Termination

DP

Definition.
Given a TRS R,
$\mathcal{F} = D \uplus C \quad D : \{f \in \mathcal{F} \mid \exists (l \to r) \in R, l(\Lambda) = f\}$
D : defined (functions) C : constructors (data)

Definition.
Given a rule $l \to r$,
Dependency pair: couple $\langle u, v \rangle$

- $u = l$,
- $v = r|_p$ such that $r(p) \in D$.

Set of dependency pairs of a TRS R: DP(R).

Termination

DP

Ex.

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & & \\ \# + x & \to & x & & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right\}$$

Termination

DP

Ex.
$D = \{0, +\} \quad C = \{\#, 1\}$

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & & \\ \# + x & \to & x & & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right\}$$

$\langle x1 + y1, x + y \rangle \qquad \langle x1 + y0, x + y \rangle$
$\langle x0 + y1, x + y \rangle \qquad \langle x0 + y0, x + y \rangle$
$\langle x0 + y0, (x+y)0 \rangle \qquad \langle x1 + y1, (x+y) + \#1 \rangle$
$\langle x1 + y1, ((x+y)+\#1)0 \rangle$

Termination

DP

Ex.
$\{f(f(x)) \to f(g(f(x)))\} \quad D = \{f\} \quad C = \{g\}$
$\langle f(f(x)), f(g(f(x))) \rangle \quad \langle f(f(x)), f(x) \rangle$

what about derivations?

Termination

DP

Ex.

$\{f(f(x)) \to f(g(f(x)))\} \quad D = \{f\} \quad C = \{g\}$

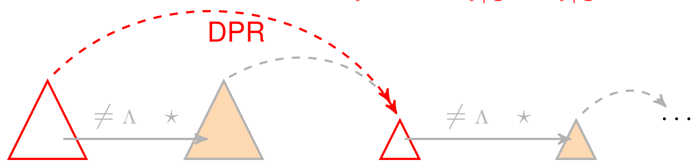
$\langle f(f(x)), f(g(f(x)))\rangle \quad \langle f(f(x)), f(x)\rangle$

Definition.

Dependency chain: sequence of DP, subst. σ such that

$$\cdots \quad \langle u_i, \quad v_i\rangle \qquad \langle u_{i+1}, \quad v_{i+1}\rangle \quad \cdots$$

$$v_i\sigma \xrightarrow{\neq\Lambda\star} u_{i+1}\sigma \quad v_{i+1}\sigma$$



Termination

DP

Ex.

$\{f(f(x)) \to f(g(f(x)))\} \quad D = \{f\} \quad C = \{g\}$

$\langle f(f(x)), f(g(f(x)))\rangle \quad \langle f(f(x)), f(x)\rangle$

Definition.

Dependency chain: sequence of DP, subst. σ such that

$$\cdots \quad \langle u_i, \quad v_i\rangle \qquad \langle u_{i+1}, \quad v_{i+1}\rangle \quad \cdots$$

$$v_i\sigma \xrightarrow{\neq\Lambda\star} u_{i+1}\sigma$$

Theorem. (A & G)

SN$(\to_R)$ $\Leftrightarrow$ no infinite chain over DP(R)

Rephrased: SN$(\to_R)$ $\Leftrightarrow$ SN$(\twoheadrightarrow_{\text{DP}(R),R})$

Termination

DP

Rk. — $R = \{f(f(x)) \to h(f(x)),\ g(x) \to f(x)\} \qquad D = \{f, g\} \quad C = \{h\}$

DP$(R) = \{\langle f(f(x)), f(x)\rangle,\ \langle g(x), f(x)\rangle\}$

SN$(\to_R)$? $\rightsquigarrow$ SN$(\xrightarrow[R]{\neq\Lambda\star} \cdot \xrightarrow[\text{DP}(R)]{})$? $\qquad$ ($\rightsquigarrow$ SN$(\twoheadrightarrow_{\text{DP}(R),R})$?)

