Termination DP

Definition.
Given a TRS R,
$\mathcal{F} = D \uplus C \quad D : \{f \in \mathcal{F} \mid \exists (l \to r) \in R, l(\Lambda) = f\}$
D : defined (functions) C : constructors (data)

Definition.
Given a rule $l \to r$,
Dependency pair: couple $\langle u, v \rangle$

- $u = l$,
- $v = r|_p$ such that $r(p) \in D$.

Set of dependency pairs of a TRS R: DP(R).

Termination DP

Ex.

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right\}$$

Termination DP

Ex.
$D = \{0, +\} \quad C = \{\#, 1\}$

$$\left\{\begin{array}{lllllll} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right\}$$

$\langle x1 + y1, x + y \rangle \qquad \langle x1 + y0, x + y \rangle$
$\langle x0 + y1, x + y \rangle \qquad \langle x0 + y0, x + y \rangle$
$\langle x0 + y0, (x+y)0 \rangle \qquad \langle x1 + y1, (x+y) + \#1 \rangle$
$\langle x1 + y1, ((x+y)+\#1)0 \rangle$

Termination DP

Ex.
$\{f(f(x)) \to f(g(f(x)))\} \quad D = \{f\} \quad C = \{g\}$
$\langle f(f(x)), f(g(f(x))) \rangle \quad \langle f(f(x)), f(x) \rangle$

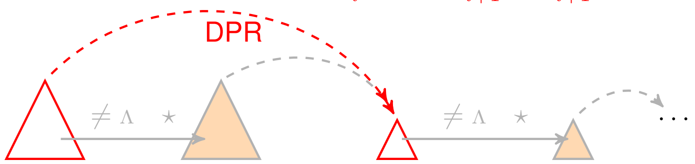
what about derivations?

Termination DP

Ex.
$\{f(f(x)) \to f(g(f(x)))\} \quad D = \{f\} \quad C = \{g\}$
$\langle f(f(x)), f(g(f(x)))\rangle \quad \langle f(f(x)), f(x)\rangle$

Definition.
Dependency chain: sequence of DP, subst. σ such that

$$\cdots \quad \langle u_i, \quad v_i\rangle \qquad \langle u_{i+1}, \quad v_{i+1}\rangle \quad \cdots$$
$$v_i\sigma \xrightarrow{\neq\Lambda\star} u_{i+1}\sigma \quad v_{i+1}\sigma$$



Termination DP

Ex.
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Theorem. (A & G)
SN$(\to_R) \Leftrightarrow$ no infinite chain over DP(R)

Rephrased: SN$(\to_R) \Leftrightarrow$ SN$(\twoheadrightarrow_{\text{DP}(R),R})$

Termination DP

Rk. — $R = \{f(f(x)) \to h(f(x)),\ g(x) \to f(x)\} \qquad D = \{f, g\} \quad C = \{h\}$
DP$(R) = \{\langle f(f(x)), f(x)\rangle,\ \langle g(x), f(x)\rangle\}$

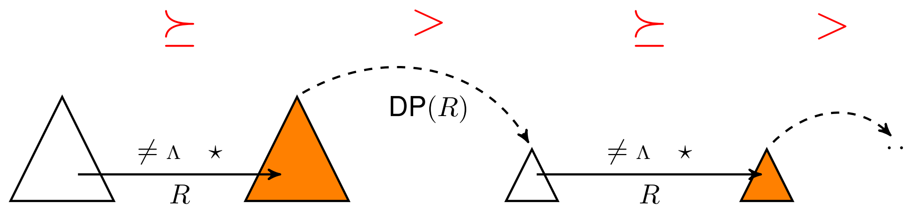
SN$(\to_R)$? $\leadsto$ SN$(\xrightarrow[R]{\neq\Lambda\star} \cdot \xrightarrow[\text{DP}(R)]{})$? $\qquad (\leadsto$ SN$(\twoheadrightarrow_{\text{DP}(R),R})$?)

With minimal chains. . .

$f(f(x))\sigma$ NOT minimal: irrelevant

SN$(\to_R) \Leftrightarrow$ SN$(\twoheadrightarrow_{\langle g(x), f(x)\rangle, R})$

Termination DP, control



Theorem. (A & G)
If $(\succeq, >)$ such that

1. $\succeq \cdot > \subseteq >$, WF$(<)$, $\succeq$ monotone, stable, $>$ stable, (monotony useless)
2. $l \succeq r$ for each $l \to r \in R$,
3. $u > v$ for each $\langle u, v\rangle \in$ DP(R),

then SN$(\to_R)$

Termination — DP, marks

Recursive calls: decrease of arguments

$\rightsquigarrow$ distinction between function *symbol* and recursive call: marks

Ex.

$$\left\{\begin{array}{lcllcl} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right\}$$

$\langle x1 \mathbin{\widehat{+}} y1, x \mathbin{\widehat{+}} y\rangle$ $\quad$ $\langle x1 \mathbin{\widehat{+}} y0, x \mathbin{\widehat{+}} y\rangle$

