Term Algebras, Rewriting

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M2IF – Automated deduction

First order terms

Signature: $(\mathcal{S}, \mathcal{F}, \tau)$

- $\mathcal{S}$: set $\neq \emptyset$ of sorts
- $\mathcal{F}$: set of symbols
- τ: function $\mathcal{F} \to \mathcal{S}^{\mathbb{N}_+}$,

$$f \in \mathcal{F} \mapsto s_1 \times \ldots \times s_n \to s$$

n: arity of f

Arity 0: constants

First order terms

$(\mathcal{S}, \mathcal{F}, \tau)$,
$X = \cup_{s \in \mathcal{S}} X_s$: set of variables

$\mathcal{T}(\mathcal{F}, X)$: smallest set such that

- $x \in X_s$ term of sort s
- $f \in \mathcal{F}$, $f : s_1 \times \ldots \times s_n \to s$,
  $t_1 : s_1, \ldots, t_n : s_n$ terms
  then $f(t_1, \ldots, t_n)$ term of sort s

Ground terms: $\mathcal{T}(\mathcal{F}, \emptyset)$

First order subterms

Terms seen as trees $\rightsquigarrow$ positions

Subterm of t at position p, $t|_p$, defined by set of positions:
$\{q \in \mathbb{N}_+^* \mid p \cdot q \in \mathcal{P}os(t)\}$ $\quad t|_p(q) = t(p \cdot q)$

Subterm relation: $t \rhd s$ $\quad$ if $\exists p \neq \Lambda$ such that $t|_p = s$ $\quad$ (proper subterm)

For $t|_p$ $(p \in \mathcal{P}os(t))$ and u of same sort, Replacement $t[u]_p$, defined by:
$\{q \in \mathbb{N}_+^* \mid q \in \mathcal{P}os(t) \bigwedge p \not\leq_{\text{pref.}} q\} \cup \{p \cdot q \mid q \in \mathcal{P}os(u)\}$

$$t[u]_p(q) = t(q) \qquad \text{if } q \in \mathcal{P}os(t) \bigwedge p \not\leq_{\text{pref.}} q$$
$$t[u]_p(p \cdot q) = u(q)$$

First order — substitutions

Substitution: application $X \to \mathcal{T}(\mathcal{F}, X)$ respecting sorts

Usually: identity except on a finite set

Notation postfix: $\sigma(x) \rightsquigarrow x\sigma$

Extended to terms by unique $H_\sigma : \mathcal{T}(\mathcal{F}, X) \to \mathcal{T}(\mathcal{F}, X)$

- $H_\sigma(x) = x$ if x not in σ's domain
- $H_\sigma(x) = x\sigma$ if x in σ's domain
- $H_\sigma(f(s_1, \ldots, s_m)) = f(H_\sigma(s_1), \ldots, H_\sigma(s_m))$ if $f(s_1, \ldots, s_m) \in \mathcal{T}(\mathcal{F}, X)$

Renaming: equivalence relation — akin to α-conversion

$$\sigma = \{x_1 \mapsto y_1, \cdots, x_n \mapsto y_n\} \qquad y_i \text{ pairwise distincts}$$

First order — substitutions

Abuse: substitution $\rightsquigarrow$ extended substitution

Matching: for s t terms, finding σ such that $s\sigma = t$

Unification: for s t terms, finding σ such that $s\sigma = t\sigma$

- Decidable (1st order, e.g. Robinson)
- If unifiable: unique most general unifier (vanilla)

A hint of semantics — $\mathcal{F}$-algebra

For $(\mathcal{S}, \mathcal{F}, \tau)$ a signature, $\mathcal{F}$-algebra:

- Support $A_s \neq \emptyset$ for each $s \in \mathcal{S}$
- Application $f_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$
  for all $f \in \mathcal{F}, f : s_1 \times \cdots \times s_n \to s$ — $\mathcal{T}(\mathcal{F}, X)$ $\mathcal{F}$-algebra

For A B $\mathcal{F}$-algebras, homomorphism from A to B:
set of applications $h_s : A_s \to B_s$ such that for all $f : s_1 \times \cdots s_n \to s$

$$\forall a_1, \cdots, a_n \in A, \quad h_s(f_A(a_1, \cdots, a_n)) = f_B(h_{s_1}(a_1), \cdots, h_{s_n}(a_n))$$

A-assignment: homomorphism from $\mathcal{T}(\mathcal{F}, X)$ to A

Congruence (equi. relation compatible with A) $\rightsquigarrow$ Quotient

A hint of semantics — $\mathcal{F}$-algebra

Equation: pair of terms of same sort, $s = t$ — set of equations E

Model: A $\mathcal{F}$-algebra, $A \models E$ if $\forall s = t \in E$, $\forall A$-assignment σ, $s\sigma = t\sigma$

$=_E$ smallest congruence on $\mathcal{T}(\mathcal{F}, X)$ such that $\forall \sigma, \forall s = t \in E, s\sigma =_E t\sigma$

$s = t$ an equation, word problem related to $s = t$: $\quad E \models^? s = t$

To solve it: equational reasoning

Starting from E, uses of: *Reflexivity, Symmetry, Transitivity, Replacement*
If derivation: $E \vdash s = t$

[(Birkoff)] if at least a ground term per sort,
$E \vdash s = t \quad \Leftrightarrow \quad E \models s = t \quad \Leftrightarrow \quad s =_E t$

Term Rewriting — system, relation

Rewriting rule: couple of terms $s \to t$ (oriented equation)

$$s \xrightarrow[l \to r, \sigma]{p} t \quad \text{if} \quad s|_p \equiv l\sigma \quad t \equiv s[r\sigma]_p$$

