_{60,Σ_A} is

$$s^*(S_{60,\Sigma_A}) = 2119/(2360 \log 60) \approx 0.219298.$$

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3 Proofs

The proofs of Theorems 4 and 5 follow the main line of the proofs provided by Henri Faure in [1, 2].

First, we build the functions $\Psi^+_{b,\sigma}(x)$, $\Psi^-_{b,\sigma}(x)$ and $\Psi_{b,\sigma}(x)$. Then, based on Theorems 1 and 2, we express $s(S_{b,\sigma})$ in terms of $\Psi_{b,\sigma}$. We perform numerical investigation of this function, make an induction hypothesis and prove it.

Similarly, we express $s^*(S_b, \sigma)$ in terms of $\Psi^-_{b,\sigma}$ and $\Psi^+_{b,\sigma}$, based on theorems Theorems 1 and 3. We make an induction hypothesis and prove it.

As in [1, 2], we introduce the function $F_n = \sum_{k=0}^{n-1} \Psi(xb^k)$ and express $\alpha = \inf_{n\geq 1}(\max_{x\in[0,1]} F_n(x)/n)$.

3.1 Function $\Psi_{84,\sigma}(x)$

Finding $\Psi^+_{b,\sigma}(x)$, $\Psi^-_{b,\sigma}(x)$ and $\Psi_{b,\sigma}(x)$ is a tedious work. These functions should be presented as a piecewise affine functions on well-defined intervals. As, for definition of $s(S_{84,\sigma})$, we need the function $\Psi_{84,\sigma}(x)$ only, we omit here, for the reasons of compactness, the intermediate expressions for $\Psi^+_{84,\sigma}(x)$ only $\Psi^-_{84,\sigma}(x)$.

The exact definition of the function $\Psi_{84,\sigma}(x)$ defined on intervals $I^1_h = [h/b^1, (h+1)/b^1]$ is presented in Table 1. Each interval I^1_h is also expressed as a set of affine subintervals. Thus, the interval $[0, 1]$ is expressed as a set of 216 affine subintervals. Figure 1 shows the function $\Psi_{84,\sigma}(x)$ visually.

Table 1: $\Psi_{84,\sigma}(x)$ defined on intervals $I^1_h = [h/b^1, (h+1)/b^1]$.

h	$\Psi_{84,\sigma}(x)$	interval I^1_h	affine subintervals
0	$83x$	$0 \leq x \leq \frac{1}{84}$	$\left(0, \frac{1}{84}\right)$

Table 1: $\Psi_{84,\sigma}(x)$ defined on intervals $I_h^1 = [h/b^1, (h+1)/b^1]$ (cont.).

h	$\Psi_{84,\sigma}(x)$	interval I_h^1	affine subintervals
1	$\max(1-x, 61x)$	$\frac{1}{84} \le x \le \frac{1}{42}$	$\left(\frac{1}{84}, \frac{1}{62}\right), \left(\frac{1}{62}, \frac{1}{42}\right)$
2	$\max(2-23x, 41x)$	$\frac{1}{42} \le x \le \frac{1}{28}$	$\left(\frac{1}{42}, \frac{1}{32}\right), \left(\frac{1}{32}, \frac{1}{28}\right)$
3	$\max(3-43x, 31x)$	$\frac{1}{28} \le x \le \frac{1}{21}$	$\left(\frac{1}{28}, \frac{3}{74}\right), \left(\frac{3}{74}, \frac{1}{21}\right)$
4	$\max(2-11x, 41x-1)$	$\frac{1}{21} \le x \le \frac{5}{84}$	$\left(\frac{1}{21}, \frac{3}{52}\right), \left(\frac{3}{52}, \frac{5}{84}\right)$
5	$\max(4-43x, 21x)$	$\frac{5}{84} \le x \le \frac{1}{14}$	$\left(\frac{5}{84}, \frac{1}{16}\right), \left(\frac{1}{16}, \frac{1}{14}\right)$
6	$\max(3-21x, 31x-1)$	$\frac{1}{14} \le x \le \frac{1}{12}$	$\left(\frac{1}{14}, \frac{1}{13}\right), \left(\frac{1}{13}, \frac{1}{12}\right)$
7	$\max(5-41x, 37x-2)$	$\frac{1}{12} \le x \le \frac{2}{21}$	$\left(\frac{1}{12}, \frac{7}{78}\right), \left(\frac{7}{78}, \frac{2}{21}\right)$
8	$\max(6-47x, 2-7x, 51x-4)$	$\frac{2}{21} \le x \le \frac{3}{28}$	$\left(\frac{2}{21}, \frac{1}{10}\right), \left(\frac{1}{10}, \frac{3}{29}\right), \left(\frac{3}{29}, \frac{3}{28}\right)$
