Termination — orderings

What kind of orderings?

- semantic orderings (interpretations)
  - Integers
  - Polynomials
  - ...
- syntactic orderings (precedences extended to terms)
  - LPO
  - MPO
  - RPO
  - ...
- By transformation

Orderings — semantic

$D \neq \emptyset$ equipped with $\geq_D$ and $>_D = \geq_D - \leq_D$
$\varphi : t \in \mathcal{T}(\mathcal{F}, \emptyset) \mapsto d \in D$
$\succeq_\varphi$ and $>_\varphi$:

$$t_1 \succeq_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) \geq_D \varphi(t_2)$$
$$t_1 >_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) >_D \varphi(t_2)$$

Well-founded if $>_D$ well-founded (converse)

Extension to terms with variables: $\varphi : t \in \mathcal{T}(\mathcal{F}, X) \mapsto d \in (X \to D) \to D$
$\succeq_\varphi$ and $>_\varphi$:

$$t_1 \succeq_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) \succeq_{D,X} \varphi(t_2) \quad (\forall \rho : (X \to D), \varphi(t_1)(\rho) \geq_D \varphi(t_2)(\rho))$$
$$t_1 >_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) >_{D,X} \varphi(t_2) \quad (\forall \rho : (X \to D), \varphi(t_1)(\rho) >_D \varphi(t_2)(\rho))$$

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$$>_\varphi \neq \succeq_\varphi - \preceq_\varphi$$

Stable-strict part, stable on all ground instance (on all extension of $\mathcal{F}$)

Orderings — semantic

Homomorphic Interpretation φ

For all $f \in \mathcal{F}$ of arity n, function $[\![f]\!]_\varphi : D^n \to D$,
for all $\rho \in X \to D$,

$$\varphi(f(t_1, \ldots, t_n))(\rho) = [\![f]\!]_\varphi(\varphi(t_1)(\rho), \ldots, \varphi(t_n)(\rho))$$
$$\varphi(x)(\rho) = \rho(x)$$

Lemma.
$(\succeq_\varphi, >_\varphi)$ stable

Lemma.
If $\forall f \in \mathcal{F}, [\![f]\!]_\varphi$ increasing (resp. strictly) in each parameter,
then $\succeq_\varphi$ (resp. $>_\varphi$) monotone

Orderings semantic

$\mathbb{Z}_\mu = \{n \in \mathbb{Z} | n \geq \mu\}$ with natural ordering

Definition.
Polynomial interpretation: homomorphic on $\mathbb{Z}_\mu$ such that
$\forall f \in \mathcal{F}$, $[\![f]\!]$ polynomial function

To go back to $\mathbb{Z}_0$: $f_0(x_1, \ldots, x_n) = f_\mu(x_1 + \mu, \ldots, x_n + \mu) - \mu$,
$\leadsto$ building $(\succeq^0_\varphi, >^0_\varphi)$ from $(\succeq^\mu_\varphi, >^\mu_\varphi)$.

Rk. — Comparison of polynomials: undecidable (Hilbert 10)
$\leadsto$ techniques not complete, here absolute positivity ($\mu = 0$, coef. > 0).

Rk. — Size: polynomial interp. $[\![f]\!](x_1, \ldots, x_n) = 1 + x_1 + \cdots + x_n$

Orderings semantic

Ex.

$$\left\{ \begin{array}{lcl} x + 0 & \rightarrow & x \\ x + s(y) & \rightarrow & s(x+y) \end{array} \right\} \qquad \begin{array}{lcl} [\![0]\!] & = & 1 \\ [\![s]\!](x) & = & x + 1 \\ [\![+]\!](x,y) & = & x + 2y \end{array}$$

$$[\![x + 0]\!] = x + 2 > [\![x]\!] = x$$
$$[\![x + s(y)]\!] = x + 2y + 2 > [\![s(x+y)]\!] = x + 2y + 1$$

Orderings semantic

Rk. — Other 'reasonable' rings (integral parts) possible: matrices, tropical algebras $(\infty, +, \min)$, arctic...