Rewriting system: set of rules

Relation $\xrightarrow[R]{}$: $s \xrightarrow[R]{} t$ iff $\exists l \to r \in R, \exists p \in \mathcal{P}os(s), \exists \sigma, s \xrightarrow[l \to r, \sigma]{p} t$

Actually, system extension by *monotony* and *stability*

Reflexive/transitive closure: $\xrightarrow[R]{}^{\star}$ Transitive closure: $\xrightarrow[R]{}^{+}$

Converse: $\xleftarrow[R]{}$ Reflexive/symmetric/transitive: $\overset{\star}{\underset{R}{\longleftrightarrow}}$

Term Rewriting — example

Binary arithmetics Encoding of 6 : $\#110$

- Variables: $X = \{x \,;\, y \cdots\}$
- Signature: $\mathcal{F} = \{\# \,;\, 0 \,;\, 1 \,;\, +\}$ (0 and 1 postfix unary, $+$ infix binary)
- Set of rules:

$$\begin{cases} \#0 \to \# & \\ \# + x \to x & x + \# \to x \\ x0 + y0 \to (x+y)0 & x1 + y0 \to (x+y)1 \\ x0 + y1 \to (x+y)1 & x1 + y1 \to ((x+y) + \#1)0 \end{cases}$$

Relation $\to$ monotone: if $s \to t$ then $C[s] \to C[t]$

stable: if $s \to t$ then $s\sigma \to t\sigma$

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$$\#10 + \#1$$

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$$\#10 + \#1 \to (\#1 + \#)1$$

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$$\#10 + \#1 \to (\#1 + \#)1 \to (\#1)1$$

Term Rewriting — example

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$$\#10 + \#1 \to (\#1 + \#)1 \to \#11$$ Stop

Term Rewriting — properties

Computing power: Turing complete

Questions:

- Existence of result ⇝ termination
- Unicity of result ⇝ confluence, convergence

Term Rewriting — termination

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- Existence of result ⇝ termination
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t normal form for R: no u such that $t \underset{R}{\to} u$

t normal form of s for R: no u such that $t \underset{R}{\to} u$ and $s \underset{R}{\to}^{\star} t$

⇝ s normalisable

System normalising: every term normalisable

System strongly normalising: every derivation ⇝ normal form
(every term accessible for $\underset{R}{\to}$)

Term Rewriting — confluence

Questions:

- Existence of result $\rightsquigarrow$ terminaison
- Unicity of result $\rightsquigarrow$ confluence, convergence

R Church-Rosser: $u \stackrel{\star}{\longleftrightarrow} v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

R confluent: $u \leftarrow^\star s \rightarrow^\star v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

R locally confluent: $u \leftarrow s \rightarrow v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

Term Rewriting — confluence

Example.

$$\begin{array}{lcl} x + 0 & \rightarrow & x \\ x + x^{-1} & \rightarrow & 0 \\ (x+y)+z & \rightarrow & x + (y+z) \end{array}$$

$$x + (x^{-1} + z) \ \leftarrow\ (x + x^{-1}) + z \ \rightarrow\ 0 + z$$

Not confluent

Term Rewriting — confluence

Questions:

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$\Updownarrow$?

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Term Rewriting — confluence

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$\not\Updownarrow$

R locally confluent: $u \leftarrow s \rightarrow v \Rightarrow \exists t,\ u \rightarrow^\star t \leftarrow^\star v$

Theorem. (Lemme de Newman)
Local confluence $\Leftrightarrow$ confluence for strongly normalising relations

Term Rewriting — confluence

Decidability

Given: finite system R
Question: R confluent?

Undecidable (red. word pb.)

Term Rewriting — confluence

Critical pair : $r\rho\sigma = (l\rho[d]_p)\sigma$ where

- $l \to r \in R, \quad g \to d \in R$
- p position non variable of l
- ρ renaming of l
- σ most general unifier of $l|_p$ and g (rule superposition)

Set of critical pairs of R: $\mathrm{CP}(R)$

Theorem.
Local confluence of pairs of $\mathrm{CP}(R)$ $\Leftrightarrow$ locale confluence of R

$\rightsquigarrow$ Decidable for strongly normalising finite systems

Termination — overview

Fundamental property:

- Inductions,
- Totality of functions,
- Preliminary,
- Selfstabilisation, liveness, etc.

In this lecture: first order term rewriting
Enough? Turing complete (cfr poly)

Termination — overview

Automation? $\rightsquigarrow$ correct, incomplete...