9	$\max(5-33x, 3-15x, 29x-2, 55x-5)$	$\frac{3}{28} \le x \le \frac{5}{42}$	$\left(\frac{3}{28}, \frac{1}{9}\right), \left(\frac{1}{9}, \frac{5}{44}\right), \left(\frac{5}{44}, \frac{3}{26}\right), \left(\frac{3}{26}, \frac{5}{42}\right)$
10	$\max(5-29x, 27x-2)$	$\frac{5}{42} \le x \le \frac{11}{84}$	$\left(\frac{5}{42}, \frac{1}{8}\right), \left(\frac{1}{8}, \frac{11}{84}\right)$
11	$\max(9-57x, 5-27x, 25x-2)$	$\frac{11}{84} \le x \le \frac{1}{7}$	$\left(\frac{11}{84}, \frac{2}{15}\right), \left(\frac{2}{15}, \frac{7}{52}\right), \left(\frac{7}{52}, \frac{1}{7}\right)$
12	$\max(8-45x, 9x, 49x-6)$	$\frac{1}{7} \le x \le \frac{13}{84}$	$\left(\frac{1}{7}, \frac{4}{27}\right), \left(\frac{4}{27}, \frac{3}{20}\right), \left(\frac{3}{20}, \frac{13}{84}\right)$
13	$\max(7-35x, 15x-1)$	$\frac{13}{84} \le x \le \frac{1}{6}$	$\left(\frac{13}{84}, \frac{4}{25}\right), \left(\frac{4}{25}, \frac{1}{6}\right)$
14	$\max(4-15x, 37x-5)$	$\frac{1}{6} \le x \le \frac{5}{28}$	$\left(\frac{1}{6}, \frac{9}{52}\right), \left(\frac{9}{52}, \frac{5}{28}\right)$
15	$\max(10-47x, 3-9x, 45x-7, 61x-10)$	$\frac{5}{28} \le x \le \frac{4}{21}$	$\left(\frac{5}{28}, \frac{7}{38}\right), \left(\frac{7}{38}, \frac{5}{27}\right), \left(\frac{5}{27}, \frac{3}{16}\right), \left(\frac{3}{16}, \frac{4}{21}\right)$
16	$\max(6-23x, 62x-11)$	$\frac{4}{21} \le x \le \frac{17}{84}$	$\left(\frac{4}{21}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{17}{84}\right)$
17	$\max(6-22x, 12x-1)$	$\frac{17}{84} \le x \le \frac{3}{14}$	$\left(\frac{17}{84}, \frac{7}{34}\right), \left(\frac{7}{34}, \frac{3}{14}\right)$
18	$\max(14-58x, 9-35x, 15x-2, 60x-12)$	$\frac{3}{14} \le x \le \frac{19}{84}$	$\left(\frac{3}{14}, \frac{5}{23}\right), \left(\frac{5}{23}, \frac{11}{50}\right), \left(\frac{11}{50}, \frac{2}{9}\right), \left(\frac{2}{9}, \frac{19}{84}\right)$
19	$\max(7-24x, 23x-4, 57x-12)$	$\frac{19}{84} \le x \le \frac{5}{21}$	$\left(\frac{19}{84}, \frac{11}{47}\right), \left(\frac{11}{47}, \frac{4}{17}\right), \left(\frac{4}{17}, \frac{5}{21}\right)$
20	$\max(8-27x, 10x-1)$	$\frac{5}{21} \le x \le \frac{1}{4}$	$\left(\frac{5}{21}, \frac{9}{37}\right), \left(\frac{9}{37}, \frac{1}{4}\right)$
21	$\max(3-6x, 25x-5)$	$\frac{1}{4} \le x \le \frac{11}{42}$	$\left(\frac{1}{4}, \frac{8}{31}\right), \left(\frac{8}{31}, \frac{11}{42}\right)$
22	$\max(17-59x, 3-6x, 24x-5, 46x-11)$	$\frac{11}{42} \le x \le \frac{23}{84}$	$\left(\frac{11}{42}, \frac{14}{53}\right), \left(\frac{14}{53}, \frac{4}{15}\right), \left(\frac{4}{15}, \frac{3}{11}\right), \left(\frac{3}{11}, \frac{23}{84}\right)$
23	$\max(12-38x, 7-20x, 12x-2, 51x-13)$	$\frac{23}{84} \le x \le \frac{2}{7}$	$\left(\frac{23}{84}, \frac{5}{18}\right), \left(\frac{5}{18}, \frac{9}{32}\right), \left(\frac{9}{32}, \frac{11}{39}\right), \left(\frac{11}{39}, \frac{2}{7}\right)$
24	$\max(7-19x, 12x-2)$	$\frac{2}{7} \le x \le \frac{25}{84}$	$\left(\frac{2}{7}, \frac{9}{31}\right), \left(\frac{9}{31}, \frac{25}{84}\right)$
25	$\max(23-72x, 11-32x, 21x-5, 60x-17)$	$\frac{25}{84} \le x \le \frac{13}{42}$	$\left(\frac{25}{84}, \frac{3}{10}\right), \left(\frac{3}{10}, \frac{16}{53}\right), \left(\frac{16}{53}, \frac{4}{13}\right), \left(\frac{4}{13}, \frac{13}{42}\right)$
26	$\max(9-24x, 33x-9)$	$\frac{13}{42} \le x \le \frac{9}{28}$	$\left(\frac{13}{42}, \frac{6}{19}\right), \left(\frac{6}{19}, \frac{9}{28}\right)$

