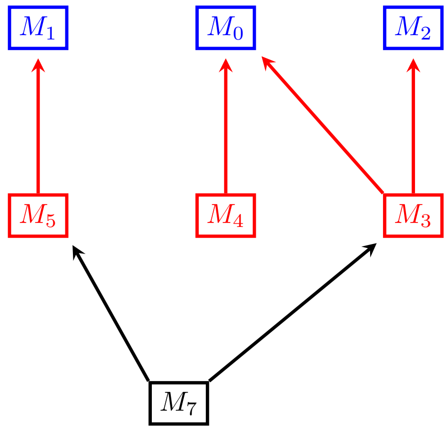
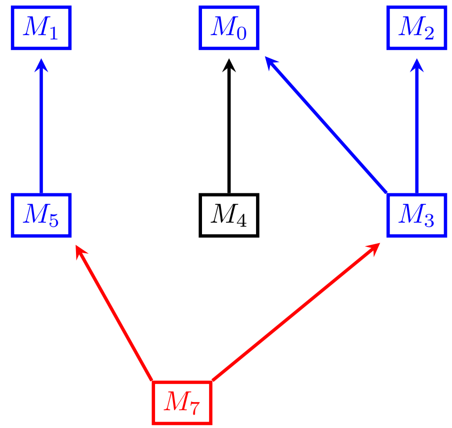
Termination incremental...



Termination modular...



Termination modularity...

Termination: not modular (Toyama)

$$R \quad : \quad \{f(0,1,x) \to f(x,x,x)$$

$$\Pi \quad : \quad \begin{cases} \pi(x,y) \to x \\ \pi(x,y) \to y \end{cases}$$

Allows for:

$$f(\pi(0,1),\pi(0,1),\pi(0,1)) \underset{\Pi}{\to}^{\star} f(0,1,\pi(0,1)) \underset{R}{\to} \dots$$

Sub-optimal solutions: simplification orderings... heavy constraints on unions...

Termination modularity...

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$$R \quad : \quad \{f(0,1,x) \to f(x,x,x)$$

$$\Pi \quad : \quad \begin{cases} \pi(x,y) \to x \\ \pi(x,y) \to y \end{cases}$$

Allows for:

$$f(\pi(0,1),\pi(0,1),\pi(0,1)) \underset{\Pi}{\to}^{\star} f(0,1,\pi(0,1)) \underset{R}{\to} \dots$$

Definition.
R $\mathcal{C}_{\mathcal{E}}$-terminating if and only if $R \cup \Pi$ terminating

Termination modularity…

$\mathcal{C}_\mathcal{E}$: good properties

$\mathcal{C}_\mathcal{E}$-Terminaison modular for:

- Disjoint unions
- Unions with shared constructors for finitely branching systems

$\rightsquigarrow$ from now on: finitely branching

$$\left\{\begin{array}{lcl} f_j(c_j,x) & \to & f_{j+1}(x,x) \\ f_j(x,y) & \to & x \\ f_j(x,y) & \to & y \end{array}\right| j \in \mathbb{N} \qquad \left\{\begin{array}{lcl} a & \to & c_j \\ \pi(x,y) & \to & x \\ \pi(x,y) & \to & y \end{array}\right| j \in \mathbb{N}$$

Termination modularity…

$\mathcal{C}_\mathcal{E}$: good properties

SN and not duplicating: $\mathcal{C}_\mathcal{E}$ SN

SN and non-deterministic projective: $\mathcal{C}_\mathcal{E}$ SN

SN simplifying: $\mathcal{C}_\mathcal{E}$ SN

Termination modularity…

Definition.
a module $[\mathcal{F}_2 \mid R_2]$ extends $R_1(\mathcal{F}_1)$:

- $\mathcal{F}_2 \cap \mathcal{F}_1 = \emptyset$,
- $R_2(\mathcal{F}_1 \cup \mathcal{F}_2)$ such that for each $l \to r \in R_2$, $\Lambda(l) \in \mathcal{F}_2$,

$\rightsquigarrow$ $R = R_1 \cup R_2$: hierarchical extension of R_1

Example.