Always difficult

- $f(f(x)) \to f(x)$
- $f(a, b, x) \to f(x, x, x)$

$\mathsf{ef}(x)[y]_{\mathtt{t}}$	$\rightarrow$	$\mathsf{ef}(x[y]_{\mathtt{t}})$
$\mathsf{Pe}(x)[y]_{\mathtt{f}}$	$\rightarrow$	$\mathsf{Pe}(x[y]_{\mathtt{t}})$
$(f \vee g)[s]_{\mathtt{f}}$	$\rightarrow$	$(f[s]_{\mathtt{f}}) \vee (g[s]_{\mathtt{f}})$
$\neg(f)[s]_{\mathtt{f}}$	$\rightarrow$	$\neg(f[s]_{\mathtt{f}})$
$(f \wedge g)[s]_{\mathtt{f}}$	$\rightarrow$	$(f[s]_{\mathtt{f}}) \wedge (g[s]_{\mathtt{f}})$
$x[i]_{\mathtt{t}}\, d$	$\rightarrow$	x
$(f \Rightarrow g)[s]_{\mathtt{f}}$	$\rightarrow$	$(f[s]_{\mathtt{f}}) \Rightarrow (g[s]_{\mathtt{f}})$
$1[x \cdot s]_{\mathtt{t}}$	$\rightarrow$	x
$\exists(f)[s]_{\mathtt{f}}$	$\rightarrow$	$\exists(f[1 \cdot (s \circ \uparrow)]_{\mathtt{f}})$
$f[i]_{\mathtt{f}}\, d$	$\rightarrow$	f
$\forall(f)[s]_{\mathtt{f}}$	$\rightarrow$	$\forall(f[1 \cdot (s \circ \uparrow)]_{\mathtt{f}})$
$(f[s]_{\mathtt{f}})[t]_{\mathtt{f}}$	$\rightarrow$	$f[s \circ t]_{\mathtt{f}}$
$(x[s]_{\mathtt{t}})[t]_{\mathtt{t}}$	$\rightarrow$	$x[s \circ t]_{\mathtt{t}}$
$id \circ s$	$\rightarrow$	s
$\uparrow \circ (x \cdot s)$	$\rightarrow$	s
$(s \circ t) \circ u$	$\rightarrow$	$s \circ (t \circ u)$
$(x \cdot s) \circ t$	$\rightarrow$	$(x[t]_{\mathtt{t}}) \cdot (s \circ t)$
$s \circ id$	$\rightarrow$	s
$1 \cdot \uparrow$	$\rightarrow$	id
$(1[s]_{\mathtt{t}}) \cdot (\uparrow \circ s)$	$\rightarrow$	s
$a, \triangledown$	$\rightarrow$	a
a, a	$\rightarrow$	a
$a \bullet \Diamond$	$\rightarrow$	a
$a \bullet a$	$\rightarrow$	a
$\neg(\neg(f))$	$\rightarrow$	f
$f \wedge f$	$\rightarrow$	f

$f \vee f$	$\rightarrow$	f
$f \Rightarrow g$	$\rightarrow$	$\neg(f) \vee g$
$\exists(f)$	$\rightarrow$	$\neg(\forall(\neg(f)))$
$(a, \{_{\mathtt{f}} \neg(f)\}) \vdash b$	$\rightarrow$	$a \vdash (\{_{\mathtt{f}} f\}, b)$
$\{_{\mathtt{f}} \neg(f)\} \vdash b$	$\rightarrow$	$\triangledown \vdash (b, \{_{\mathtt{f}} f\})$
$a \vdash (\{_{\mathtt{f}} \neg(f)\}, b)$	$\rightarrow$	$(a, \{_{\mathtt{f}} f\}) \vdash b$
$a \vdash \{_{\mathtt{f}} \neg(f)\}$	$\rightarrow$	$(a, \{_{\mathtt{f}} f\}) \vdash \triangledown$
$(a, \{_{\mathtt{f}} f \wedge g\}) \vdash b$	$\rightarrow$	$(a, \{_{\mathtt{f}} f\}, \{_{\mathtt{f}} g\}) \vdash b$
$\{_{\mathtt{f}} f \wedge g\} \vdash b$	$\rightarrow$	$(\{_{\mathtt{f}} f\}, \{_{\mathtt{f}} g\}) \vdash b$
$a \vdash (\{_{\mathtt{f}} f \vee g\}, b)$	$\rightarrow$	$a \vdash (\{_{\mathtt{f}} f\}, \{_{\mathtt{f}} g\}, b)$
$a \vdash \{_{\mathtt{f}} f \vee g\}$	$\rightarrow$	$a \vdash (\{_{\mathtt{f}} f\}, \{_{\mathtt{f}} g\})$
$\{_{\mathtt{s}} a \vdash (\{_{\mathtt{f}} f \wedge g\}, b)\}$	$\rightarrow$	$\{_{\mathtt{s}} a \vdash (\{_{\mathtt{f}} f\}, b)\} \bullet \{_{\mathtt{s}} a \vdash (\{_{\mathtt{f}} g\}, b)\}$
$\{_{\mathtt{s}} a \vdash \{_{\mathtt{f}} f \wedge g\}\}$	$\rightarrow$	$\{_{\mathtt{s}} a \vdash \{_{\mathtt{f}} f\}\} \bullet \{_{\mathtt{s}} a \vdash \{_{\mathtt{f}} g\}\}$
$\{_{\mathtt{s}} (a, \{_{\mathtt{f}} f \vee g\}) \vdash b\}$	$\rightarrow$	$\{_{\mathtt{s}} (a, \{_{\mathtt{f}} f\}) \vdash b\} \bullet \{_{\mathtt{s}} (a, \{_{\mathtt{f}} g\}) \vdash b\}$
$\{_{\mathtt{s}} \{_{\mathtt{f}} f \vee g\} \vdash b\}$	$\rightarrow$	$\{_{\mathtt{s}} \{_{\mathtt{f}} f\} \vdash b\} \bullet \{_{\mathtt{s}} \{_{\mathtt{f}} g\} \vdash b\}$
$\{_{\mathtt{s}} (a, \{_{\mathtt{f}} f\}) \vdash (\{_{\mathtt{f}} f\}, b)\}$	$\rightarrow$	$\Diamond$
$\{_{\mathtt{s}} (a, \{_{\mathtt{f}} f\}) \vdash \{_{\mathtt{f}} f\}\}$	$\rightarrow$	$\Diamond$
$\{_{\mathtt{s}} \{_{\mathtt{f}} f\} \vdash (b, \{_{\mathtt{f}} f\})\}$	$\rightarrow$	$\Diamond$
$\{_{\mathtt{s}} \{_{\mathtt{f}} f\} \vdash \{_{\mathtt{f}} f\}\}$	$\rightarrow$	$\Diamond$
$\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} (a, f) \vdash (g, b)\}$	$\rightarrow$	$\{_{\mathtt{s}} a \vdash b\}$
$\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} (a, f) \vdash b\}$	$\rightarrow$	$\{_{\mathtt{s}} a \vdash b\}$
$\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} a \vdash (b, f)\}$	$\rightarrow$	$\{_{\mathtt{s}} a \vdash b\}$
$\{_{\mathtt{s}} a \vdash \triangledown\} \bullet \{_{\mathtt{s}} (a, f) \vdash b\}$	$\rightarrow$	$\{_{\mathtt{s}} a \vdash \triangledown\}$
$\{_{\mathtt{s}} a \vdash (b, f)\} \bullet \{_{\mathtt{s}} \triangledown \vdash b\}$	$\rightarrow$	$\{_{\mathtt{s}} \triangledown \vdash b\}$
$\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} \triangledown \vdash b\}$	$\rightarrow$	$\{_{\mathtt{s}} \triangledown \vdash b\}$
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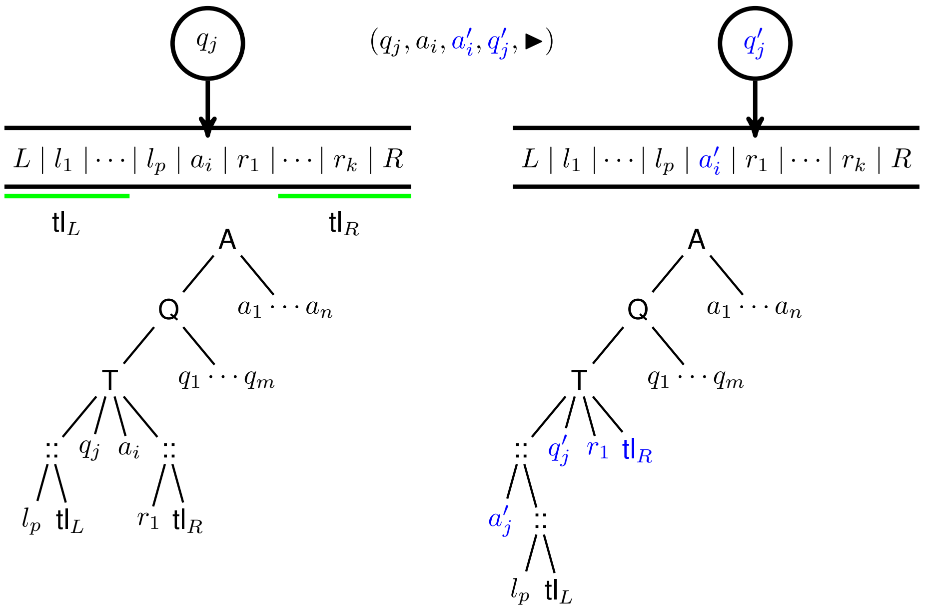
$\{_{\mathtt{s}} a \vdash b\} \bullet \{_{\mathtt{s}} \triangledown \vdash \triangledown\}$	$\rightarrow$	$\{_{\mathtt{s}} \triangledown \vdash \triangledown\}$