Table 1: $\Psi_{84,\sigma}(x)$ defined on intervals $I_h^1 = [h/b^1, (h+1)/b^1]$ (cont.).

h	$\Psi_{84,\sigma}(x)$	interval I_h^1	affine subintervals
27	$9-23x$	$\frac{9}{28} \le x \le \frac{1}{3}$	$\left(\frac{9}{28}, \frac{1}{3}\right)$
28	$25x-7$	$\frac{1}{3} \le x \le \frac{29}{84}$	$\left(\frac{1}{3}, \frac{29}{84}\right)$
29	$\max(22-59x, 13-33x, 21x-6)$	$\frac{29}{84} \le x \le \frac{5}{14}$	$\left(\frac{29}{84}, \frac{9}{26}\right), \left(\frac{9}{26}, \frac{19}{54}\right), \left(\frac{19}{54}, \frac{5}{14}\right)$
30	$\max(19-49x, 10-24x, 9x-2, 61x-21)$	$\frac{5}{14} \le x \le \frac{31}{84}$	$\left(\frac{5}{14}, \frac{9}{25}\right), \left(\frac{9}{25}, \frac{4}{11}\right), \left(\frac{4}{11}, \frac{19}{52}\right), \left(\frac{19}{52}, \frac{31}{84}\right)$
31	$\max(10-23x, 33x-11)$	$\frac{31}{84} \le x \le \frac{8}{21}$	$\left(\frac{31}{84}, \frac{3}{8}\right), \left(\frac{3}{8}, \frac{8}{21}\right)$
32	$\max(21-51x, 8-17x, 37x-13)$	$\frac{8}{21} \le x \le \frac{11}{28}$	$\left(\frac{8}{21}, \frac{13}{34}\right), \left(\frac{13}{34}, \frac{7}{18}\right), \left(\frac{7}{18}, \frac{11}{28}\right)$
33	$\max(20-47x, 68x-26)$	$\frac{11}{28} \le x \le \frac{17}{42}$	$\left(\frac{11}{28}, \frac{2}{5}\right), \left(\frac{2}{5}, \frac{17}{42}\right)$
34	$\max(8-16x, 18x-6, 47x-18)$	$\frac{17}{42} \le x \le \frac{5}{12}$	$\left(\frac{17}{42}, \frac{7}{17}\right), \left(\frac{7}{17}, \frac{12}{29}\right), \left(\frac{12}{29}, \frac{5}{12}\right)$
35	$\max(12-25x, 13x-4)$	$\frac{5}{12} \le x \le \frac{3}{7}$	$\left(\frac{5}{12}, \frac{8}{19}\right), \left(\frac{8}{19}, \frac{3}{7}\right)$
36	$\max(14-29x, 17x-6)$	$\frac{3}{7} \le x \le \frac{37}{84}$	$\left(\frac{3}{7}, \frac{10}{23}\right), \left(\frac{10}{23}, \frac{37}{84}\right)$
37	$\max(31-67x, 41x-17, 61x-26)$	$\frac{37}{84} \le x \le \frac{19}{42}$	$\left(\frac{37}{84}, \frac{4}{9}\right), \left(\frac{4}{9}, \frac{9}{20}\right), \left(\frac{9}{20}, \frac{19}{42}\right)$
38	$\max(12-23x, 27x-11, 40x-17)$	$\frac{19}{42} \le x \le \frac{13}{28}$	$\left(\frac{19}{42}, \frac{23}{50}\right), \left(\frac{23}{50}, \frac{6}{13}\right), \left(\frac{6}{13}, \frac{13}{28}\right)$
39	$\max(9-16x, 18x-7)$	$\frac{13}{28} \le x \le \frac{10}{21}$	$\left(\frac{13}{28}, \frac{8}{17}\right), \left(\frac{8}{17}, \frac{10}{21}\right)$
40	$\max(33-66x, 22-43x, 7x-2, 40x-18)$	$\frac{10}{21} \le x \le \frac{41}{84}$	$\left(\frac{10}{21}, \frac{11}{23}\right), \left(\frac{11}{23}, \frac{12}{25}\right), \left(\frac{12}{25}, \frac{16}{33}\right), \left(\frac{16}{33}, \frac{41}{84}\right)$
41	$\max(23-44x, 7x-2)$	$\frac{41}{84} \le x \le \frac{1}{2}$	$\left(\frac{41}{84}, \frac{25}{51}\right), \left(\frac{25}{51}, \frac{1}{2}\right)$
42	$\max(5-7x, 44x-21)$	$\frac{1}{2} \le x \le \frac{43}{84}$	$\left(\frac{1}{2}, \frac{26}{51}\right), \left(\frac{26}{51}, \frac{43}{84}\right)$
43	$\max(22-40x, 26x-12)$	$\frac{43}{84} \le x \le \frac{11}{21}$	$\left(\frac{43}{84}, \frac{17}{33}\right), \left(\frac{17}{33}, \frac{11}{21}\right)$
44	$\max(21-37x, 14x-6, 59x-30)$	$\frac{11}{21} \le x \le \frac{15}{28}$	$\left(\frac{11}{21}, \frac{9}{17}\right), \left(\frac{9}{17}, \frac{8}{15}\right), \left(\frac{8}{15}, \frac{15}{28}\right)$
45	$\max(15-25x, 21x-10, 54x-28)$	$\frac{15}{28} \le x \le \frac{23}{42}$	$\left(\frac{15}{28}, \frac{25}{46}\right), \left(\frac{25}{46}, \frac{6}{11}\right), \left(\frac{6}{11}, \frac{23}{42}\right)$
46	$\max(18-30x, 8x-3, 26x-13)$	$\frac{23}{42} \le x \le \frac{47}{84}$	$\left(\frac{23}{42}, \frac{21}{38}\right), \left(\frac{21}{38}, \frac{5}{9}\right), \left(\frac{5}{9}, \frac{47}{84}\right)$
47	$\max(34-58x, 11-17x, 29x-15)$	$\frac{47}{84} \le x \le \frac{4}{7}$	$\left(\frac{47}{84}, \frac{23}{41}\right), \left(\frac{23}{41}, \frac{13}{23}\right), \left(\frac{13}{23}, \frac{4}{7}\right)$
48	$\max(17-27x, 11x-5, 54x-30)$	$\frac{4}{7} \le x \le \frac{7}{12}$	$\left(\frac{4}{7}, \frac{11}{19}\right), \left(\frac{11}{19}, \frac{25}{43}\right), \left(\frac{25}{43}, \frac{7}{12}\right)$
49	$\max(12-18x, 16x-8)$	$\frac{7}{12} \le x \le \frac{25}{42}$	$\left(\frac{7}{12}, \frac{10}{17}\right), \left(\frac{10}{17}, \frac{25}{42}\right)$
50	$\max(42-68x, 27x-15)$	$\frac{25}{42} \le x \le \frac{17}{28}$	$\left(\frac{25}{42}, \frac{3}{5}\right), \left(\frac{3}{5}, \frac{17}{28}\right)$
51	$\max(36-57x, 51x-30, 64x-38)$	$\frac{17}{28} \le x \le \frac{13}{21}$	$\left(\frac{17}{28}, \frac{11}{18}\right), \left(\frac{11}{18}, \frac{8}{13}\right), \left(\frac{8}{13}, \frac{13}{21}\right)$
52	$\max(14-20x, 9-12x, 39x-23)$	$\frac{13}{21} \le x \le \frac{53}{84}$	$\left(\frac{13}{21}, \frac{5}{8}\right), \left(\frac{5}{8}, \frac{32}{51}\right), \left(\frac{32}{51}, \frac{53}{84}\right)$

Table 1: $\Psi_{84,\sigma}(x)$ defined on intervals $I_h^1 = [h/b^1, (h+1)/b^1]$ (cont.).