Rk. — Same idea, other functions: exponential

Ex.

$$\left\{ \begin{array}{lcl} --x & \rightarrow & x \\ -(x \vee y) & \rightarrow & (-x) \wedge (-y) \\ -(x \wedge y) & \rightarrow & (-x) \vee (-y) \\ x \wedge (y \vee z) & \rightarrow & (x \wedge y) \vee (x \wedge z) \\ (y \vee z) \wedge x & \rightarrow & (x \wedge y) \vee (x \wedge z) \end{array} \right\} \qquad \begin{array}{lcl} [\![\text{cte}]\!] & = & 2 \\ [\![\vee]\!](x,y) & = & x + y + 1 \\ [\![\wedge]\!](x,y) & = & xy \\ [\![-]\!](x) & = & 2^x \end{array}$$

Orderings semantic

Rk. — Other 'reasonable' rings (integral parts) possible: matrices, tropical algebras $(\infty, +, \min)$, arctic...

Rk. — Same idea, other functions: ordinals

Ex.

$$\left\{ \begin{array}{lcl} Dt \rightarrow & & 1 \\ D(\text{cte}) & \rightarrow & 0 \\ D(x+y) & \rightarrow & D(x) + D(y) \\ D(x \times y) & \rightarrow & (y \times D(x)) + (x \times D(y)) \\ D(x-y) & \rightarrow & D(x) - D(y) \end{array} \right\} \qquad \begin{array}{lcl} [\![D]\!](x) & = & \omega^x \\ [\![t]\!] & = & 1 \\ [\![\text{cte}]\!] & = & 1 \\ [\![*]\!](x,y) & = & x + y \end{array}$$

Orderings semantic

Rk. — Other 'reasonable' rings (integral parts) possible: matrices, tropical algebras $(\infty, +, \min)$, arctic...

Rk. — Same idea, other functions: exponential, ordinals...

Rk. — DP $\rightsquigarrow$ weak monotony? forget variables!

$$\left\{ \begin{array}{lllllll} \#0 & \to & \# & & & \\ \# + x & \to & x & & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & & x1 + y1 & \to & ((x+y)+\#1)0 \end{array} \right\}$$

$[\![\#]\!] = 0$ $\quad [\![0]\!](x) = x+1$ $\quad [\![\widehat{0}]\!](x) = 0$ non mono

$[\![1]\!](x) = x+1$ $\quad [\![+]\!](x,y) = x$ non mono $\quad [\![\widehat{+}]\!](x,y) = x$ non mono

Orderings syntactic

Orderings on symbols (precedences) extended to terms

$s < t$ if s consists of subterms smaller (for ordering),
in a structure of symbols smaller (for precedence)

Comparison of subterms $\rightsquigarrow$ lexicographic extension, multiset extension
Ordering pairs extended to lists and multisets of terms

Definition.

Lexicographic extension: $s :: l >^{\mathsf{lex}} t :: l' \left\{ \begin{array}{l} s > t \text{ and } |l| = |l'|, \\ s = t \text{ and } l >^{\mathsf{lex}} l' \end{array} \right.$

Theorem.
$>^{\mathsf{lex}}$ well-founded if $>$ well-founded
Stable, monotone

Orderings syntactic

Definition.
Multiset M: application $M_E : E \to \mathbb{N}$ such that $\{e \in E | M_E(e) \neq 0\}$ finite

Notation:

- $e \in M$ if $M_E(e) \geq 1$,
- $M \subseteq N$ if $\forall e, M_E(e) \leq N_E(e)$,
- $M' = M \setminus N$ def. $M'_E(e) = \max(0, M_E(e) - N_E(e))$

Definition.
$(\succeq^{\mathsf{mul}}, >^{\mathsf{mul}})$

- $M \succeq^{\mathsf{mul}} M$,
- $M \succeq^{\mathsf{mul}} N \wedge e \succeq e' \;\Rightarrow\; M \cup \{e\} \succeq^{\mathsf{mul}} N \cup \{e'\}$,
- $M \succeq^{\mathsf{mul}} N \wedge e > e_1, \ldots, e > e_{k \geq 0} \;\Rightarrow$
  $M \cup \{e\} >^{\mathsf{mul}} N \cup \{e_1, \ldots, e_k\}$,
- $M >^{\mathsf{mul}} N \wedge e \succeq e' \;\Rightarrow\; M \cup \{e\} >^{\mathsf{mul}} N \cup \{e'\}$.

