CSc 30400 Introduction to Theory of Computer Science
CSc 1st Homework Set

A. Sets

1. Exercise 0.3 of the book: Let $A = \{x, y, z\}$ and $B = \{x, y\}$
   a. Is A a subset of B?
   b. Is B a subset of A?
   c. What is $A \cup B$?
   d. What is $A \cap B$?
   e. What is $A \times B$?
   f. What is the power set of B?
2. Exercises 0.4, 0.5 of the book: If A has a elements and B has b elements, how may elements are in $A \times B$? How many elements are in the power set of A? Explain your answers.
3. In the following sentences write "true" if the sentence is true; otherwise give a counterexample. The sets X, Y, Z could be any subsets of a universal set U. Assume that the universe for Cartesian products is $U \times U$. To show that a set A is a subset of a set B, show that every member of A is also a member of B. To show that $A = B$ show that $A \subseteq B$ and $B \subseteq A$. If one direction is the reverse of the other, you don't need to prove the reverse again. Just write "similar proof".
   a. Either $X \subseteq Y$ or $Y \subseteq X$
   b. $(X \cap Y) \setminus Z = X \cap (Y \setminus Z)$
   c. $(X \cup Y) \setminus Z = X \cup (Y \setminus Z)$
   d. $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$
   e. $(X \times Y)^c = X^c \times Y^c$

B. Relations and Functions

1. Is this a function? If yes is it one-to-one? Onto? Explain your answer. Remember that a function is a relationship in which any given input has a **unique** output. A function is called 1-1 (*injection*) if every output corresponds to a unique input. A function is called onto (*surjection*) if every element of the range is the image of some element of the domain.
   a. $f_1(x) = 2x$, $f_1 : \mathbb{Z} \to E$, where $E = \{x | x \text{ is an even integer}\}$

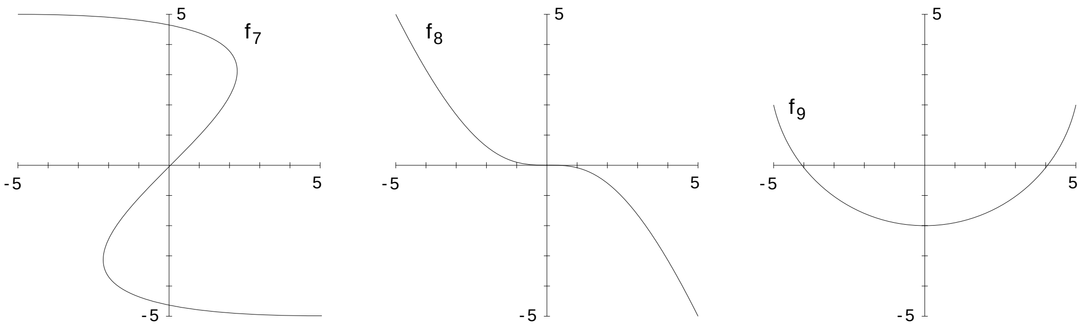

Figure 1: f_7, f_8, f_9 of exercise B(1)g

b. $f_2(x) = x^2$, $f_2 : \mathbb{N} \to \mathbb{N}$

c. $f_3(x) = |x|$, $f_3 : \mathbb{Z} \to \mathbb{N}$

d. $f_4(x) = \begin{cases} -x, & x \leq 0 \\ x, & x \geq 0 \end{cases}$, $f_4 : \mathbb{Z} \to \mathbb{N}$

e. $f_5(x) = \begin{cases} -x, & x \leq 1 \\ x, & x \geq 1 \end{cases}$, $f_5 : \mathbb{Z} \to \mathbb{N}$

f. $f_6 = \sin x$, $f_6 : \mathbb{R} \to [0, 1]$

g. $f_7, f_8, f_9 : [-5, 5] \to [-5, 5]$, see figure 1

2. Determine whether the given relations are equivalence relations on $\{1, 2, 3, 4, 5\}$. If yes list the equivalence classes. Drawing a graph will help you. Draw a graph with 5 vertices. Draw a directed edge from u to v if (u, v) is true -belongs in the domain of the relation. If every vertex has a loop (reflexive), for every edge (u, v) (v, u) is also an edge (symmetric) and for every pair of edges (u, v) and (v, w) (u, w) is also an edge (transitive), then the relation is an equivalence relation. The equivalence classes are the connected components of the graph.

   a. {(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,1)}
   b. {(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,1), (3,5), (5,3)}
   c. {(1,1), (2,2), (3,3), (4,4)}
   d. {(1,1), (2,2), (3,3), (4,4), (5,5), (1,5), (5,1), (3,5), (5,3), (1,3), (3,1)}

3. As mentioned in the previous question a binary relation $R : A \times A \to \{true, false\}$ can be modeled by a multi digraph (a directed graph with loops). The vertices of the graph will be the members of A and the edges will be pairs belonging in the domain $A \times A$. Observe that if the relation is an equivalence relation then the connected components of the graph (the connected "pieces" which form the graph) will be complete digraphs with a loop on every vertex (complete digraph is a

graph where every ordered pair of vertices (u, v) has a directed edge uv). Can you explain why?

C. Graphs

1. Exercise 0.8 from book: Consider the undirected graph $G(V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{12, 23, 13, 24, 14\}$. Draw the graph G. What is the degree of node 1? Of node 3? Indicate a path from node 3 to node 4.
2. Exercise 0.12 from book

Figure 2: The pigeonhole principle

a. Find the largest and the smallest possible degree of a node in a connected graph.

b. The pigeonhole principle is the following: If you have n pigeons (items) and $n - 1$ holes then at least 2 pigeons are going to be in the same hole (see figure 2). Show by using question C(2)a and the pigeonhole principle that every connected graph with 2 or more nodes contains 2 nodes that have equal degrees.

c. Show that this holds for every graph (for a disconnected graph show that at least one connected component has 2 nodes with equal degrees unless every connected component is a single vertex -concider what happens in that case).

D. BOOLEAN LOGIC

1. Show by constructing a truth table that de Morgan's Law is a tautology. In other words, you should prove that $\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$. Complete the truth table that is shown below and conclude that for every possible combination of the values of p and q the value of the

sentence above is always true.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$
$true$	$true$						
$true$	$false$						
$false$	$true$						
$false$	$false$						

E. Proofs

1. Prove (by construction) that the square of any integer number $\mod 3$ is not equal to 2. In other words prove that for every integer n, n^2 is either of the form $n^2 = 3k$ or $n^2 = 3k + 1$.
2. In class we proved that $\sqrt{2}$ is an irrational number.
   a. Prove that $\sqrt{3}$ is also an irrational number.
   b. Prove that for any prime p, $\sqrt{p}$ is an irrational number.
   c. Is this statement true for every natural number? Show a counterexample. Why the proof for prime numbers doesn't work in this case?
   d. Can you define the subset of the natural numbers for which the statement is true?
3. Prove by construction that for every natural number $n \geq 5$ there exists a graph of n nodes that is 4-regular.
4. Prove by induction that $1 + 3 + 5 + \ldots + 2n - 1 = n^2$
5. Prove by induction that any graph with $n \geq 3$ vertices of min. degree 2 contains at least one cycle (inductive step: remove one vertex of degree 2 and see what happens).