h	$\Psi_{84,\sigma}(x)$	interval I_h^1	affine subintervals
53	$\max(30-45x, 21x-12, 46x-28)$	$\frac{53}{84} \le x \le \frac{9}{14}$	$\left(\frac{53}{84}, \frac{7}{11}\right), \left(\frac{7}{11}, \frac{16}{25}\right), \left(\frac{16}{25}, \frac{9}{14}\right)$
54	$\max(26-38x, 13x-7, 33x-20, 59x-37)$	$\frac{9}{14} \le x \le \frac{55}{84}$	$\left(\frac{9}{14}, \frac{11}{17}\right), \left(\frac{11}{17}, \frac{13}{20}\right), \left(\frac{13}{20}, \frac{17}{26}\right), \left(\frac{17}{26}, \frac{55}{84}\right)$
55	$18-25x$	$\frac{55}{84} \le x \le \frac{2}{3}$	$\left(\frac{55}{84}, \frac{2}{3}\right)$
56	$23x-14$	$\frac{2}{3} \le x \le \frac{19}{28}$	$\left(\frac{2}{3}, \frac{19}{28}\right)$
57	$\max(24-33x, 24x-15)$	$\frac{19}{28} \le x \le \frac{29}{42}$	$\left(\frac{19}{28}, \frac{13}{19}\right), \left(\frac{13}{19}, \frac{29}{42}\right)$
58	$\max(43-60x, 16-21x, 32x-21, 72x-49)$	$\frac{29}{42} \le x \le \frac{59}{84}$	$\left(\frac{29}{42}, \frac{9}{13}\right), \left(\frac{9}{13}, \frac{37}{53}\right), \left(\frac{37}{53}, \frac{7}{10}\right), \left(\frac{7}{10}, \frac{59}{84}\right)$
59	$\max(10-12x, 19x-12)$	$\frac{59}{84} \le x \le \frac{5}{7}$	$\left(\frac{59}{84}, \frac{22}{31}\right), \left(\frac{22}{31}, \frac{5}{7}\right)$
60	$\max(38-51x, 10-12x, 20x-13, 38x-26)$	$\frac{5}{7} \le x \le \frac{61}{84}$	$\left(\frac{5}{7}, \frac{28}{39}\right), \left(\frac{28}{39}, \frac{23}{32}\right), \left(\frac{23}{32}, \frac{13}{18}\right), \left(\frac{13}{18}, \frac{61}{84}\right)$
61	$\max(35-46x, 19-24x, 6x-3, 59x-42)$	$\frac{61}{84} \le x \le \frac{31}{42}$	$\left(\frac{61}{84}, \frac{8}{11}\right), \left(\frac{8}{11}, \frac{11}{15}\right), \left(\frac{11}{15}, \frac{39}{53}\right), \left(\frac{39}{53}, \frac{31}{42}\right)$
62	$\max(20-25x, 6x-3)$	$\frac{31}{42} \le x \le \frac{3}{4}$	$\left(\frac{31}{42}, \frac{23}{31}\right), \left(\frac{23}{31}, \frac{3}{4}\right)$
63	$\max(9-10x, 27x-19)$	$\frac{3}{4} \le x \le \frac{16}{21}$	$\left(\frac{3}{4}, \frac{28}{37}\right), \left(\frac{28}{37}, \frac{16}{21}\right)$
64	$\max(29-36x, 11x-7, 24x-17)$	$\frac{16}{21} \le x \le \frac{65}{84}$	$\left(\frac{16}{21}, \frac{36}{47}\right), \left(\frac{36}{47}, \frac{10}{13}\right), \left(\frac{10}{13}, \frac{65}{84}\right)$
65	$\max(48-60x, 13-15x, 35x-26, 58x-44)$	$\frac{65}{84} \le x \le \frac{11}{14}$	$\left(\frac{65}{84}, \frac{7}{9}\right), \left(\frac{7}{9}, \frac{39}{50}\right), \left(\frac{39}{50}, \frac{18}{23}\right), \left(\frac{18}{23}, \frac{11}{14}\right)$
66	$\max(11-12x, 22x-16)$	$\frac{11}{14} \le x \le \frac{67}{84}$	$\left(\frac{11}{14}, \frac{27}{34}\right), \left(\frac{27}{34}, \frac{67}{84}\right)$
67	$\max(51-62x, 23x-17)$	$\frac{67}{84} \le x \le \frac{17}{21}$	$\left(\frac{67}{84}, \frac{4}{5}\right), \left(\frac{4}{5}, \frac{17}{21}\right)$
68	$\max(51-61x, 38-45x, 9x-6, 47x-37)$	$\frac{17}{21} \le x \le \frac{23}{28}$	$\left(\frac{17}{21}, \frac{13}{16}\right), \left(\frac{13}{16}, \frac{22}{27}\right), \left(\frac{22}{27}, \frac{31}{38}\right), \left(\frac{31}{38}, \frac{23}{28}\right)$
69	$\max(32-37x, 15x-11)$	$\frac{23}{28} \le x \le \frac{5}{6}$	$\left(\frac{23}{28}, \frac{43}{52}\right), \left(\frac{43}{52}, \frac{5}{6}\right)$
70	$\max(14-15x, 35x-28)$	$\frac{5}{6} \le x \le \frac{71}{84}$	$\left(\frac{5}{6}, \frac{21}{25}\right), \left(\frac{21}{25}, \frac{71}{84}\right)$
71	$\max(43-49x, 9-9x, 45x-37)$	$\frac{71}{84} \le x \le \frac{6}{7}$	$\left(\frac{71}{84}, \frac{17}{20}\right), \left(\frac{17}{20}, \frac{23}{27}\right), \left(\frac{23}{27}, \frac{6}{7}\right)$
72	$\max(23-25x, 27x-22, 57x-48)$	$\frac{6}{7} \le x \le \frac{73}{84}$	$\left(\frac{6}{7}, \frac{45}{52}\right), \left(\frac{45}{52}, \frac{13}{15}\right), \left(\frac{13}{15}, \frac{73}{84}\right)$
73	$\max(25-27x, 29x-24)$	$\frac{73}{84} \le x \le \frac{37}{42}$	$\left(\frac{73}{84}, \frac{7}{8}\right), \left(\frac{7}{8}, \frac{37}{42}\right)$
74	$\max(50-55x, 27-29x, 15x-12, 33x-28)$	$\frac{37}{42} \le x \le \frac{25}{28}$	$\left(\frac{37}{42}, \frac{23}{26}\right), \left(\frac{23}{26}, \frac{39}{44}\right), \left(\frac{39}{44}, \frac{8}{9}\right), \left(\frac{8}{9}, \frac{25}{28}\right)$
75	$\max(47-51x, 7x-5, 47x-41)$	$\frac{25}{28} \le x \le \frac{19}{21}$	$\left(\frac{25}{28}, \frac{26}{29}\right), \left(\frac{26}{29}, \frac{9}{10}\right), \left(\frac{9}{10}, \frac{19}{21}\right)$
76	$\max(35-37x, 41x-36)$	$\frac{19}{21} \le x \le \frac{11}{12}$	$\left(\frac{19}{21}, \frac{71}{78}\right), \left(\frac{71}{78}, \frac{11}{12}\right)$
77	$\max(30-31x, 23x-20)$	$\frac{11}{12} \le x \le \frac{13}{14}$	$\left(\frac{11}{12}, \frac{25}{27}\right), \left(\frac{25}{27}, \frac{13}{14}\right)$
78	$\max(58-61x, 27x-24)$	$\frac{13}{14} \le x \le \frac{79}{84}$	$\left(\frac{13}{14}, \frac{41}{44}\right), \left(\frac{41}{44}, \frac{79}{84}\right)$

Table 1: $\Psi_{84,\sigma}(x)$ defined on intervals $I_h^1 = [h/b^1, (h+1)/b^1]$ (cont.).

h	$\Psi_{84,\sigma}(x)$	interval I_h^1	affine subintervals
79	$\max(55-57x, 13x-11, 31x-28)$	$\frac{79}{84} \le x \le \frac{20}{21}$	$\left(\frac{79}{84}, \frac{33}{35}\right), \left(\frac{33}{35}, \frac{17}{18}\right), \left(\frac{17}{18}, \frac{20}{21}\right)$
80	$\max(52-53x, 31-31x, 23x-21)$	$\frac{20}{21} \le x \le \frac{27}{28}$	$\left(\frac{20}{21}, \frac{21}{22}\right), \left(\frac{21}{22}, \frac{26}{27}\right), \left(\frac{26}{27}, \frac{27}{28}\right)$
81	$\max(60-61x, 31-31x, 33x-31)$	$\frac{27}{28} \le x \le \frac{41}{42}$	$\left(\frac{27}{28}, \frac{29}{30}\right), \left(\frac{29}{30}, \frac{31}{32}\right), \left(\frac{31}{32}, \frac{41}{42}\right)$
82	$\max(51-51x, x)$	$\frac{41}{42} \le x \le \frac{83}{84}$	$\left(\frac{41}{42}, \frac{51}{52}\right), \left(\frac{51}{52}, \frac{83}{84}\right)$
83	$83-83x$	$\frac{83}{84} \le x \le 1$	$\left(\frac{83}{84}, 1\right)$