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$\langle x0 \mathbin{\widehat{+}} y0, (x+y)\widehat{0}\rangle$ $\quad$ $\langle x1 \mathbin{\widehat{+}} y1, (x+y) \mathbin{\widehat{+}} \#1\rangle$

$\langle x1 \mathbin{\widehat{+}} y1, ((x+y)+\#1)\widehat{0}\rangle$

Termination — DP, marks

$[\![\#]\!] = 0$	$[\![0]\!](x) = x + 1$	$\neq$	$[\![\widehat{0}]\!](x) = 0$	non mono
$[\![1]\!](x) = x + 1$	$[\![+]\!](x, y) = x$	non mono	$[\![\widehat{+}]\!](x, y) = x$	non mono

Ex.

$$\left\{\begin{array}{lcllcl} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right\}$$

$\langle x1 \mathbin{\widehat{+}} y1, x \mathbin{\widehat{+}} y\rangle$ $\quad$ $\langle x1 \mathbin{\widehat{+}} y0, x \mathbin{\widehat{+}} y\rangle$

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$\langle x0 \mathbin{\widehat{+}} y0, (x+y)\widehat{0}\rangle$ $\quad$ $\langle x1 \mathbin{\widehat{+}} y1, (x+y) \mathbin{\widehat{+}} \#1\rangle$

$\langle x1 \mathbin{\widehat{+}} y1, ((x+y)+\#1)\widehat{0}\rangle$

Termination — DP, marks

Ex.

$\{f(f(x)) \to f(g(f(x)))\}$

$[\![g]\!](x) = 0$

$[\![f]\!](x) = 1$

$[\![\widehat{f}]\!](x) = x$

Termination — DP, graphs

Already noticed: not anything after $\langle u_i, v_i\rangle$

Coarse: $v_i\sigma \xrightarrow[R]{\neq\Lambda\star} u_j\sigma \Rightarrow v_i(\Lambda) \equiv u_j(\Lambda)$

For R finite, DP(R) finite $\rightsquigarrow$ finite graph of the relation linking DP

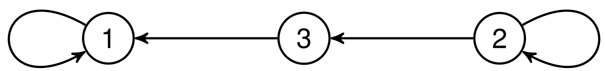
Ex.

$$\left\{\begin{array}{rcl} x - 0 & \to & x \\ s(x) - s(y) & \to & x - y \\ 0 \div s(y) & \to & 0 \\ s(x) \div s(y) & \to & s((x-y) \div s(y)) \end{array}\right.$$

$\langle s(x) \mathbin{\widehat{-}} s(y), x \mathbin{\widehat{-}} y\rangle$ 1

$\langle s(x) \mathbin{\widehat{\div}} s(y), (x-y) \mathbin{\widehat{\div}} s(y)\rangle$ 2

$\langle s(x) \mathbin{\widehat{\div}} s(y), x \mathbin{\widehat{-}} y\rangle$ 3



Termination — DP, graphs

Now: FINITE sytems

Chain $\mapsto$ path $\leadsto$ chain $\infty \mapsto$ path ∞, here strongly connected part (SCP)

Chains in SCP independently controlled

Ex.

$R = \{f(f(x)) \to h(f(x)),\ g(x) \to f(x)\}$

SN$(\to_R) \Leftrightarrow$ SN$(\twoheadrightarrow_{\langle g(x), f(x)\rangle, R})$

No circuit: trivial problem, OK

Termination — DP, graphs

- One relation per SCP $\leadsto$ one ordering (proof) per SCP
- SCP $\neq$ composante
- SCP $\neq$ elementary circuits

  $\{f(0) \to g(1) \quad f(1) \to g(0) \quad g(x) \to f(x)\}$

Termination — DP, graphs

- One relation per SCP $\leadsto$ one ordering (proof) per SCP
- SCP $\neq$ component
- SCP $\neq$ elementary circuits

  $\{f(0) \to g(1) \quad f(1) \to g(0) \quad g(x) \to f(x)\}$

- Automation: graph NOT computable $\leadsto$ approximations
  - Coarse: head symbol
  - Finer: discriminate with constructor cap (REN/CAP)

    CAP: fresh variable for each defined symbol, REN: renaming,

    s connectable to t if REN(CAP(s)) and t unify

Theorem.

Correct: $\langle u, v\rangle \longrightarrow \langle u', v'\rangle$ only if v connectable to u'

Termination — DP, graphs

Theorem. (A,G & O)

R TRS, G graph of R, $G = \bigcup_{i=0}^{k-1} G_i$ where $G_i \subseteq$ DP(R) SCP,

then $(\forall i \in [0..k-1]$ SN$(\twoheadrightarrow_{G_i,R})) \Leftrightarrow$ SN$(\twoheadrightarrow_{\mathrm{DP}(R),R})$

All SCP: utmost expensive $\leadsto$ by components?

Corollaire.

If $\forall G_i^{max}, \exists(\succeq, >)$ with usual good properties such that:

- $\forall \langle u, v\rangle \in G_i^{max}, u > v$
- $\forall (l \to r) \in R, l \succeq r$

then SN$(\twoheadrightarrow_{\mathrm{DP}(R),R})$