$\mathcal{F}_0 \quad \{\#, 0, 1\}$

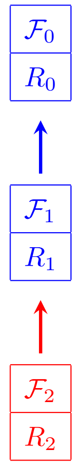
$R_0 \quad \{\#0 \to \#$

$\mathcal{F}_1 \quad \{+\}$

$$R_1 \quad \left\{\begin{array}{lcllcl} \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array}\right.$$

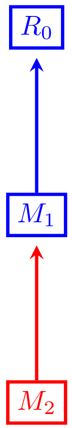
$\mathcal{F}_2 \quad \{\times\}$

$$R_2 \quad \left\{\begin{array}{lcllcl} \# \times x & \to & \# & x \times \# & \to & \# \\ x0 \times y & \to & (x \times y)0 & x1 \times y & \to & (x \times y)0 + y \end{array}\right.$$

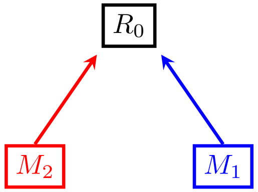


Goal: proving termination incrementally

Hierarchical with common subsystem (Ohlebusch)



Composable systems (Gramlich, Kurihara & Ohuchi. . .)



Termination modularity. . .

Definition.
$R_1(\mathcal{F}_1) \leftarrow [\mathcal{F}_2 \mid R_2]$.
$\langle l, r' \rangle$ dependency pair of module $[\mathcal{F}_2 \mid R_2]$:

- $l \rightarrow r \in R_2$,
- r' subterm of r such that $\Lambda(r') \in \mathcal{F}_2$

DP(M): dependency pairs of all rules in M

Rk. — $R(\mathcal{F}) \quad \rightsquigarrow \quad [\mathcal{F}_C \mid \emptyset] \leftarrow [\mathcal{F}_D \mid R]$.

For $[\mathcal{F}_C \mid \emptyset] \leftarrow [\mathcal{F}_D \mid R]$, same as Arts & Giesl's

Example.

$$R_\# \left\{ \#0 \rightarrow \# \right.$$

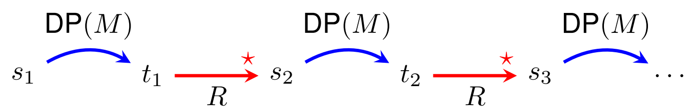
$$R_+ \left\{ \begin{array}{llllll} \# + x & \rightarrow & x & x + \# & \rightarrow & x \\ x0 + y0 & \rightarrow & (x+y)0 & x1 + y0 & \rightarrow & (x+y)1 \\ x0 + y1 & \rightarrow & (x+y)1 & x1 + y1 & \rightarrow & ((x+y) + \#1)0 \end{array} \right.$$

$$R_\times \left\{ \begin{array}{llllll} \# \times x & \rightarrow & \# & x \times \# & \rightarrow & \# \\ x0 \times y & \rightarrow & (x \times y)0 & x1 \times y & \rightarrow & (x \times y)0 + y \end{array} \right.$$

$$\mathrm{DP}(M_\times) : \left\{ \begin{array}{l} \langle x0 \times y, x \times y \rangle \\ \langle x1 \times y, x \times y \rangle \end{array} \right\}$$

Termination relative chains

Definition.
Dependency chain of M over R:



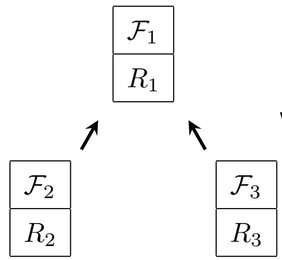
Minimal: for all $\langle s, t \rangle$ with $s\sigma = f(u_1, \dots, u_n)$, for all i, u_i strongly normalisable (for R)

Theorem 1.
$R_1(\mathcal{F}_1) \leftarrow [\mathcal{F}_2 \mid R_2]$.

1. R_1 $\mathcal{C}_\mathcal{E}$-terminating,
2. no infinite chain of $[\mathcal{F}_2 \mid R_2]$ over $R_1 \cup R_2 \cup \Pi$,

then $R_1 \cup R_2$ $\mathcal{C}_\mathcal{E}$-terminating

Theorem 1.

