Term Rewriting

Rewriting rule : couple of terms $s \to t$ (oriented equation)

$$s \xrightarrow[l\to r,\sigma]{p} t \quad \text{if} \quad s|_p \equiv l\sigma \quad t \equiv s[r\sigma]_p$$

Rewriting system : set of rules

Relation $\underset{R}{\to}$: $s \underset{R}{\to} t$ iff $\exists l \to r \in R, \exists p \in \mathcal{P}os(s), \exists \sigma, s \xrightarrow[l\to r,\sigma]{p} t$

Actually, system extension by *monotony* and *stability*

Reflexive/transitive closure : $\underset{R}{\to}^{\star}$ Transitive closure : $\underset{R}{\to}^{+}$

Converse : $\underset{R}{\leftarrow}$ Reflexive/symmetric/transitive : $\underset{R}{\overset{\star}{\leftrightarrow}}$

Rewriting Example

Binary arithmetics

6 seen as : $\#110$. Set of variables : $X = \{x \,;\, y \ldots\}$;

Signature : $\mathcal{F} = \{\# : 0 \,;\, 0 : 1 \,;\, 1 : 1 \,;\, + : 2\}$;

Set of rules.

$$\begin{cases} \#0 \to \# & \\ \# + x \to x & x + \# \to x \\ x0 + y0 \to (x+y)0 & x1 + y0 \to (x+y)1 \\ x0 + y1 \to (x+y)1 & x1 + y1 \to ((x+y) + \#1)0 \end{cases}$$

Relation $\to$ monotonic : if $s \to t$ then $C[s] \to C[t]$;

stable : if $s \to t$ then $s\sigma \to t\sigma$.

$\#10 + \#1 \to (\#1 + \#)1 \to (\#1)1$ STOP

Term Rewriting

Computing power : Turing complete

Questions :

- Existence of result ⇝ termination
- Unicity of result ⇝ confluence, convergence

Term Rewriting

Questions :

- Existence of result ⇝ termination
- Unicity of result ⇝ confluence, convergence

t normal form for R : no u such that $t \underset{R}{\to} u$

t normal form of s for R : no u such that $t \underset{R}{\to} u$ and $s \underset{R}{\to}^{\star} t$

⇝ s normalisable

System normalising : every term normalisable

System strongly normalising : every derivation ⇝ normal form

Term Rewriting

Questions :

- Existence of result $\rightsquigarrow$ termination
- Unicity of result $\rightsquigarrow$ confluence, convergence

R Church-Rosser : $u \overset{\star}{\leftrightarrow} v \Rightarrow \exists t,\ u \rightarrow^{\star} t \longleftarrow^{\star} v$

R confluent : $u \longleftarrow^{\star} s \rightarrow^{\star} v \Rightarrow \exists t,\ u \rightarrow^{\star} t \longleftarrow^{\star} v$

R locally confluent : $u \longleftarrow s \rightarrow v \Rightarrow \exists t,\ u \rightarrow^{\star} t \longleftarrow^{\star} v$

Term Rewriting

Example

$$\begin{array}{lcl} x \cdot e & \rightarrow & x \\ x \cdot I(x) & \rightarrow & e \\ (x \cdot y) \cdot z & \rightarrow & x \cdot (y \cdot z) \end{array}$$

$$x \cdot (I(x) \cdot z) \longleftarrow (x \cdot I(x)) \cdot z \longrightarrow e \cdot z$$

Term Rewriting

R Church-Rosser : $u \overset{\star}{\leftrightarrow} v \Rightarrow \exists t,\ u \rightarrow^{\star} t \longleftarrow^{\star} v$

$\Updownarrow$?

R confluent : $u \longleftarrow^{\star} s \rightarrow^{\star} v \Rightarrow \exists t,\ u \rightarrow^{\star} t \longleftarrow^{\star} v$

$\Updownarrow$ ~~?~~ X

R locally confluent : $u \longleftarrow s \rightarrow v \Rightarrow \exists t,\ u \rightarrow^{\star} t \longleftarrow^{\star} v$

Questions :

- Existence of result $\rightsquigarrow$ termination
- Unicity of result $\rightsquigarrow$ confluence, convergence

But. . .

Theorem. (Newman's lemma)
Local confluence $\Leftrightarrow$ confluence for strongly normalising relations

Term Rewriting

Decidability

Given : finite system R

Question : R confluent ?

Undecidable (red. word pb.)

Term Rewriting

Critical pair : $r\rho\sigma = (l\rho[d]_p)\sigma$ where

- $l \to r \in R, \quad g \to d \in R$
- p position non variable of l
- ρ renaming of l
- σ most general unifier of $l|_p$ and g (rule superposition)

Set of critical pairs of R : $\mathrm{PC}(R)$

Theorem.
Local confluence of pairs of $\mathrm{PC}(R)$ $\Leftrightarrow$ local confluence of R

$\leadsto$ Decidable for strongly normalising finite systems

CP-Lemma

Theorem.
Local confluence of pairs of $\mathrm{PC}(R)$ $\Leftrightarrow$ local confluence of R

Proof : $\Leftarrow$ trivial.