Orderings syntactic

Orderings on symbols (precedences) extended to terms

Definition.
Precedence: preordering on $\mathcal{F}$.

Admissible status function for precedence $\geq_{\mathsf{P}}$:
application $\mathrm{ST} : \mathcal{F} \to \{\mathsf{lex}, \mathsf{mul}\}$ such that

1. $f =_{\mathsf{P}} g \;\Rightarrow\; \mathrm{ST}(f) = \mathrm{ST}(g)$,
2. $f =_{\mathsf{P}} g$ and $\mathrm{ST}(f) = \mathsf{lex} = \mathrm{ST}(g)$ then f and g same arity

Orderings — syntactic

RPO: $s \succeq_{\text{RPO}} t$ if and only if

- $s = x \in X$ and $t = x$ or
- $s = f(s_1, \ldots, s_n)$ with $f \in \mathcal{F}$ and
  - $s_i \succeq_{\text{RPO}} t$ for an i, $1 \leq i \leq n$ or
  - $t = g(t_1, \ldots, t_m)$ with $g \in \mathcal{F}$ and
    - $f \succ g$ and for all j, $1 \leq j \leq m$, $s \succ_{\text{RPO}} t_j$ or
    - $f \simeq g$ and
      - $\text{ST}(f) = \text{mul}$ and $\{s_1, \ldots, s_n\}(\succeq_{\text{RPO}})_{\text{mul}}\{t_1, \ldots, t_m\}$ or
      - $\text{ST}(f) = \text{lex}$ thus $n = m$ and $(s_1, \ldots, s_n)(\succeq_{\text{RPO}})_{\text{lex}}(t_1, \ldots, t_m)$ with for all j, $1 \leq j \leq m$, $s \succ_{\text{RPO}} t_j$

$s \succ_{\text{RPO}} t$ if $s \succeq_{\text{RPO}} t$ and $t \not\succeq_{\text{RPO}} s$

Stable, monotone, well-founded if precedence well-founded

Orderings — syntactic

Ex.

$$\left\{\begin{array}{lcl} \text{Ack}(0, x) & \rightarrow & s(x) \\ \text{Ack}(s(x), 0) & \rightarrow & \text{Ack}(x, s(0)) \\ \text{Ack}(s(x), s(y)) & \rightarrow & \text{Ack}(x, \text{Ack}(s(x), y)) \end{array}\right\}$$

RPO with $\text{Ack} >_{\text{P}} s$ and $\text{ST}(\text{Ack}) = \text{lex}$

Rq. — RPO $\supseteq \triangleright$ (simplification ordering): benefit with DP?

$\rightsquigarrow$ change the relation!

Orderings — monotony?

Definition.
AFS A (Argument Filtering System): TRS over $\mathcal{T}(\mathcal{F}' \supseteq \mathcal{F}(\cup \widehat{\mathcal{F}}), X)$ such that $\forall (l \rightarrow r) \in A$

- $l = f(x_1, \ldots, x_n), f \in \mathcal{F}$ and x_i pairwise distinct
- ONE occurrence of f (left-hand side),
- $r = \begin{cases} x_i \in \{x_1, \ldots, x_n\}, \\ f'(x_{i_1}, \ldots, x_{i_k}),\ f' \in \mathcal{F}' \setminus \mathcal{F},\ (x_{i_1}, \ldots, x_{i_k}) \trianglelefteq (x_1, \ldots, x_n). \end{cases}$

Idea: projection to (sub)term with fewer arguments

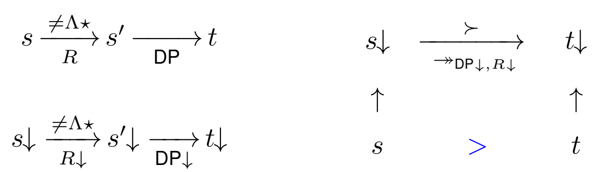
Rq. — AFS strongly normalising, confluent

Notation: $t{\downarrow}_A$ the normal form of t for A

Orderings — monotony?