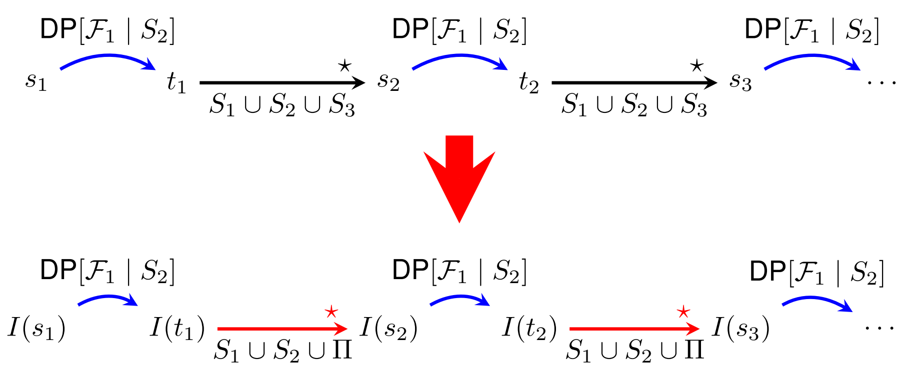


with $\mathcal{F}_3 \cap \mathcal{F}_2 = \emptyset$

1. $R_1 \cup R_2$ $\mathcal{C}_\mathcal{E}$-terminating,
2. no infinite chain of $[\mathcal{F}_3 \mid R_3]$ over $R_1 \cup R_3 \cup \Pi$, ← No R_2

then $R_1 \cup R_2 \cup R_3$ $\mathcal{C}_\mathcal{E}$-terminating

Lemma.
$\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, $S_1(\mathcal{F}_1)$, $S_2(\mathcal{F}_1)$, $[\mathcal{F}_1 \mid \emptyset] \leftarrow [\mathcal{F}_2 \mid S_3]$



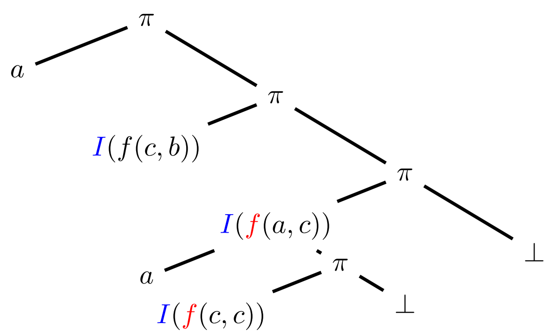
$I(x) : T(\mathcal{F}_1 \cup \mathcal{F}_2, X) \to T_\infty(\mathcal{F}_1 \cup \{\pi : 2\} \cup \{\bot : 0\}, X)$

$$
\begin{array}{rcl}
I(x) & = & x \text{ if } x \in X, \\
I(f(t_1 \dots t_n)) & = & \begin{cases} f(I(t_1) \dots I(t_n)) \text{ if } f \in \mathcal{F}_1, \\ \mathsf{I}^\star(\mathsf{S}(f(t_1 \dots t_n))) \text{ if } f \in \mathcal{F}_2, \end{cases} \\
\text{where} \quad \mathsf{S}(t) & = & \{I(t') / \ t \xrightarrow[S_1 \cup S_2 \cup S_3]{} t'\}, \\
\mathsf{I}^\star(\emptyset) & = & \bot, \\
\mathsf{I}^\star(\{a\} \cup \mathsf{Set}) & = & \pi(a, \mathsf{I}^\star(\mathsf{Set})) \text{ where for all } e \in \mathsf{Set}, a < e.
\end{array}
$$

Rk. — For t, $\forall t_i \lhd t, t_i(\Lambda) \in \mathcal{F}_2 \Rightarrow t_i$ mortal for $S_1 \cup S_2 \cup S_3$
$\Rightarrow I(t)$ well defined

$$S_1 \cup S_2 : \begin{cases} a & \to & c \\ b & \to & c \\ g(c) & \to & c \end{cases} \qquad S_3 : \begin{cases} f(x,y) & \to & x \\ f(x,x) & \to & g(x) \end{cases}$$

$I(f(a,b)) =$



$t_1 = f(a, h(\ldots)) \quad h \in \mathcal{F}_2$

$t_1 \xrightarrow[S_3]{} t_2$

$I(t_1) \xrightarrow[\Pi_1]{} \xrightarrow[\Pi_2]{}^{\star} I(t_2)$

Notation: $I(\sigma)$: $\quad x \in X \mapsto I(x\sigma)$.