$\Rightarrow$: u, s, t terms such that $u \to^p_{l\to r,\sigma} s$ and $u \to^q_{g\to d,\tau} t$

- p not prefix of q (and q not prefix of p)
  $\leadsto$ commutation, no problem
- q inside substitution ($q = p \cdot q_1 \cdot q_2$ and $l(q_1) \in X$)
  $\leadsto$ rewrite ALL instances of such q by $g \to d$ then at p by $l \to r$,
  (same as first $l \to r$ then all $g \to d$)
- $p,\ q$ critical ($g\tau$ somewhere in $l\sigma$ outside σ)
  $\leadsto$ assumption local confluence of CP

Back to Termination then…

- Existence of result $\leadsto$ termination
- Unicity of result $\leadsto$ confluence, convergence

Termination

Fundamental property :

- Inductions,
- Totality of functions,
- Preliminary,
- Selfstabilisation, liveness, etc.

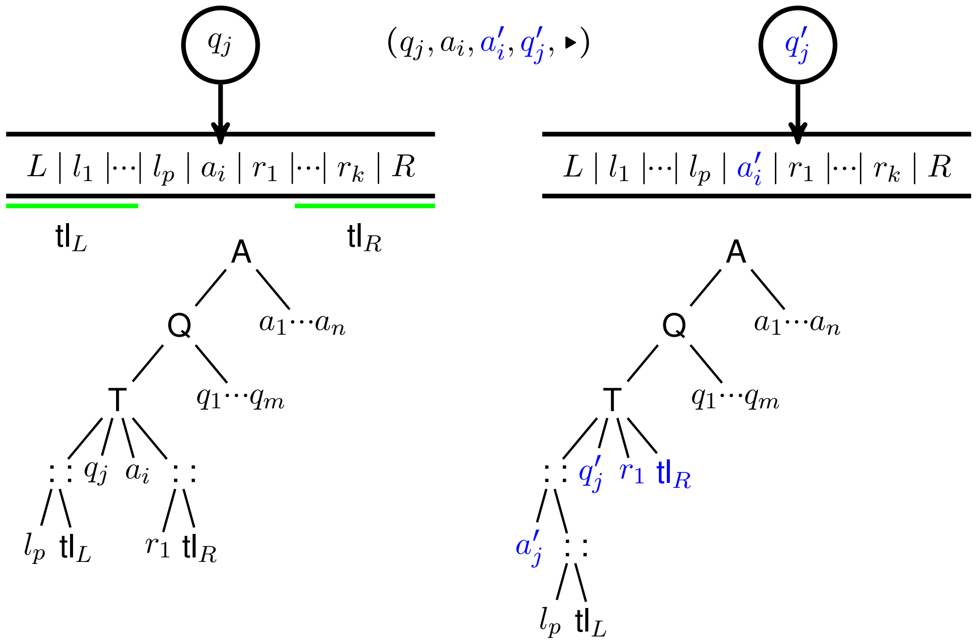
In this lecture : first order term rewriting
Enough ? Turing complete

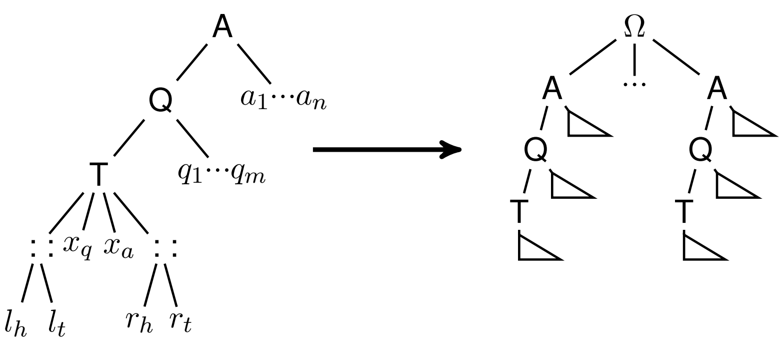
Termination

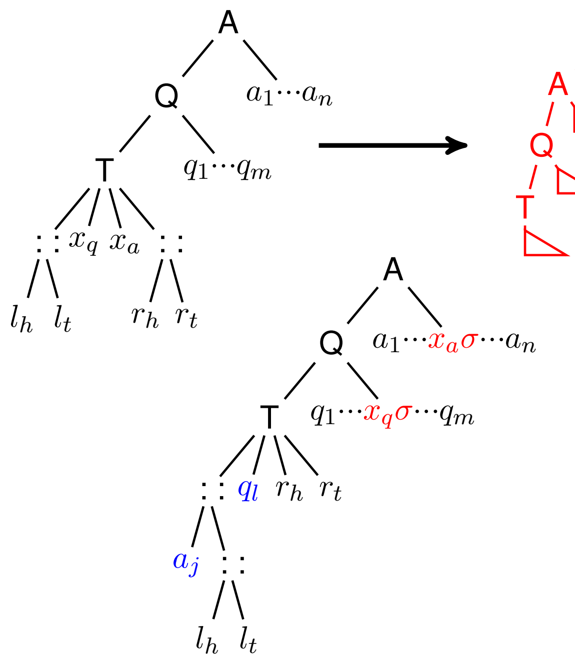
Automation ? $\rightsquigarrow$ correct, incomplete…