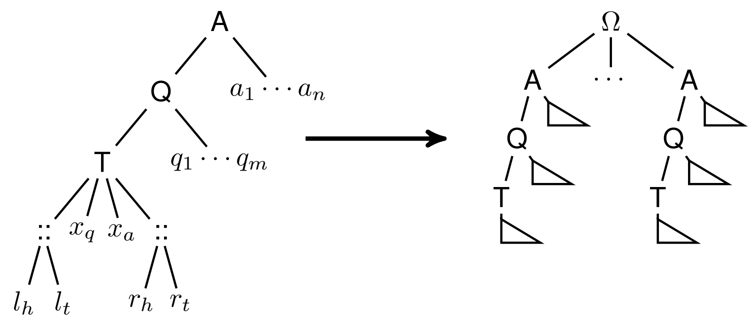
Termination overview

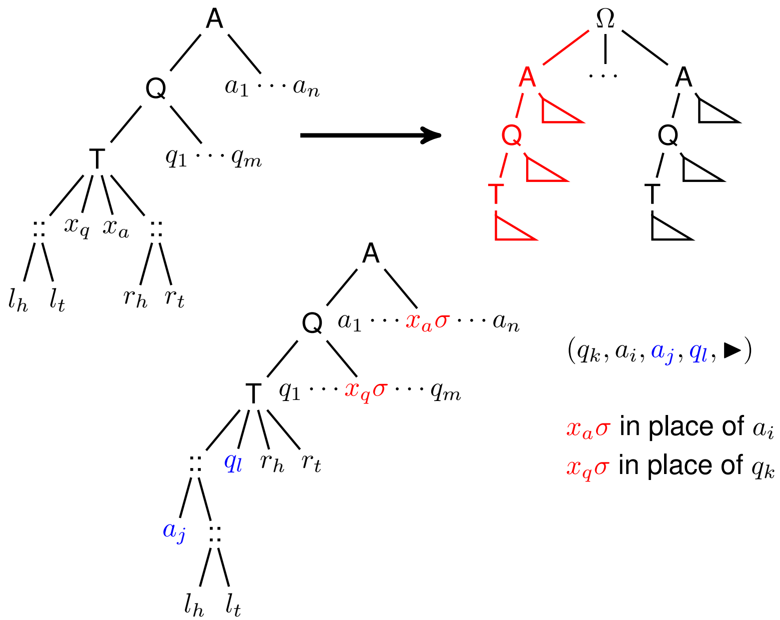
Automation? ⇝ correct, incomplete...