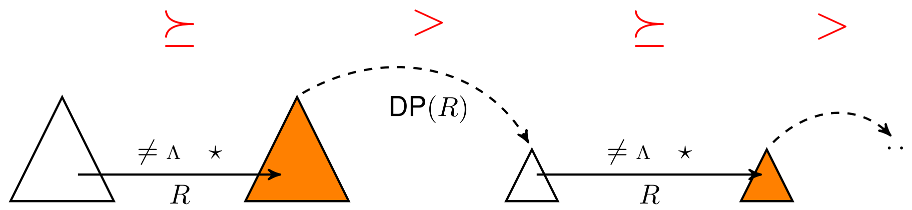
With minimal chains...

$f(f(x))\sigma$ NOT minimal: irrelevant

SN$(\to_R)$ $\Leftrightarrow$ SN$(\twoheadrightarrow_{\langle g(x), f(x)\rangle, R})$

Termination

DP, control



Theorem. (A & G)

If $(\succeq, >)$ such that

1. $\succeq \cdot > \subseteq >$, WF$(<)$, $\succeq$ monotone, stable, $>$ stable, (monotony useless)
2. $l \succeq r$ for each $l \to r \in R$,
3. $u > v$ for each $\langle u, v\rangle \in$ DP(R),

then SN$(\to_R)$

Termination — DP, marks

Recursive calls: decrease of arguments

$\rightsquigarrow$ distinction between function *symbol* and recursive call: marks

Ex.

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & & \\ \# + x & \to & x & & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right\}$$

$\langle x1 \,\widehat{+}\, y1, x \,\widehat{+}\, y\rangle$ $\quad$ $\langle x1 \,\widehat{+}\, y0, x \,\widehat{+}\, y\rangle$

$\langle x0 \,\widehat{+}\, y1, x \,\widehat{+}\, y\rangle$ $\quad$ $\langle x0 \,\widehat{+}\, y0, x \,\widehat{+}\, y\rangle$

$\langle x0 \,\widehat{+}\, y0, (x+y)\widehat{0}\rangle$ $\quad$ $\langle x1 \,\widehat{+}\, y1, (x+y) \,\widehat{+}\, \#1\rangle$

$\langle x1 \,\widehat{+}\, y1, ((x+y)+\#1)\widehat{0}\rangle$

Termination — DP, marks

$[\![\#]\!] = 0$ $\quad$ $[\![0]\!](x) = x + 1$ $\quad$ $\neq$ $\quad$ $[\![\widehat{0}]\!](x) = 0$ $\quad$ non mono

$[\![1]\!](x) = x + 1$ $\quad$ $[\![+]\!](x,y) = x$ $\quad$ non mono $\quad$ $[\![\widehat{+}]\!](x,y) = x$ $\quad$ non mono

Ex.

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & & \\ \# + x & \to & x & & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right\}$$

$\langle x1 \,\widehat{+}\, y1, x \,\widehat{+}\, y\rangle$ $\quad$ $\langle x1 \,\widehat{+}\, y0, x \,\widehat{+}\, y\rangle$

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$\langle x0 \,\widehat{+}\, y0, (x+y)\widehat{0}\rangle$ $\quad$ $\langle x1 \,\widehat{+}\, y1, (x+y) \,\widehat{+}\, \#1\rangle$

$\langle x1 \,\widehat{+}\, y1, ((x+y)+\#1)\widehat{0}\rangle$

Termination — DP, marks

Ex.

$\{f(f(x)) \to f(g(f(x)))\}$

$[\![g]\!](x) = 0$

$[\![f]\!](x) = 1$

$[\![\widehat{f}]\!](x) = x$

Termination — DP, graphs

Already noticed: not anything after $\langle u_i, v_i\rangle$

Coarse: $v_i\sigma \xrightarrow[R]{\neq\Lambda\star} u_j\sigma \Rightarrow v_i(\Lambda) \equiv u_j(\Lambda)$

For R finite, DP(R) finite $\rightsquigarrow$ finite graph of the relation linking DP

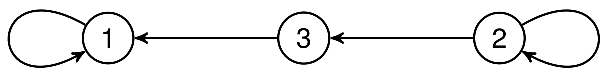
Ex.

$$\left\{\begin{array}{rcl} x - 0 & \to & x \\ s(x) - s(y) & \to & x - y \\ 0 \div s(y) & \to & 0 \\ s(x) \div s(y) & \to & s((x-y) \div s(y)) \end{array}\right.$$