Lemma 1.
$\forall t \in T(\mathcal{F}_1, X), \forall \sigma$ substitution (with $t \in Dom(I)$) $\quad I(t\sigma) = tI(\sigma)$

Ind. on t.

Lemma 2.
$\forall t_1, \ldots, t_n \in T(\mathcal{F}_1 \cup \mathcal{F}_2, X)$, $\forall C$ context over $\mathcal{F}_1$ with n holes,
$I(C[t_1, \ldots, t_n]) = C[I(t_1), \ldots, I(t_n)]$

Ind. on C.

Lemma 3.
$\forall s, t \in T(\mathcal{F}_1 \cup \mathcal{F}_2, X), l \to r \in S_1 \cup S_2, \qquad s \xrightarrow[l \to r]{p} t \Rightarrow I(s) \xrightarrow[S_1 \cup S_2 \cup \Pi]{}^{+} I(t)$

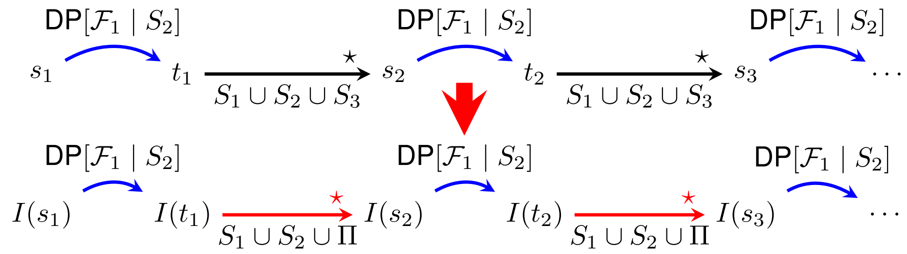
- No $f \in \mathcal{F}_2$ from Λ to p, OK by lemmas 1 and 2
- $\exists$ smallest p' such that $s(p') \in \mathcal{F}_2$,
  through context ok by lemma 2 and
  from def. of I, $I(r\sigma) \in$ S$(l\sigma)$ hence OK

Lemma 4.
$\forall s, t \in T(\mathcal{F}_1 \cup \mathcal{F}_2, X), l \to r \in S_3, \qquad s \xrightarrow[l \to r]{p} t \Rightarrow I(s) \xrightarrow[\Pi]{}^{+} I(t)$

Same

Lemma.

$\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, $S_1(\mathcal{F}_1)$, $S_2(\mathcal{F}_1)$, $[\mathcal{F}_1 \mid \emptyset] \leftarrow [\mathcal{F}_2 \mid S_3]$



$\langle u_i, v_i \rangle \xrightarrow[S_1 \cup S_2 \cup S_3]{\neq\Lambda\star} \langle u_{i+1}, v_{i+1} \rangle \xrightarrow[S_1 \cup S_2 \cup S_3]{\neq\Lambda\star} \cdots \qquad \sigma$

Choose $\sigma' = I(\sigma)$ (minimal chain hence σ ok)

DP over $\mathcal{F}_1$ hence no pb.

By lemmas 3 and 4: $v_i\sigma' \xrightarrow[S_1 \cup S_2 \cup \Pi]{\neq\Lambda\star} u_{i+1}\sigma'$

Theorem.

$R_1(\mathcal{F}_1) \leftarrow [\mathcal{F}_2 \mid R_2]$.