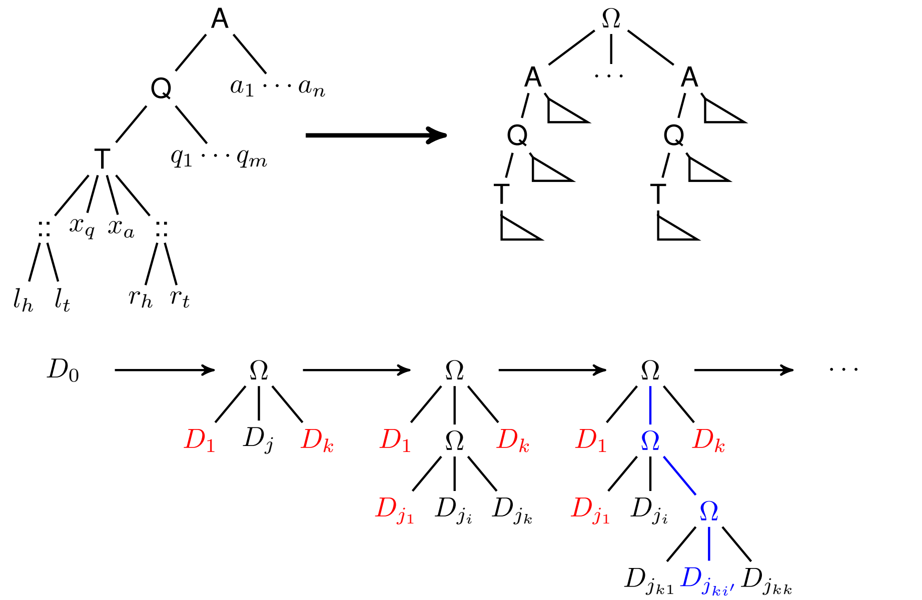
Always difficult

- $f(f(x)) \rightarrow f(x)$
- $f(a,b,x) \rightarrow f(x,x,x)$
- > 50 rules + equational theories
- > 1800 rules (> 1000 symbols)
- $\left\{ \begin{array}{lcl} a(a(x)) & \rightarrow & b(c(x)) \\ b(b(x)) & \rightarrow & a(c(x)) \\ c(c(x)) & \rightarrow & a(b(x)) \end{array} \right.$
- Syracuse...









Termination proof

Termination of TRS = well foundness of relation (WF) $\rightsquigarrow$ « measure »

How? $\mathcal{R} \subseteq O \wedge \text{WF}(O) \Rightarrow \text{WF}(\mathcal{R})$

$\mathcal{R}, O, f$ such that $s\ \mathcal{R}^+\ t \Rightarrow f(s)\ O^+\ f(t) \wedge \text{WF}(O) \Rightarrow \text{WF}(\mathcal{R})$.

$(\text{WF}(O) \Rightarrow \text{WF}(O^+) \Rightarrow \text{WF}(O^+ \circ f) \Rightarrow \text{WF}(\mathcal{R}^+))$

Problem transformation until: pb. trivial, pb. inclusion

$$\text{RULE NAME(PARAM)}\ \frac{p_1 \dots p_n}{p}\ \text{CONDITIONS}$$

Transf. correct and complete: criterion $\rightsquigarrow$ termination constraints

Inclusion: ordering constraints

Termination proof

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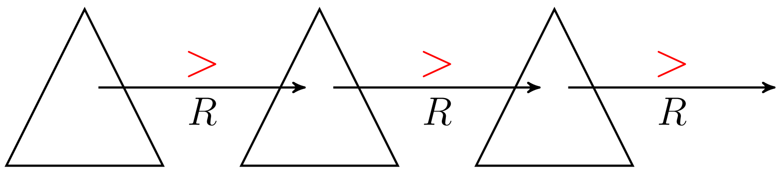
$$\dfrac{\dfrac{<\ \dfrac{<_1 \text{ (poly. interp.)}}{\{\dots\langle u'_n, v'_n\rangle\dots\}\ \text{SN?}} \qquad \dfrac{<\ \dfrac{<_2 \text{ (RPO)}}{\{\dots\langle u''_k, v''_k\rangle\dots\}\ \text{SN?}} \qquad \dfrac{}{\{\dots\langle u''_j, v''_j\rangle\dots\}\ \text{SN?}}}{\{\dots\langle u'_i, v'_i\rangle\dots\}\ \text{SN?}}\ \text{GRAPH}}{\mathcal{R}_{\text{DP}} = \{\dots\langle u_i, v_i\rangle\dots\}\ \text{SN?}}\ \text{GRAPH}}{\mathcal{R}_{\text{init}} = \{\dots l_i \to r_i \dots\}\ \text{SN?}}\ \text{DP}$$

Termination — proof

Inclusion $\to_R \subseteq >$: $s > t$ for all $s \to_R t$



Infinitely many $s \to t \rightsquigarrow$ automation?

Test finite?

Termination — Manna-Ness

Idea: ordering on rules stable through closures giving relation

Theorem. (Lankford)

- R a TRS $\{l_1 \to r_1, \ldots, l_n \to r_n\}$,
- $<$ such that WF($<$), $<$ stable and monotone,

then $(\forall i, l_i > r_i) \Rightarrow$ SN($\to_R$).