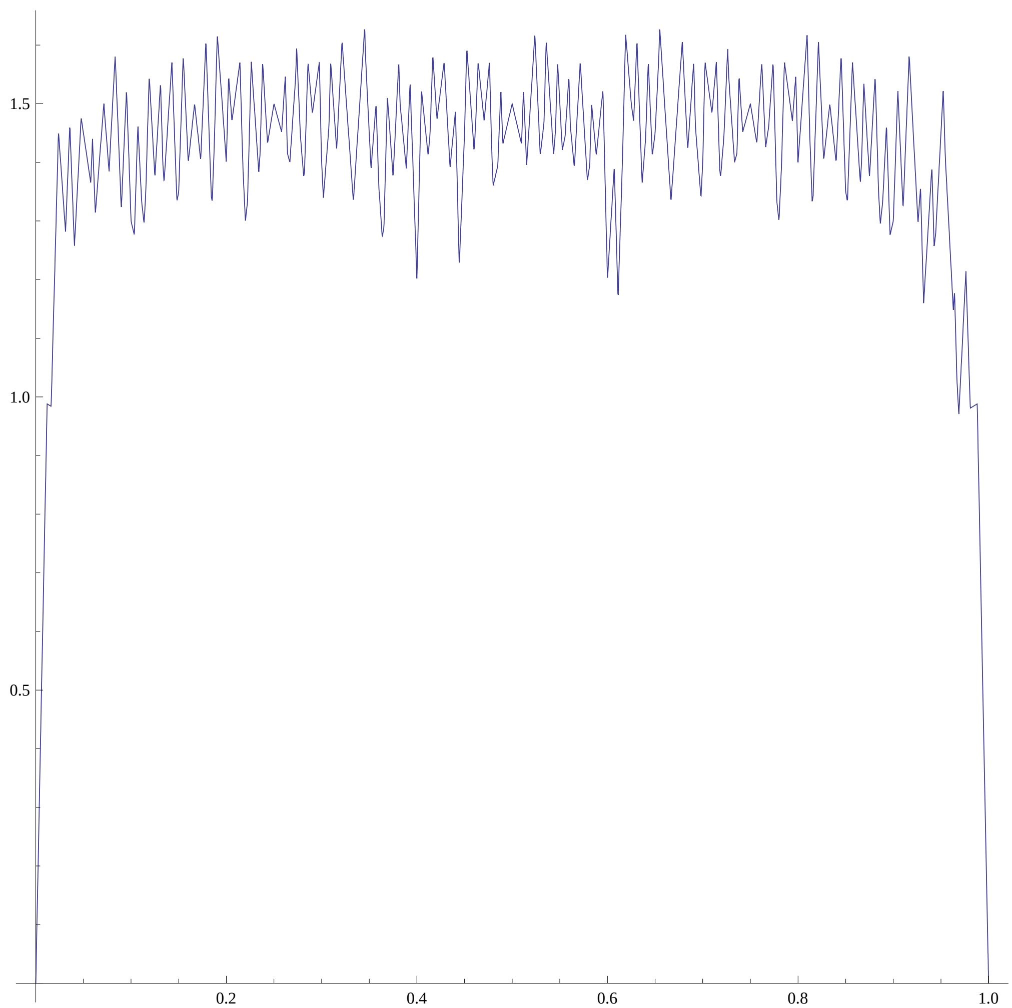

Fig. 1. Function $\Psi_{84,\sigma}(x)$, defined on interval $[0,1]$

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3.2 Proof of Theorem 4

Numerical investigations shows that there are three dominant intervals when $n = 1$: J^1_{28}, J^1_{52} and J^1_{55}. But, for higher n, there are exactly two dominant intervals. For example, when $n = 2$, the dominant intervals are J^2_{2420} and J^2_{4636}. Further numerical investigations allow us to make the following induction hypothesis: for any $n > 1$, the index h_n of dominant intervals $J^n_{h_n}$ can be expressed as follows: either

$$h_n = -\frac{16}{83} + \frac{509}{83}3^{n+1}28^{n-1}$$

or

$$h_n = \frac{16}{83} + \frac{797}{83}3^{n}28^{n-1}$$

In these intervals, F_n is the affine function in form $p_n(x - h_n/84^n) + q_n$, where the coefficients p_n and q_n are either

$$p_n = -\frac{61}{83} + \frac{703384^n}{83}; q_n = \frac{130n}{83} + \frac{1434^{2-n}21^{-n}}{6889} + \frac{11715}{192892}$$

or

$$p_n = -\frac{23}{83} + \frac{699584^n}{83}; q_n = \frac{130n}{83} + \frac{1434^{2-n}21^{-n}}{6889} + \frac{11715}{192892};$$

In both cases, $\max x \in J^n_{h_n} F_n(x) = q_n$.

Our induction hypothesis can be easily checked for $n = 1$. Let us suppose that it holds for an arbitrary $n > 1$. To check that it holds for $n+1$, we need to add $\Psi(xb^n)$ to $F_n(x)$ on $J^n_{h_n}$ and check that $F_{n+1}(x)$ is still dominant on $J^{n+1}_{h_{n+1}}$. We performed this checking for each affine subinterval of definition of the function $\Psi_{84,\sigma}(x)$, and found that, effectively, our induction hypothesis holds: the intervals $J^{n+1}_{h_{n+1}}$ are dominant.

There, we have proved that the

$$d_n = (\max_{x\in[0,1]} F_n(x)/n) = q_n$$

and

$$\alpha_{84,\sigma} = \inf_{n/geq1} d_n/n = \lim_{n\to\infty} d_n/n = 130/83.$$

Consequently,

$$s(S_{84,\sigma}) = 130/(83 \log 84) \approx 0.353494.$$

3.3 Functions $\Psi^{+}_{60,\sigma}(x)$ and $\Psi^{-}_{60,\sigma}(x)$

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For definition of $s^*(S_{60,\Sigma_A})$, we need the function $\Psi^{+}_{60,\sigma}(x)$ and $\Psi^{-}_{60,\sigma}(x)$.

The exact definition of the function $\Psi^{+}_{60,\sigma}(x)$ defined on intervals $I^1_h = [h/b^1, (h+1)/b^1]$ is presented in Table 2. Each interval I^1_h is also expressed as a set of affine subintervals. Thus, the interval $[0,1]$ is expressed as a set of 102 affine subintervals. $\Psi^{-}_{60,\sigma}(x) = 0$ on $[0,1]$. Figure 2 shows the function $\Psi^{+}_{60,\sigma}(x)$ visually.