$R{\downarrow} = \{l{\downarrow} \rightarrow r{\downarrow} \mid l \rightarrow r \in R\}$

$\text{DP}{\downarrow}(R) = \{\langle u{\downarrow}, v{\downarrow}\rangle \mid \langle u, v\rangle \in \text{DP}(R)\}$ $\neq \text{DP}(R{\downarrow})$



Orderings monotony?

Ex.

$$\begin{cases} x - 0 & \to & x \\ s(x) - s(y) & \to & x - y \\ 0 \div s(y) & \to & 0 \\ s(x) \div s(y) & \to & s((x-y) \div s(y)) \end{cases}$$

$\langle s(x) \widehat{-} s(y), x \widehat{-} y \rangle$ 1

$\langle s(x) \widehat{\div} s(y), (x - y) \widehat{\div} s(y) \rangle$ 2

$\langle s(x) \widehat{\div} s(y), x \widehat{-} y \rangle$ 3

$A = \{x - y \to x\}$, RPO with $\div >_p s$

Orderings monotony?

u11(Aout(v_0))	$\to$	Aout(v_0)	u22(Aout(v_0))	$\to$	Aout(v_0)
u21(Aout(v_0), v_1)	$\to$	u22(Ain(v_1, v_0))	Ain(O, v_0)	$\to$	Aout($S(v_0)$)
Ain($S(v_0), O$)	$\to$	u11(Ain($v_0, S(O)$))			
Ain($S(v_0), S(v_1)$)	$\to$	u21(Ain($S(v_0), v_1$), v_0)			

$\langle$u21$^{\#}$(Aout(V_0), V_1), Ain$^{\#}$(V_1, V_0)$\rangle$

$\langle$u21$^{\#}$(Aout(V_0), V_1), u22$^{\#}$(Ain(V_1, V_0))$\rangle$

$\langle$Ain$^{\#}$($S(V_0), O$), u11$^{\#}$(Ain($V_0, S(O)$))$\rangle$

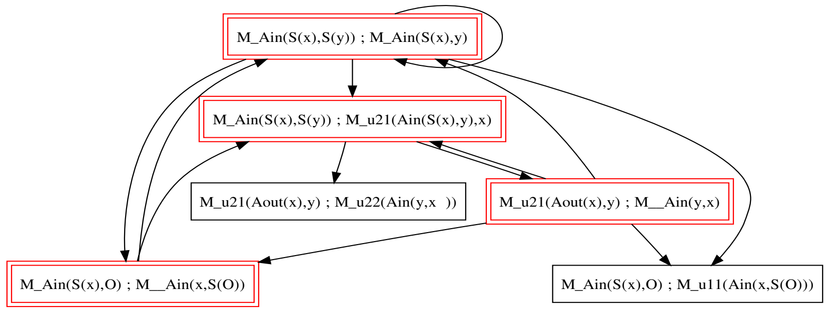
$\langle$Ain$^{\#}$($S(V_0), O$), Ain$^{\#}$($V_0, S(O)$)$\rangle$

$\langle$Ain$^{\#}$($S(V_0), S(V_1)$), u21$^{\#}$(Ain($S(V_0), V_1$), V_0)$\rangle$

$\langle$Ain$^{\#}$($S(V_0), S(V_1)$), Ain$^{\#}$($S(V_0), V_1$)$\rangle$

Orderings monotony?

u11(Aout(v_0))	$\to$	Aout(v_0)	u22(Aout(v_0))	$\to$	Aout(v_0)
u21(Aout(v_0), v_1)	$\to$	u22(Ain(v_1, v_0))	Ain(O, v_0)	$\to$	Aout($S(v_0)$)
Ain($S(v_0), O$)	$\to$	u11(Ain($v_0, S(O)$))			
Ain($S(v_0), S(v_1)$)	$\to$	u21(Ain($S(v_0), v_1$), v_0)			