Ex.

$$\left\{ \begin{array}{lcl} x + 0 & \to & x \\ x + s(y) & \to & s(x+y) \end{array} \right\} \quad [\![s]\!] >_e [\![t]\!] \text{ with } \begin{array}{lcl} [\![0]\!] & = & 1 \\ [\![s]\!](x) & = & x+1 \\ [\![+]\!](x,y) & = & x+2y \end{array}$$

$$[\![x+0]\!] = x + 2 >_e [\![x]\!] = x$$
$$[\![x+s(y)]\!] = x + 2y + 2 >_e [\![s(x+y)]\!] = x + 2y + 1$$

Termination — Manna-Ness

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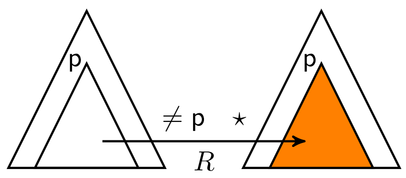
Heavy constraints:

- Complexity ($\rightsquigarrow$ relation),
- Orderings

In practice: simplification orderings (easier for WF)

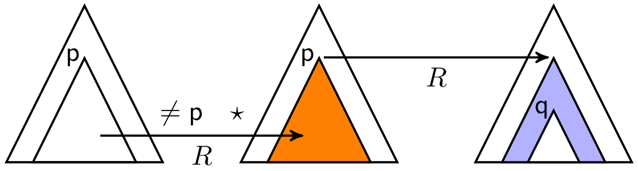
$f(f(x)) \to f(g(f(x)))$? stuck!

Termination — DP



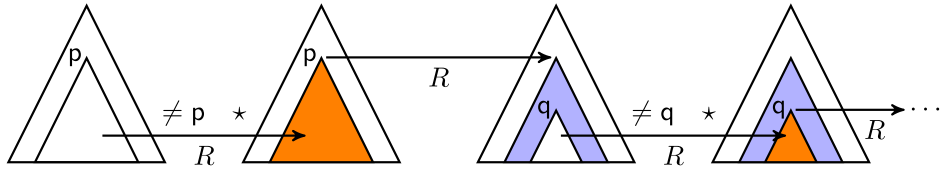
Termination

DP



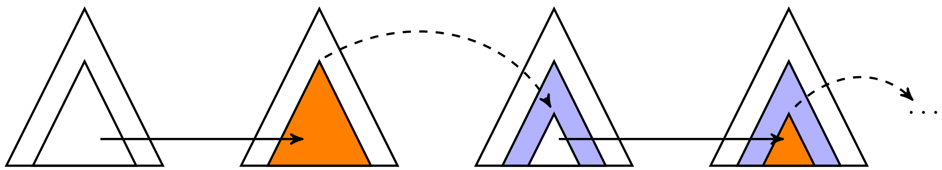
Termination

DP



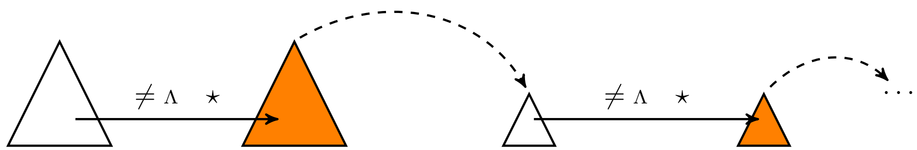
Termination

DP

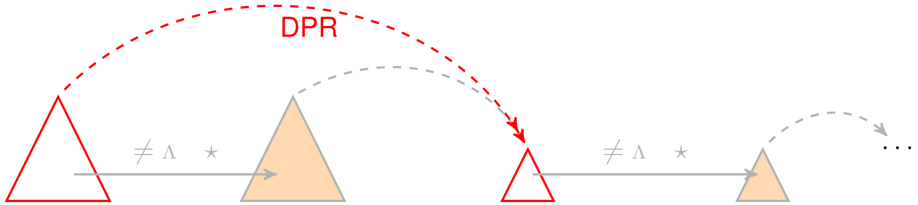


Termination

DP



Termination



Termination

DP

Définition.

Given a TRS R,

$\mathcal{F} = D \uplus C \quad D : \{f \in \mathcal{F} \mid \exists (l \to r) \in R, l(\Lambda) = f\}$

D : defined (functions) C : constructors (data)

Définition.

Given a rule $l \to r$,

Dependency pair: couple $\langle u, v \rangle$

- $u = l$,
- $v = r|_p$ such that $r(p) \in D$.

Set of dependency pairs of a TRS R: DP(R).

Termination

DP

Ex.

$$\left\{ \begin{array}{lllllll} \#0 & \to & \# & & & & \\ \# + x & \to & x & & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & & x1 + y1 & \to & ((x+y) + \#1)0 \end{array} \right\}$$

Termination

DP

Ex.

$D = \{0, +\} \quad C = \{\#, 1\}$

$$\left\{ \begin{array}{lllllll} \#0 & \to & \# & & & & \\ \# + x & \to & x & & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & & x1 + y1 & \to & ((x+y) + \#1)0 \end{array} \right\}$$

$\langle x1 + y1, x + y \rangle \quad \langle x1 + y0, x + y \rangle$

$\langle x0 + y1, x + y \rangle \quad \langle x0 + y0, x + y \rangle$

$\langle x0 + y0, (x + y)0 \rangle \quad \langle x1 + y1, (x + y) + \#1 \rangle$

$\langle x1 + y1, ((x + y) + \#1)0 \rangle$

Termination

DP

Ex.

$\{f(f(x)) \to f(g(f(x)))\}$

Termination DP

Ex.

$\{f(f(x)) \to f(g(f(x)))\} \quad D = \{f\} \quad C = \{g\}$

$\langle f(f(x)), f(g(f(x)))\rangle \quad \langle f(f(x)), f(x)\rangle$

what about derivations?

Termination DP

Ex.

$\{f(f(x)) \to f(g(f(x)))\} \quad D = \{f\} \quad C = \{g\}$

$\langle f(f(x)), f(g(f(x)))\rangle \quad \langle f(f(x)), f(x)\rangle$

Définition.