Table 2: $\Psi^{+}_{60,\sigma}(x)$ defined on intervals $I^1_h = [h/b^1, (h+1)/b^1]$.

h	$\Psi^{+}_{60,\sigma}(x)$	interval I^1_h	affine subintervals
0	$59x$	$0 \le x \le \frac{1}{60}$	$\left(0, \frac{1}{60}\right)$
1	$\max(1-x, 44x)$	$\frac{1}{60} \le x \le \frac{1}{30}$	$\left(\frac{1}{60}, \frac{1}{45}\right), \left(\frac{1}{45}, \frac{1}{30}\right)$
2	$\max(2-16x, 29x)$	$\frac{1}{30} \le x \le \frac{1}{20}$	$\left(\frac{1}{30}, \frac{2}{45}\right), \left(\frac{2}{45}, \frac{1}{20}\right)$
3	$\max(3-31x, 19x)$	$\frac{1}{20} \le x \le \frac{1}{15}$	$\left(\frac{1}{20}, \frac{3}{50}\right), \left(\frac{3}{50}, \frac{1}{15}\right)$
4	$\max(19x, 57x-3)$	$\frac{1}{15} \le x \le \frac{1}{12}$	$\left(\frac{1}{15}, \frac{3}{38}\right), \left(\frac{3}{38}, \frac{1}{12}\right)$
5	$2-3x$	$\frac{1}{12} \le x \le \frac{1}{10}$	$\left(\frac{1}{12}, \frac{1}{10}\right)$
6	$2-3x$	$\frac{1}{10} \le x \le \frac{7}{60}$	$\left(\frac{1}{10}, \frac{7}{60}\right)$
7	$2-3x$	$\frac{7}{60} \le x \le \frac{2}{15}$	$\left(\frac{7}{60}, \frac{2}{15}\right)$
8	$\max(2-3x, 19x-1)$	$\frac{2}{15} \le x \le \frac{3}{20}$	$\left(\frac{2}{15}, \frac{3}{22}\right), \left(\frac{3}{22}, \frac{3}{20}\right)$
9	$5-21x$	$\frac{3}{20} \le x \le \frac{1}{6}$	$\left(\frac{3}{20}, \frac{1}{6}\right)$
10	$\max(2-3x, 36x-5)$	$\frac{1}{6} \le x \le \frac{11}{60}$	$\left(\frac{1}{6}, \frac{7}{39}\right), \left(\frac{7}{39}, \frac{11}{60}\right)$
11	$\max(6-24x, 2-3x)$	$\frac{11}{60} \le x \le \frac{1}{5}$	$\left(\frac{11}{60}, \frac{4}{21}\right), \left(\frac{4}{21}, \frac{1}{5}\right)$
12	$\max(7x, 36x-6)$	$\frac{1}{5} \le x \le \frac{13}{60}$	$\left(\frac{1}{5}, \frac{6}{29}\right), \left(\frac{6}{29}, \frac{13}{60}\right)$
13	$\max(7-24x, 16x-2, 29x-5)$	$\frac{13}{60} \le x \le \frac{7}{30}$	$\left(\frac{13}{60}, \frac{9}{40}\right), \left(\frac{9}{40}, \frac{3}{13}\right), \left(\frac{3}{13}, \frac{7}{30}\right)$
14	$9-31x$	$\frac{7}{30} \le x \le \frac{1}{4}$	$\left(\frac{7}{30}, \frac{1}{4}\right)$
15	$33x-7$	$\frac{1}{4} \le x \le \frac{4}{15}$	$\left(\frac{1}{4}, \frac{4}{15}\right)$
16	$\max(9-27x, 3-5x, 24x-5)$	$\frac{4}{15} \le x \le \frac{17}{60}$	$\left(\frac{4}{15}, \frac{3}{11}\right), \left(\frac{3}{11}, \frac{8}{29}\right), \left(\frac{8}{29}, \frac{17}{60}\right)$
17	$\max(12-36x, 3-5x)$	$\frac{17}{60} \le x \le \frac{3}{10}$	$\left(\frac{17}{60}, \frac{9}{31}\right), \left(\frac{9}{31}, \frac{3}{10}\right)$

Table 2: $\Psi^{+}_{60,\sigma}(x)$ defined on intervals $I^1_h = [h/b^1, (h+1)/b^1]$ (cont.).

