Equational theories — Back to $\mathcal{F}$-algebras...

Precongruence: relation $\sim$ on $\mathcal{F}$-algebra A compatible with structure.

$\forall f : s_1 \times \cdots \times s_n \to s, \forall t_1, \ldots, t_n, u_1, \ldots, u_n$ such that $t_i, u_i \in A_i$,
$t_1 \sim u_1 \wedge t_n \sim u_n \Longrightarrow f_A(t_1, \ldots, t_n) \sim f_A(u_1, \ldots, u_n)$

Congruence: precongruence + equivalence

$\rightsquigarrow$ quotient

For E set of equations on $\mathcal{T}((,\mathcal{F}), X)$,
$=_E$ smallest congruence such that $\forall \sigma, \forall s = t \in E, s\sigma =_E t\sigma$

Equational theory — AC

A particular one: AC

Efficiency... Power of expression...

Axioms: $f \in \mathcal{F}_{\mathrm{AC}} \left\{ \begin{array}{l} f(x,y) = f(y,x) \\ f(f(x,y),z) = f(x, f(y,z)) \end{array} \right\}$ $=_{\mathrm{AC}}$

Many changes (unification... termination...)

Flattened variadic terms: $a + (b + c) = (a + b) + c = a + b + c$

Equational theory — AC

Variadic:

- $x \in X \Rightarrow x \in T_{var}(\mathcal{F}, X)$
- $f \notin \mathcal{F}_{\mathrm{AC}}$ of arity n,
  $t_1, \ldots, t_n \in T_{var}(\mathcal{F}, X) \Rightarrow f(t_1, \ldots, t_n) \in T_{var}(\mathcal{F}, X)$
- $f \in \mathcal{F}_{\mathrm{AC}}, t_1, \ldots, t_n \in T_{var}(\mathcal{F}, X), n \geq 2 \Rightarrow f(t_1, \ldots, t_n) \in T_{var}(\mathcal{F}, X)$

- Commutativity $\rightsquigarrow$ permutation
- Associativity $\rightsquigarrow$ insertion arg

Equational theory — AC

Flattening: *flat*: $T_{var} \to T_{var}$

- $f \in \mathcal{F} \setminus \mathcal{F}_{\mathrm{AC}}$, *flat*$(f(t_1, \ldots, t_n)) = f($*flat*$(t_1), \ldots,$ *flat*$(t_n))$
- $g \in \mathcal{F}_{\mathrm{AC}}$, *flat*$(g(s_1, \ldots, s_n)) = g(t_{1,1}, \ldots, t_{1,k_1}, \ldots, t_{n,1}, \ldots, t_{n,k_n})$ where
  - $s_i(\Lambda) \neq g$: $k_i = 1, t_{i,1} = s_i$
  - $s_i(\Lambda) = g$: $g(t_{i,1}, \ldots, t_{i,k_i}) =$ *flat*(s_i)

Notation *flat*(s): $\overline{s}$

AC: congruence of permutation on flattened terms

Equational theory — AC-rewriting, Peterson & Stickel

s rewrites to t modulo AC (at pos. p):

$$\exists C,\ p,\ \sigma,\ l \to r \in R \text{ such that } s =_{\mathrm{AC}} C[l\sigma]_p \quad C[r\sigma]_p =_{\mathrm{AC}} t$$

R/AC

Impractical (size of class...) $\rightsquigarrow$ weaker relation

$s \xrightarrow[\mathrm{AC}\backslash R]{} t$ iff $\exists p,\ \sigma,\ l \to r \in R, s|_p =_{\mathrm{AC}} l\sigma \quad t \equiv s[r\sigma]_p$

$a + b \to c$ what with $(a + c) + b \ldots$ coherence?

$\rightsquigarrow$ extension rules: for $1_1 + l_2 \to r$ add $l_1 + (l_2 + x) \to r + x$

Equational theory — AC-rewriting, Peterson & Stickel

Here: AC-extended on flattened terms (à la P & S): $s \to t$

- $s|_p =_{\mathrm{AC}} \overline{l\sigma}, \quad t = \overline{s[r\sigma]_p}$,
- $s|_p =_{\mathrm{AC}} \overline{f(l_1, \ldots, l_n, x)\sigma}, \quad t = \overline{s[f(r, x)\sigma]_p}$ $\qquad l = f(l_1, \ldots, l_n)$

Theorem.
R/AC terminating iff AC*flat* $\backslash\ R$ terminating

(exercise)

AC-Termination — orderings

Must be compatible

$$\begin{array}{ccccccc} s & \succ & t & \qquad & s & \geq & t \\ ||_{\mathrm{AC}} & & ||_{\mathrm{AC}} & & ||_{\mathrm{AC}} & & ||_{\mathrm{AC}} \\ s' & \succ & t' & & s' & \geq & t' \end{array}$$