Dependency chain: sequence of DP, subst. σ such that

$$\cdots \quad \langle u_i, \quad v_i\rangle \qquad \langle u_{i+1}, \quad v_{i+1}\rangle \quad \cdots$$

$$v_i\sigma \xrightarrow{\neq\Lambda\star} u_{i+1}\sigma$$

Termination DP

Ex.

$\{f(f(x)) \to f(g(f(x)))\} \quad D = \{f\} \quad C = \{g\}$

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Termination DP

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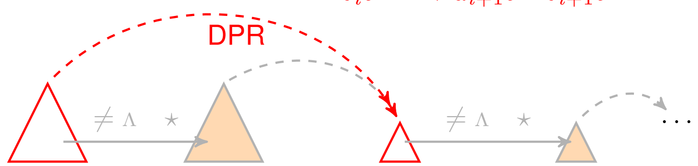

Termination

DP

Ex.
$\{f(f(x)) \to f(g(f(x)))\}$ $D = \{f\}$ $C = \{g\}$
$\langle f(f(x)), f(g(f(x)))\rangle$ $\langle f(f(x)), f(x)\rangle$

Définition.
Dependency chain: sequence of DP, subst. σ such that

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$$v_i\sigma \xrightarrow{\neq\Lambda\star} u_{i+1}\sigma$$

Theorem. (A & G)
$\mathsf{SN}(\to_R) \Leftrightarrow$ no infinite chain over $\mathsf{DP}(R)$

Rephrased: $\mathsf{SN}(\to_R) \Leftrightarrow \mathsf{SN}(\twoheadrightarrow_{\mathsf{DP}(R),R})$

Termination

DP

Rk. — $R = \{f(f(x)) \to h(f(x)),\ g(x) \to f(x)\}$ $D = \{f, g\}$ $C = \{h\}$
$\mathsf{DP}(R) = \{\langle f(f(x)), f(x)\rangle,\ \langle g(x), f(x)\rangle\}$

$\mathsf{SN}(\to_R)? \rightsquigarrow \mathsf{SN}(\xrightarrow[R]{\neq\Lambda\star} \cdot \xrightarrow[\mathsf{DP}(R)]{})?$ $(\rightsquigarrow \mathsf{SN}(\twoheadrightarrow_{\mathsf{DP}(R),R})?)$

With minimal chains...

$f(f(x))\sigma$?

Termination

DP

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$\mathsf{SN}(\to_R)? \rightsquigarrow \mathsf{SN}(\xrightarrow[R]{\neq\Lambda\star} \cdot \xrightarrow[\mathsf{DP}(R)]{})?$ $(\rightsquigarrow \mathsf{SN}(\twoheadrightarrow_{\mathsf{DP}(R),R})?)$

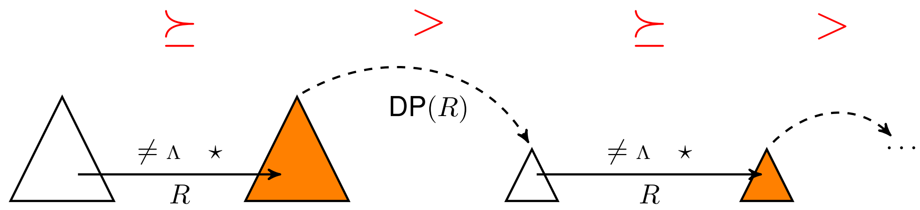
With minimal chains...

$f(f(x))\sigma$ NOT minimal: irrelevant

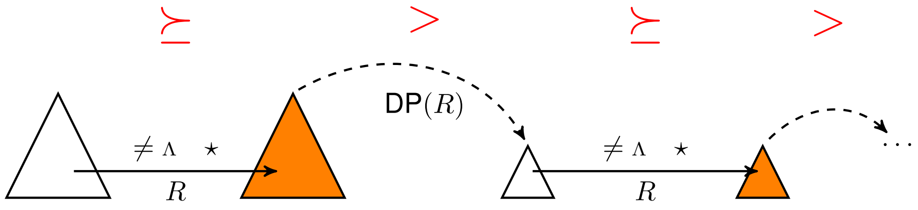
$\mathsf{SN}(\to_R) \Leftrightarrow \mathsf{SN}(\twoheadrightarrow_{\langle g(x), f(x)\rangle,R})$

Termination

DP, control



Termination — DP, control

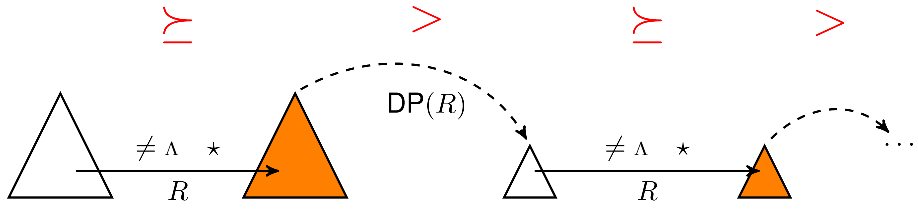


Theorem. (A & G)
If $(\succeq, >)$ such that

1. $\succeq \cdot > \subseteq >$, WF($<$), $\succeq$ monotone, stable, $>$ stable,
2. $l \succeq r$ for each $l \to r \in R$,
3. $u > v$ for each $\langle u, v\rangle \in$ DP(R),

then SN($\to_R$)

Termination — DP, control



Theorem. (A & G)
If $(\succeq, >)$ such that

1. $\succeq \cdot > \subseteq >$, WF($<$), $\succeq$ monotone, stable, $>$ stable, (monotony useless)
2. $l \succeq r$ for each $l \to r \in R$,
3. $u > v$ for each $\langle u, v\rangle \in$ DP(R),

then SN($\to_R$)

Termination — DP, marks

Recursive calls: decrease of arguments

$\leadsto$ distinction between function *symbol* and recursive call: marks

Ex.