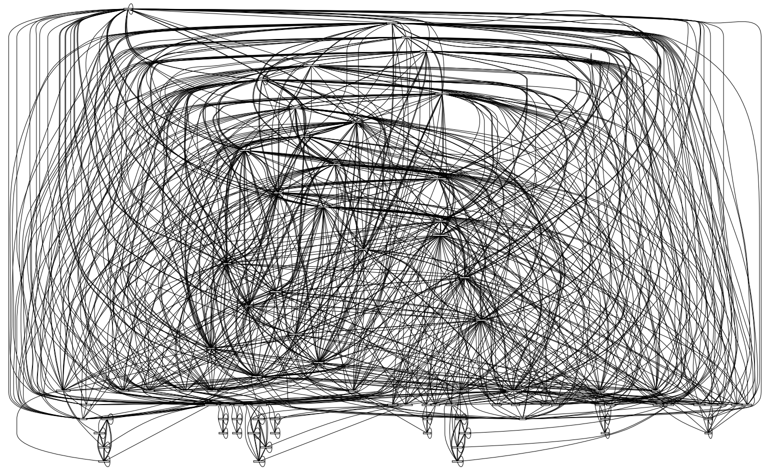
Orderings monotony?

u11(Aout(v_0))	$\to$	Aout(v_0)	u22(Aout(v_0))	$\to$	Aout(v_0)
u21(Aout(v_0), v_1)	$\to$	u22(Ain(v_1, v_0))	Ain(O, v_0)	$\to$	Aout($S(v_0)$)
Ain($S(v_0), O$)	$\to$	u11(Ain($v_0, S(O)$))			
Ain($S(v_0), S(v_1)$)	$\to$	u21(Ain($S(v_0), v_1$), v_0)			

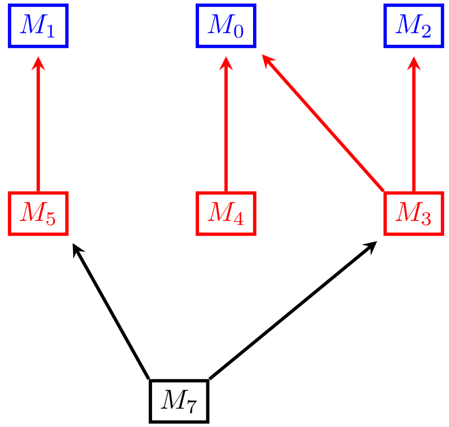
1 [[Aout]]$(X_0) = 0$; [[u11]]$(X_0) = 0$; [[u21]]$(X_0, X_1) = 0$;
[[Ain]]$(X_0, X_1) = 0$; [[u22]]$(X_0) = 0$; $[\![O]\!] = 0$;
$[\![S]\!](X_0) = X_0 + 1$; [[u21$^{\#}$]]$(X_0, X_1) = X_1$; [[Ain$^{\#}$]]$(X_0, X_1) = X_0$;

2 AFS: u21 $\to 1$; Ain $\to 0$; Aout $\to][$; u22 $\to 0$;
RPO with Aout $< O$ and u11 $< S$ (lex status)

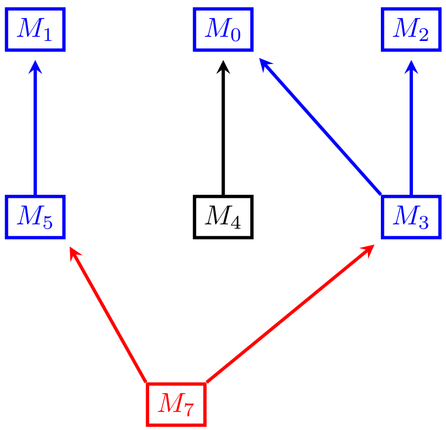
Termination real life graphs...



Termination incremental...



Termination modular...



Termination modularity...

Termination: not modular (Toyama)

$$R \quad : \quad \{f(0,1,x) \to f(x,x,x)\}$$

$$\Pi \quad : \quad \begin{cases} \pi(x,y) \to x \\ \pi(x,y) \to y \end{cases}$$

Allows for:

$$f(\pi(0,1),\pi(0,1),\pi(0,1)) \xrightarrow[\Pi]{}^{\star} f(0,1,\pi(0,1)) \xrightarrow[R]{} \ldots$$

Sub-optimal solutions: simplification orderings... heavy constraints on unions...