h	$\Psi^{+}_{60,\sigma}(x)$	interval I^1_h	affine subintervals
18	$\max(3-5x, 24x-6)$	$\frac{3}{10} \le x \le \frac{19}{60}$	$\left(\frac{3}{10}, \frac{9}{29}\right)$, $\left(\frac{9}{29}, \frac{19}{60}\right)$
19	$\max(13-36x, 11x-2)$	$\frac{19}{60} \le x \le \frac{1}{3}$	$\left(\frac{19}{60}, \frac{15}{47}\right)$, $\left(\frac{15}{47}, \frac{1}{3}\right)$
20	$14x-3$	$\frac{1}{3} \le x \le \frac{7}{20}$	$\left(\frac{1}{3}, \frac{7}{20}\right)$
21	$\max(18-46x, 5-9x)$	$\frac{7}{20} \le x \le \frac{11}{30}$	$\left(\frac{7}{20}, \frac{13}{37}\right)$, $\left(\frac{13}{37}, \frac{11}{30}\right)$
22	$\max(5-9x, 7x-1)$	$\frac{11}{30} \le x \le \frac{23}{60}$	$\left(\frac{11}{30}, \frac{3}{8}\right)$, $\left(\frac{3}{8}, \frac{23}{60}\right)$
23	$7x-1$	$\frac{23}{60} \le x \le \frac{2}{5}$	$\left(\frac{23}{60}, \frac{2}{5}\right)$
24	$7-13x$	$\frac{2}{5} \le x \le \frac{5}{12}$	$\left(\frac{2}{5}, \frac{5}{12}\right)$
25	$\max(7-13x, 27x-10)$	$\frac{5}{12} \le x \le \frac{13}{30}$	$\left(\frac{5}{12}, \frac{17}{40}\right)$, $\left(\frac{17}{40}, \frac{13}{30}\right)$
26	$\max(16-33x, 12-24x, 14x-5)$	$\frac{13}{30} \le x \le \frac{9}{20}$	$\left(\frac{13}{30}, \frac{4}{9}\right)$, $\left(\frac{4}{9}, \frac{17}{38}\right)$, $\left(\frac{17}{38}, \frac{9}{20}\right)$
27	$\max(14x-5, 36x-15)$	$\frac{9}{20} \le x \le \frac{7}{15}$	$\left(\frac{9}{20}, \frac{5}{11}\right)$, $\left(\frac{5}{11}, \frac{7}{15}\right)$
28	$\max(13-24x, 14x-5)$	$\frac{7}{15} \le x \le \frac{29}{60}$	$\left(\frac{7}{15}, \frac{9}{19}\right)$, $\left(\frac{9}{19}, \frac{29}{60}\right)$
29	$24-46x$	$\frac{29}{60} \le x \le \frac{1}{2}$	$\left(\frac{29}{60}, \frac{1}{2}\right)$
30	$46x-22$	$\frac{1}{2} \le x \le \frac{31}{60}$	$\left(\frac{1}{2}, \frac{31}{60}\right)$
31	$\max(9-14x, 26x-12)$	$\frac{31}{60} \le x \le \frac{8}{15}$	$\left(\frac{31}{60}, \frac{21}{40}\right)$, $\left(\frac{21}{40}, \frac{8}{15}\right)$
32	$\max(20-34x, 3-3x)$	$\frac{8}{15} \le x \le \frac{11}{20}$	$\left(\frac{8}{15}, \frac{17}{31}\right)$, $\left(\frac{17}{31}, \frac{11}{20}\right)$
33	$\max(3-3x, 26x-13)$	$\frac{11}{20} \le x \le \frac{17}{30}$	$\left(\frac{11}{20}, \frac{16}{29}\right)$, $\left(\frac{16}{29}, \frac{17}{30}\right)$
34	$\max(21-34x, 17-27x, 13x-6)$	$\frac{17}{30} \le x \le \frac{7}{12}$	$\left(\frac{17}{30}, \frac{4}{7}\right)$, $\left(\frac{4}{7}, \frac{23}{40}\right)$, $\left(\frac{23}{40}, \frac{7}{12}\right)$
35	$13x-6$	$\frac{7}{12} \le x \le \frac{3}{5}$	$\left(\frac{7}{12}, \frac{3}{5}\right)$
36	$6-7x$	$\frac{3}{5} \le x \le \frac{37}{60}$	$\left(\frac{3}{5}, \frac{37}{60}\right)$
37	$\max(6-7x, 9x-4)$	$\frac{37}{60} \le x \le \frac{19}{30}$	$\left(\frac{37}{60}, \frac{5}{8}\right)$, $\left(\frac{5}{8}, \frac{19}{30}\right)$
38	$\max(9x-4, 26x-15)$	$\frac{19}{30} \le x \le \frac{13}{20}$	$\left(\frac{19}{30}, \frac{11}{17}\right)$, $\left(\frac{11}{17}, \frac{13}{20}\right)$
39	$11-14x$	$\frac{13}{20} \le x \le \frac{2}{3}$	$\left(\frac{13}{20}, \frac{2}{3}\right)$
40	$\max(9-11x, 29x-18)$	$\frac{2}{3} \le x \le \frac{41}{60}$	$\left(\frac{2}{3}, \frac{27}{40}\right)$, $\left(\frac{27}{40}, \frac{41}{60}\right)$
41	$23-31x$	$\frac{41}{60} \le x \le \frac{7}{10}$	$\left(\frac{41}{60}, \frac{7}{10}\right)$
42	$\max(29x-19, 36x-24)$	$\frac{7}{10} \le x \le \frac{43}{60}$	$\left(\frac{7}{10}, \frac{5}{7}\right)$, $\left(\frac{5}{7}, \frac{43}{60}\right)$
43	$\max(19-24x, 16x-10)$	$\frac{43}{60} \le x \le \frac{11}{15}$	$\left(\frac{43}{60}, \frac{29}{40}\right)$, $\left(\frac{29}{40}, \frac{11}{15}\right)$

Table 2: $\Psi^{+}_{60,\sigma}(x)$ defined on intervals $I^1_h = [h/b^1, (h+1)/b^1]$ (cont.).

h	$\Psi^{+}_{60,\sigma}(x)$	interval I^1_h	affine subintervals
44	$34 - 44x$	$\frac{11}{15} \leq x \leq \frac{3}{4}$	$\left(\frac{11}{15}, \frac{3}{4}\right)$
45	$44x - 32$	$\frac{3}{4} \leq x \leq \frac{23}{30}$	$\left(\frac{3}{4}, \frac{23}{30}\right)$
46	$\max(14 - 16x, 24x - 17)$	$\frac{23}{30} \leq x \leq \frac{47}{60}$	$\left(\frac{23}{30}, \frac{31}{40}\right)$, $\left(\frac{31}{40}, \frac{47}{60}\right)$
47	$\max(30 - 36x, 7 - 7x)$	$\frac{47}{60} \leq x \leq \frac{4}{5}$	$\left(\frac{47}{60}, \frac{23}{29}\right)$, $\left(\frac{23}{29}, \frac{4}{5}\right)$
48	$\max(7 - 7x, 29x - 22)$	$\frac{4}{5} \leq x \leq \frac{49}{60}$	$\left(\frac{4}{5}, \frac{29}{36}\right)$, $\left(\frac{29}{36}, \frac{49}{60}\right)$
49	$\max(27 - 31x, 9x - 6)$	$\frac{49}{60} \leq x \leq \frac{5}{6}$	$\left(\frac{49}{60}, \frac{33}{40}\right)$, $\left(\frac{33}{40}, \frac{5}{6}\right)$
50	$21x - 16$	$\frac{5}{6} \leq x \leq \frac{17}{20}$	$\left(\frac{5}{6}, \frac{17}{20}\right)$
51	$18 - 19x$	$\frac{17}{20} \leq x \leq \frac{13}{15}$	$\left(\frac{17}{20}, \frac{13}{15}\right)$
52	$\max(18 - 19x, 4x - 2, 36x - 30)$	$\frac{13}{15} \leq x \leq \frac{53}{60}$	$\left(\frac{13}{15}, \frac{20}{23}\right)$, $\left(\frac{20}{23}, \frac{7}{8}\right)$, $\left(\frac{7}{8}, \frac{53}{60}\right)$
53	$\max(23 - 24x, 4x - 2)$	$\frac{53}{60} \leq x \leq \frac{9}{10}$	$\left(\frac{53}{60}, \frac{25}{28}\right)$, $\left(\frac{25}{28}, \frac{9}{10}\right)$
54	$4x - 2$	$\frac{9}{10} \leq x \leq \frac{11}{12}$	$\left(\frac{9}{10}, \frac{11}{12}\right)$
55	$4x - 2$	$\frac{11}{12} \leq x \leq \frac{14}{15}$	$\left(\frac{11}{12}, \frac{14}{15}\right)$
56	$\max(54 - 56x, 24 - 24x)$	$\frac{14}{15} \leq x \leq \frac{19}{20}$	$\left(\frac{14}{15}, \frac{15}{16}\right)$, $\left(\frac{15}{16}, \frac{19}{20}\right)$
57	$\max(24 - 24x, 21x - 19)$	$\frac{19}{20} \leq x \leq \frac{29}{30}$	$\left(\frac{19}{20}, \frac{43}{45}\right)$, $\left(\frac{43}{45}, \frac{29}{30}\right)$
58	$\max(39 - 39x, x)$	$\frac{29}{30} \leq x \leq \frac{59}{60}$	$\left(\frac{29}{30}, \frac{39}{40}\right)$, $\left(\frac{39}{40}, \frac{59}{60}\right)$
59	$59 - 59x$	$\frac{59}{60} \leq x \leq 1$	$\left(\frac{59}{60}, 1\right)$

3.4 Proof of Theorem 5

In this case, $\Psi^{-}_{60,\sigma}(x) = 0$ on $[0,1]$. Consequently, $\Psi_{60,\sigma}(x) = \Psi^{+}_{60,\sigma}(x)$.