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y) + \#1)0 \end{array}\right\}$$

$\langle x1 \widehat{+} y1, x \widehat{+} y\rangle$ $\quad$ $\langle x1 \widehat{+} y0, x \widehat{+} y\rangle$

$\langle x0 \widehat{+} y1, x \widehat{+} y\rangle$ $\quad$ $\langle x0 \widehat{+} y0, x \widehat{+} y\rangle$

$\langle x0 \widehat{+} y0, (x+y)\widehat{0}\rangle$ $\quad$ $\langle x1 \widehat{+} y1, (x+y) \widehat{+} \#1\rangle$

$\langle x1 \widehat{+} y1, ((x+y) + \#1)\widehat{0}\rangle$

Termination — DP, marks

$[\![\#]\!] = 0$	$[\![0]\!](x) = x + 1$	$\neq$	$[\![\widehat{0}]\!](x) = 0$	non mono
$[\![1]\!](x) = x + 1$	$[\![+]\!](x, y) = x$	non mono	$[\![\widehat{+}]\!](x, y) = x$	non mono

Ex.

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y) + \#1)0 \end{array}\right\}$$

$\langle x1 \widehat{+} y1, x \widehat{+} y\rangle$ $\quad$ $\langle x1 \widehat{+} y0, x \widehat{+} y\rangle$

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$\langle x1 \widehat{+} y1, ((x+y) + \#1)\widehat{0}\rangle$

Termination — DP, marks

Ex.

$\{f(f(x)) \to f(g(f(x)))\}$

$[\![g]\!](x) = 0$

$[\![f]\!](x) = 1$

$[\![\widehat{f}]\!](x) = x$

Termination — DP, graphs

Already noticed: not anything after $\langle u_i, v_i\rangle$

Coarse: $v_i\sigma \xrightarrow[R]{\neq\Lambda\star} u_j\sigma \Rightarrow v_i(\Lambda) \equiv u_j(\Lambda)$

For R finite, DP(R) finite $\leadsto$ finite graph of the relation linking DP

Termination — DP, graphs

Already noticed: not anything after $\langle u_i, v_i\rangle$

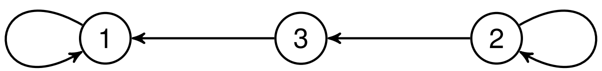
Coarse: $v_i\sigma \xrightarrow[R]{\neq\Lambda\star} u_j\sigma \Rightarrow v_i(\Lambda) \equiv u_j(\Lambda)$

For R finite, DP(R) finite $\leadsto$ finite graph of the relation linking DP

Ex.

$$\left\{\begin{array}{rcl} x - 0 & \to & x \\ s(x) - s(y) & \to & x - y \\ 0 \div s(y) & \to & 0 \\ s(x) \div s(y) & \to & s((x-y) \div s(y)) \end{array}\right.$$

$\langle s(x) \,\widehat{-}\, s(y), x \,\widehat{-}\, y\rangle$	1
$\langle s(x) \,\widehat{\div}\, s(y), (x-y) \,\widehat{\div}\, s(y)\rangle$	2
$\langle s(x) \,\widehat{\div}\, s(y), x \,\widehat{-}\, y\rangle$	3



Termination — DP, graphs

Now: FINITE sytems

Chain $\mapsto$ path $\leadsto$ chain $\infty \mapsto$ path ∞, here strongly connected part (SCP)

Chains in SCP independently controlled

Ex.

$R = \{f(f(x)) \to h(f(x)),\ g(x) \to f(x)\}$

SN$(\to_R) \Leftrightarrow$ SN$(\twoheadrightarrow_{\langle g(x), f(x)\rangle, R})$

No circuit: trivial problem, OK

Termination — DP, graphs

- One relation per SCP $\rightsquigarrow$ one ordering (proof) per SCP
- SCP $\neq$ composante
- SCP $\neq$ elementary circuits

  $\{f(0) \to g(1) \quad f(1) \to g(0) \quad g(x) \to f(x)\}$

Termination — DP, graphs

- One relation per SCP $\rightsquigarrow$ one ordering (proof) per SCP
- SCP $\neq$ composante
- SCP $\neq$ elementary circuits

  $\{f(0) \to g(1) \quad f(1) \to g(0) \quad g(x) \to f(x)\}$

- Automation: graph NOT computable $\rightsquigarrow$ approximations
  - Coarse: head symbol
  - Finer: discriminate with constructor cap (REN/CAP)
    CAP: fresh variable for each defined symbol, REN: renaming,
    s connectable to t if REN(CAP(s)) and t unify

Theorem.
Correct: $\langle u, v\rangle \longrightarrow \langle u', v'\rangle$ if v connectable to u'

Termination — DP, graphs

Theorem. (A,G & O)
R TRS, G graph of R, $G = \bigcup_{i=0}^{k-1} G_i$ where $G_i \subseteq$ DP(R) SCP,
then $(\forall i \in [0..k-1]\ \mathsf{SN}(\twoheadrightarrow_{G_i,R})) \Leftrightarrow \mathsf{SN}(\twoheadrightarrow_{\mathsf{DP}(R),R})$

All SCP: utmost expensive $\rightsquigarrow$ by components?

Corollaire.
If $\forall G_i^{max}, \exists(\succeq, >)$ with usual good properties such that:

- $\forall \langle u, v\rangle \in G_i^{max}, u > v$
- $\forall (l \to r) \in R, l \succeq r$

then $\mathsf{SN}(\twoheadrightarrow_{\mathsf{DP}(R),R})$