- R_1 $\mathcal{C}_\mathcal{E}$-terminating,
- no infinite chain of $[\mathcal{F}_2 \mid R_2]$ over $R_1 \cup R_2 \cup \Pi$,

then $R_1 \cup R_2$ $\mathcal{C}_\mathcal{E}$-terminating

Infinite reduction: infinite chain of $[\mathcal{F}_1 \cup \mathcal{F}_2 \mid R_1 \cup R_2]$ over $R_1 \cup R_2$

1. ~~only pairs of $[\mathcal{F}_2 \mid R_2]$~~
2. only pairs of $[\mathcal{F}_1 \mid R_1]$
3. a few pairs of $[\mathcal{F}_2 \mid R_2]$ then only pairs of $[\mathcal{F}_1 \mid R_1]$

Infinite chain of $[\mathcal{F}_1 \mid R_1]$ over $R_1 \cup R_2$

Lemma $S_1 = S_2 = R_1$

$S_3 = R_2$

Infinite chain of $[\mathcal{F}_1 \cup \{\pi\,;\bot\} \mid R_1 \cup \Pi]$ over $R_1 \cup \Pi$

DP$(\Pi) = \emptyset$

~~Infinite chain of $[\mathcal{F}_1 \mid R_1]$ sur $R_1 \cup \Pi$~~

Theorem.

$R_1(\mathcal{F}_1) \leftarrow [\mathcal{F}_3 \mid R_3]$, $R_1(\mathcal{F}_1) \leftarrow [\mathcal{F}_2 \mid R_2]$ with $\mathcal{F}_3 \cap \mathcal{F}_2 = \emptyset$

- $R_1 \cup R_2$ $\mathcal{C}_\mathcal{E}$-terminating,
- no infinite chain of $[\mathcal{F}_3 \mid R_3]$ over $R_1 \cup R_3 \cup \Pi$,

then $R_1 \cup R_2 \cup R_3$ $\mathcal{C}_\mathcal{E}$-terminating

1. only pairs of $[\mathcal{F}_3 \mid R_3]$

Lemma with $S_1 := R_1, S_2 := R_3 \cup \Pi, S_3 := R_2 \quad \mathcal{F}_1 := \mathcal{F}_1 \cup \mathcal{F}_3, \mathcal{F}_2 := \mathcal{F}_2$

Theorem.

$R_1(\mathcal{F}_1) \longleftarrow [\mathcal{F}_3 \mid R_3]$, $R_1(\mathcal{F}_1) \longleftarrow [\mathcal{F}_2 \mid R_2]$ with $\mathcal{F}_3 \cap \mathcal{F}_2 = \emptyset$

- $R_1 \cup R_2$ $\mathcal{C}_\mathcal{E}$-terminating,
- no infinite chain of $[\mathcal{F}_3 \mid R_3]$ over $R_1 \cup R_3 \cup \Pi$,

then $R_1 \cup R_2 \cup R_3$ $\mathcal{C}_\mathcal{E}$-terminating

1. ~~only pairs of $[\mathcal{F}_3 \mid R_3]$~~
2. only pairs of $[\mathcal{F}_1 \cup \mathcal{F}_2 \mid R_1 \cup R_2]$
3. a few pairs of $[\mathcal{F}_3 \mid R_3]$ then only pairs of $[\mathcal{F}_1 \cup \mathcal{F}_2 \mid R_1 \cup R_2]$

Lemma with $S_1 := \emptyset, S_2 := R_1 \cup R_2, S_3 := R_3 \cup \Pi \quad \mathcal{F}_1 := \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{F}_2 := \mathcal{F}_3$

Theorem.

$R_1(\mathcal{F}_1) \longleftarrow [\mathcal{F}_3 \mid R_3]$, $R_1(\mathcal{F}_1) \longleftarrow [\mathcal{F}_2 \mid R_2]$ with $\mathcal{F}_3 \cap \mathcal{F}_2 = \emptyset$