Numerical investigations shows that there are are exactly two dominant intervals when $n = 1$: J^1_{21} and J^1_{39}. When $n = 2$, the dominant intervals are J^2_{1239} and J^2_{2361}. Further numerical investigations allow us to make the following induction hypothesis: for any $n > 1$, the index h_n of dominant intervals $J^n_{h_n}$ can be expressed as follows: either

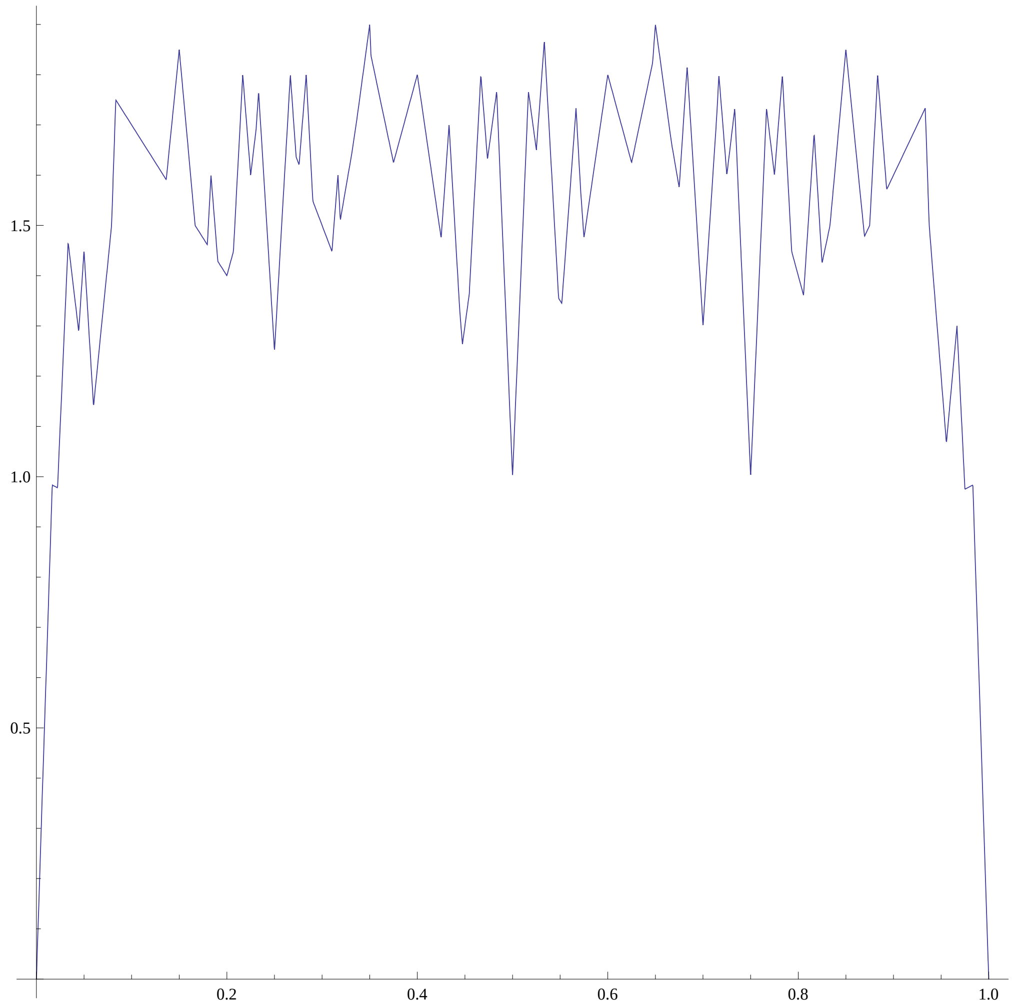

Fig. 2. Function $\Psi^{+}_{60,\sigma}(x)$, defined on interval $[0,1]$. Please note that $\Psi^{-}_{60,\sigma}(x)=0$ on $[0,1]$.

$$h_n = \frac{21}{59} + \frac{203}{59} 2^{2n-1} 3^n 5^{n-1}$$

or

$$h_n = -\frac{21}{59} + \frac{43}{59} 2^{2n-1} 3^{n+2} 5^{n-1}$$

In these intervals, F_n is the affine function in form $p_n(x - h_n/60^n) + q_n$, where the coefficients p_n and q_n are

$$p_n = \frac{14}{59}\left(-1 + 11960^n\right); q_n = \frac{2119n}{1180} - \frac{7992^{1-2n}3^{1-n}5^{-n}}{3481} + \frac{1771}{13924},$$

and, $\max x \in J^n_{h_n} F_n(x) = q_n$.

Our induction hypothesis can be easily checked for $n = 1$. Let us suppose that it holds for an arbitrary $n > 1$. To check that it holds for $n+1$, we need to add $\Psi^{+}(xb^n)$ to $F_n(x)$ on $J^n_{h_n}$ and check that $F_{n+1}(x)$ is still dominant on $J^{n+1}_{h_{n+1}}$. We performed this checking for each affine subinterval of definition of the function $\Psi^+_{60,\sigma}(x)$, and found that, effectively, our induction hypothesis holds: the intervals $J^{n+1}_{h_{n+1}}$ are dominant.

There, we have proved that the

$$d_n = (\max_{x\in[0,1]} F_n(x)/n) = q_n$$

and

$$\alpha^+_{60,\sigma} = \inf_{n/geq1} d_n/n = \lim_{n\to\infty} d_n/n = 2119/1180; \alpha^-_{60,\sigma} = 0.$$

Consequently,

$$s^*(S_{60,\sigma}) = (\alpha^+_{60,\sigma} + \alpha^-_{60,\sigma})/(2\log 60) = 2119/(2360\log 60) \approx 0.219298.$$

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