AC-Termination — semantical orderings

Must be compatible... Polynomials

$$\begin{array}{rcl} P(X, Y) & = & P(Y, X) \\ P(P(X, Y), Z) & = & P(X, P(Y, Z)) \end{array}$$

If over commutative ring: $aXY + b(X + Y) + c$ with $b^2 = b + ac$

In practice: linear/simple, possibly simple-mixed

$$\left\{ \begin{array}{lcl} x + 0 & \to & x \\ x + s(y) & \to & s(x + y) \end{array} \right. \qquad \mathcal{F}_{\mathrm{AC}} = \{+\} \qquad \begin{array}{lcl} [\![+]\!](x, y) & = & yx + 2(y + x) + 2 \\ [\![s]\!](x) & = & x + 1 \\ [\![0]\!] & = & 0 \end{array}$$

AC-Termination — syntactical orderings

Path orderings

$f(x,y) >_{RPO} g(x,y) \qquad f >_{\text{prec}} g$

But $f(x,y,z) <_{RPO} f(g(x,y),z)$

Not monotone! $\rightsquigarrow$ normalisation by other systems..., new orderings...

APO [Bachmair & Plaisted]

Associative Path Cond.: $\forall f \in \mathcal{F}_{\mathrm{AC}} \begin{cases} f \text{ minimal in } \mathcal{F} \quad \text{or} \\ \exists g \in \mathcal{F}_{\mathrm{AC}} \text{ s.t. } f \text{ minimal } \in \mathcal{F} \setminus \{g\} \end{cases}$

Normalisation D convergent:

$\forall f >_{\text{prec}} g \in \mathcal{F}_{\mathrm{AC}} \quad f(g(x,y),z) \to g(f(x,z),f(y,z))$

APO: $s >_{APO} t \iff \overline{s\downarrow_D} >_{RPO} \overline{t\downarrow_D}$ (monotone, simplifying)

A path ordering for AC without normalisation system: ACRPO [Rubio]

AC-Termination — AC-DP

For now without marks

Definition.
For a $f(t_1,\ldots,t_n) \to r$,
AC-dependency pairs: vanilla DP + DP from $f(t_1,\ldots,t_n,x) \to \overline{f(r,x)}$
with fresh x (when necessary)

$$\begin{array}{lll} x+0 & \to & x \\ x+s(y) & \to & s(x+y) \end{array} \qquad \begin{array}{lll} x\times 0 & \to & 0 \\ x\times s(y) & \to & (x\times y)+x \end{array}$$

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$\langle x+s(y), x+y\rangle$
$\langle x\times s(y), x\times y\rangle$
$\langle x\times s(y), (x\times y)+x\rangle$

$\langle x+s(y)+z, x+y\rangle$
$\langle x+s(y)+z, s(x+y)+z\rangle$
$\langle x\times s(y)\times z, x\times y\rangle$
$\langle x\times s(y)\times z, ((x\times y)+x)\rangle$
$\langle x\times s(y)\times z, ((x\times y)+x)\times z\rangle$

AC-Termination — AC-DP

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$\langle x+s(y), x+y\rangle$
$\langle x\times s(y), x\times y\rangle$
$\langle x\times s(y), (x\times y)+x\rangle$
$\langle x\times 0\times z, 0\times z\rangle$

$\langle x+s(y)+z, x+y\rangle$
$\langle x+s(y)+z, s(x+y)+z\rangle$
$\langle x\times s(y)\times z, x\times y\rangle$
$\langle x\times s(y)\times z, ((x\times y)+x)\rangle$
$\langle x\times s(y)\times z, ((x\times y)+x)\times z\rangle$

AC-Termination

Minimality? $u = f(u_1, \cdots, u_n) \trianglelefteq t$ immortal

- u immortal
- u_i all mortal
- f defined
- If $f \in \mathcal{F}_{\mathrm{AC}}$, $u = f(u', u'')$ with $f(u')$ immortal and $f(u' \setminus \{u_i\})$ mortal

$\rightsquigarrow$ parameters unavoidable

AC-Termination

Lemma.
t immortal for R (modulo AC) then $\exists p, p', C, C', \langle s_1, s_2\rangle, \sigma$ s.t.

$$t \xrightarrow[R]{>p}{}^{\star} C[s_1\sigma]_p \xrightarrow[R]{p} C[C'[s_2\sigma]'_p]_p \text{ with } s_2\sigma \text{ immortal}$$

Theorem.
R AC-terminating iff $\twoheadrightarrow_{\mathrm{DP}_{\mathrm{AC}}(R), R}$ AC-terminating

AC and modules? sweet as. . .

Marking DP? nightmarish: marked symbols $\notin \mathcal{F}_{\mathrm{AC}}$