- $R_1 \cup R_2$ $\mathcal{C}_\mathcal{E}$-terminating,
- no infinite chain of $[\mathcal{F}_3 \mid R_3]$ over $R_1 \cup R_3 \cup \Pi$,

then $R_1 \cup R_2 \cup R_3$ $\mathcal{C}_\mathcal{E}$-terminating

1. ~~only pairs of $[\mathcal{F}_3 \mid R_3]$~~
2. ~~only pairs of $[\mathcal{F}_1 \cup \mathcal{F}_2 \mid R_1 \cup R_2]$~~
3. ~~a few pairs of $[\mathcal{F}_3 \mid R_3]$ then only pairs of $[\mathcal{F}_1 \cup \mathcal{F}_2 \mid R_1 \cup R_2]$~~

Lemma with $S_1 := \emptyset, S_2 := R_1 \cup R_2, S_3 := R_3 \cup \Pi \quad \mathcal{F}_1 := \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{F}_2 := \mathcal{F}_3$

Theorem.

$R_1(\mathcal{F}_1) \longleftarrow [\mathcal{F}_3 \mid R_3]$, $R_1(\mathcal{F}_1) \longleftarrow [\mathcal{F}_2 \mid R_2]$ with $\mathcal{F}_3 \cap \mathcal{F}_2 = \emptyset$

- $R_1 \cup R_2$ terminating,
- no infinite chain of $[\mathcal{F}_3 \mid R_3]$ over $R_1 \cup R_3 \cup \Pi$,

then $R_1 \cup R_2 \cup R_3$ $\mathcal{C}_\mathcal{E}$-terminating

INCORRECT: $R_1 := \emptyset, R_2 := f(0,1,x) \to f(x,x,x), R_3 := \Pi$

Termination — modularity and orderings

Definition.
$(\geq, \succ)$ $\mathcal{C}_\mathcal{E}$-compatible (over $T(\mathcal{F}, X)$): $\exists(\geq', \succ')$ over $T(\mathcal{F} \cup \{\pi\}, X)$ s. t.

- $(\geq', \succ')|_{T(\mathcal{F},X)} = (\geq, \succ)$,
- $\pi(s,t) \geq' s$, $\pi(s,t) \geq' t$ $\quad \forall s,t \in T(\mathcal{F} \cup \{\pi\}, X)$

Theorem.
$R \subset \succ$ for $(\geq, \succ)$ $\mathcal{C}_\mathcal{E}$-compatible, stable, monotone $\Rightarrow R$ $\mathcal{C}_\mathcal{E}$-terminating

Theorem.

$(\geq, \succ)$ $\mathcal{C}_\mathcal{E}$-compatible, stable, weakly monotone s.t.: $\begin{cases} R \subseteq \geq, \\ \text{DP}(R) \subseteq \succ, \end{cases}$

$\Rightarrow R$ $\mathcal{C}_\mathcal{E}$-terminating

Termination modularity and orderings

Lexicographic composition: OK, defined by AFS: OK

Theorem.
$R_1(\mathcal{F}_1) \leftarrow [\mathcal{F}_2 \mid R_2]$

- R_1 $\mathcal{C}_\mathcal{E}$-terminating
- $\exists(\geq, \succ)$ $\mathcal{C}_\mathcal{E}$-compatible, stable, weakly monotone such that:
  - $R_1 \cup R_2 \subseteq \geq$,
  - DP($[\mathcal{F}_2 \mid R_2]$) $\subseteq \succ$,

then $R_1 \cup R_2$ $\mathcal{C}_\mathcal{E}$-terminating

Termination modularity and orderings

Lexicographic composition: OK, defined by AFS: OK

Theorem.

$R_1(\mathcal{F}_1) \leftarrow [\mathcal{F}_3 \mid R_3]$

$R_1(\mathcal{F}_1) \leftarrow [\mathcal{F}_2 \mid R_2]$

with $\mathcal{F}_3 \cap \mathcal{F}_2 = \emptyset$

- $R_1 \cup R_2$ $\mathcal{C}_\mathcal{E}$-terminating,
- $\exists(\geq, \succ)$ $\mathcal{C}_\mathcal{E}$-compatible, stable, weakly monotone such that:
  - $R_1 \cup R_3 \subseteq \geq$,
  - DP($[\mathcal{F}_3 \mid R_3]$) $\subseteq \succ$,

then $R_1 \cup R_2 \cup R_3$ $\mathcal{C}_\mathcal{E}$-terminating