Termination — modularity...

Termination: not modular (Toyama)

$$R \quad : \quad \{f(0,1,x) \to f(x,x,x)\}$$

$$\Pi \quad : \quad \left\{ \begin{array}{l} \pi(x,y) \to x \\ \pi(x,y) \to y \end{array} \right.$$

Allows for:

$$f(\pi(0,1),\pi(0,1),\pi(0,1)) \xrightarrow[\Pi]{}^{\star} f(0,1,\pi(0,1)) \xrightarrow[R]{} \dots$$

Definition.
R $\mathcal{C}_\mathcal{E}$-terminating if and only if $R \cup \Pi$ terminating

Termination — modularity...

$\mathcal{C}_\mathcal{E}$: good properties

$\mathcal{C}_\mathcal{E}$-Terminaison modular for:

- Disjoint unions
- Unions with shared constructors for finitely branching systems

⇝ from now on: finitely branching

$$\left\{ \begin{array}{lcl} f_j(c_j,x) & \to & f_{j+1}(x,x) \\ f_j(x,y) & \to & x \\ f_j(x,y) & \to & y \end{array} \right| j \in \mathbb{N} \qquad \left\{ \begin{array}{lcl} a & \to & c_j \\ \pi(x,y) & \to & x \\ \pi(x,y) & \to & y \end{array} \right| j \in \mathbb{N}$$

Termination — modularity...

$\mathcal{C}_\mathcal{E}$: good properties

SN and not duplicating: $\mathcal{C}_\mathcal{E}$ SN

SN and non-deterministic projective: $\mathcal{C}_\mathcal{E}$ SN

SN simplifying: $\mathcal{C}_\mathcal{E}$ SN

Termination — modularity...

Definition.
a module $[\mathcal{F}_2 \mid R_2]$ extends $R_1(\mathcal{F}_1)$:

- $\mathcal{F}_2 \cap \mathcal{F}_1 = \emptyset$,
- $R_2(\mathcal{F}_1 \cup \mathcal{F}_2)$ such that for each $l \to r \in R_2$, $\Lambda(l) \in \mathcal{F}_2$,

⇝ $R = R_1 \cup R_2$: hierarchical extension of R_1

Example.

$\mathcal{F}_0 \quad \{\#, 0, 1\}$

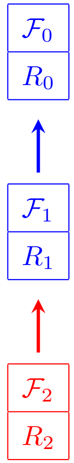
$R_0 \quad \{\#0 \to \#$

$\mathcal{F}_1 \quad \{+\}$

$$R_1 \left\{ \begin{array}{llllll} \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y)+\#1)0 \end{array} \right.$$

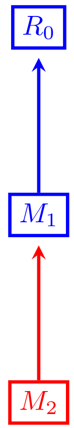
$\mathcal{F}_2 \quad \{\times\}$

$$R_2 \left\{ \begin{array}{llllll} \# \times x & \to & \# & x \times \# & \to & \# \\ x0 \times y & \to & (x \times y)0 & x1 \times y & \to & (x \times y)0 + y \end{array} \right.$$

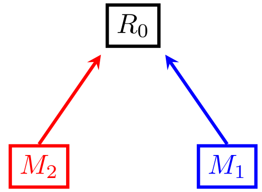


Goal: proving termination incrementally

Hierarchical with common subsystem (Ohlebusch)



Composable systems (Gramlich, Kurihara & Ohuchi. . .)



Termination modularity. . .

Definition.
$R_1(\mathcal{F}_1) \leftarrow [\mathcal{F}_2 \mid R_2]$.
$\langle l, r' \rangle$ dependency pair of module $[\mathcal{F}_2 \mid R_2]$:

- $l \to r \in R_2$,
- r' subterm of r such that $\Lambda(r') \in \mathcal{F}_2$

DP(M): dependency pairs of all rules in M

Rk. — $R(\mathcal{F}) \quad \rightsquigarrow \quad [\mathcal{F}_C \mid \emptyset] \leftarrow [\mathcal{F}_D \mid R]$.

For $[\mathcal{F}_C \mid \emptyset] \leftarrow [\mathcal{F}_D \mid R]$, same as Arts & Giesl's