$\langle s(x) \,\widehat{-}\, s(y), x \,\widehat{-}\, y\rangle$ 1

$\langle s(x) \,\widehat{\div}\, s(y), (x-y) \,\widehat{\div}\, s(y)\rangle$ 2

$\langle s(x) \,\widehat{\div}\, s(y), x \,\widehat{-}\, y\rangle$ 3



Termination

DP, graphs

Now: FINITE sytems

Chain $\mapsto$ path $\rightsquigarrow$ chain $\infty \mapsto$ path ∞, here strongly connected part (SCP)

Chains in SCP independently controlled

Ex.

$R = \{f(f(x)) \to h(f(x)),\ g(x) \to f(x)\}$

$\mathsf{SN}(\to_R) \Leftrightarrow \mathsf{SN}(\twoheadrightarrow_{\langle g(x), f(x)\rangle, R})$

No circuit: trivial problem, OK

Termination

DP, graphs

- One relation per SCP $\rightsquigarrow$ one ordering (proof) per SCP
- SCP $\neq$ composante
- SCP $\neq$ elementary circuits

  $\{f(0) \to g(1) \quad f(1) \to g(0) \quad g(x) \to f(x)\}$

Termination

DP, graphs

- One relation per SCP $\rightsquigarrow$ one ordering (proof) per SCP
- SCP $\neq$ component
- SCP $\neq$ elementary circuits

  $\{f(0) \to g(1) \quad f(1) \to g(0) \quad g(x) \to f(x)\}$
- Automation: graph NOT computable $\rightsquigarrow$ approximations
  - Coarse: head symbol
  - Finer: discriminate with constructor cap (REN/CAP)

    CAP: fresh variable for each defined symbol, REN: renaming, s connectable to t if REN(CAP(s)) and t unify

Theorem.

Correct: $\langle u, v\rangle \longrightarrow \langle u', v'\rangle$ only if v connectable to u'

Termination

DP, graphs

Theorem. (A,G & O)

R TRS, G graph of R, $G = \bigcup_{i=0}^{k-1} G_i$ where $G_i \subseteq \mathsf{DP}(R)$ SCP, then $(\forall i \in [0..k-1]\ \mathsf{SN}(\twoheadrightarrow_{G_i, R})) \Leftrightarrow \mathsf{SN}(\twoheadrightarrow_{\mathsf{DP}(R), R})$

All SCP: utmost expensive $\rightsquigarrow$ by components?

Corollaire.

If $\forall G_i^{max}, \exists(\succeq, >)$ with usual good properties such that:

- $\forall \langle u, v\rangle \in G_i^{max}, u > v$
- $\forall (l \to r) \in R, l \succeq r$

then $\mathsf{SN}(\twoheadrightarrow_{\mathsf{DP}(R), R})$

Termination — orderings

What kind of orderings?

- semantic orderings (interpretations)
  - Integers
  - Polynomials
  - ...
- syntactic orderings (precedences extended to terms)
  - LPO
  - MPO
  - RPO
  - ...
- By transformation

Orderings — semantic

$D \neq \emptyset$ equipped with $\geq_D$ and $>_D = \geq_D - \leq_D$

$\varphi : t \in \mathcal{T}(\mathcal{F}, \emptyset) \mapsto d \in D$

$\succeq_\varphi$ and $>_\varphi$:

$$t_1 \succeq_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) \geq_D \varphi(t_2)$$
$$t_1 >_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) >_D \varphi(t_2)$$

Well-founded if $>_D$ well-founded (converse)

Extension to terms with variables: $\varphi : t \in \mathcal{T}(\mathcal{F}, X) \mapsto d \in (X \to D) \to D$

$\succeq_\varphi$ and $>_\varphi$:

$$t_1 \succeq_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) \succeq_{D,X} \varphi(t_2) \quad (\forall \rho : (X \to D), \varphi(t_1)(\rho) \geq_D \varphi(t_2)(\rho))$$
$$t_1 >_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) >_{D,X} \varphi(t_2) \quad (\forall \rho : (X \to D), \varphi(t_1)(\rho) >_D \varphi(t_2)(\rho))$$