Always difficult

- $f(f(x)) \to f(x)$
- $f(a, b, x) \to f(x, x, x)$
- > 50 rules + equational theories
- > 1800 rules (> 1000 symbols)
- $\left\{ \begin{array}{lcl} a(a(x)) & \to & b(c(x)) \\ b(b(x)) & \to & a(c(x)) \\ c(c(x)) & \to & a(b(x)) \end{array} \right.$
- Syracuse…



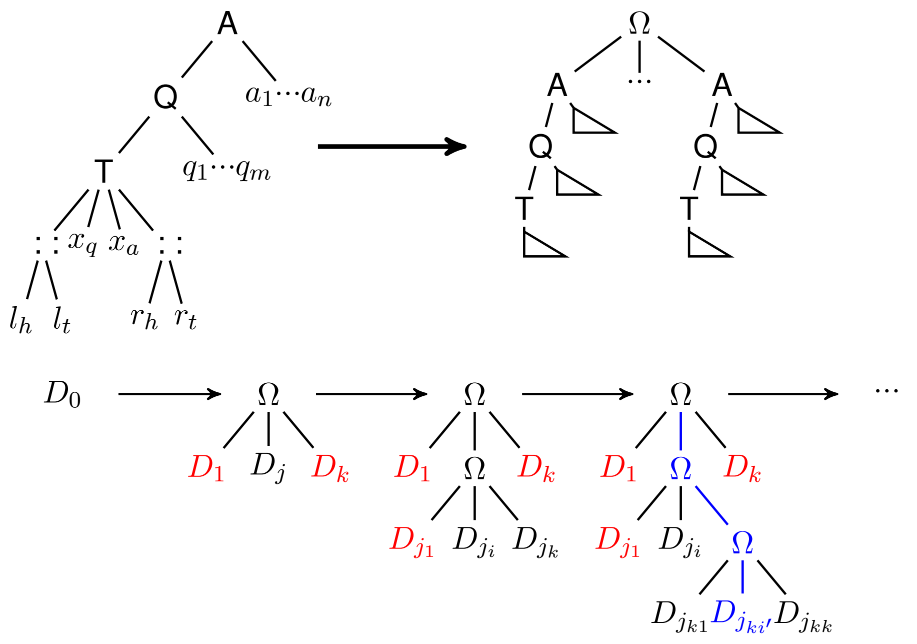




$(q_k, a_i, a_j, q_l, ▸)$

$x_a\sigma$ in place of a_i

$x_q\sigma$ in place of q_k



Termination

Termination of TRS = well-foundness of relation (WF) $\rightsquigarrow$ « measure »

How ? $\mathcal{R} \subseteq O \wedge \text{WF}(O) \Rightarrow \text{WF}(\mathcal{R})$

$\mathcal{R}, O, f$ such that $s\ \mathcal{R}^+\ t \Rightarrow f(s)\ O^+\ f(t) \wedge \text{WF}(O) \Rightarrow \text{WF}(\mathcal{R})$.

$(\text{WF}(O) \Rightarrow \text{WF}(O^+) \Rightarrow \text{WF}(O^+ \circ f) \Rightarrow \text{WF}(\mathcal{R}^+))$

Problem transformation until : pb. trivial, pb. inclusion

$$\text{RULE NAME(PARAM)}\ \frac{p_1 \dots p_n}{p}\ \text{CONDITIONS}$$

Transf. correct and complete : criterion $\rightsquigarrow$ termination constraints

Inclusion : ordering constraints

Termination

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Problem transformation until : pb. trivial, pb. inclusion

$<_2$ (RPO)

$<$ $\{\dots\langle u''_k, v''_k\rangle\dots\}$ SN ? $\{\dots\langle u''_j, v''_j\rangle\dots\}$ SN ?

$<_1$ (poly. interp.)

$<$ $\{\dots\langle u'_n, v'_n\rangle\dots\}$ SN ? $\{\dots\langle u'_i, v'_i\rangle\dots\}$ SN ? GRAPH

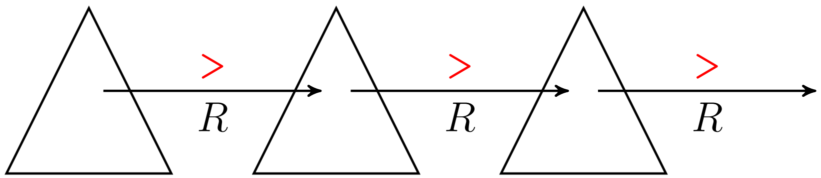
GRAPH

$\mathcal{R}_{\text{DP}} = \{\dots\langle u_i, v_i\rangle\dots\}$ SN ? DP

$\mathcal{R}_{\text{init}} = \{\dots l_i \to r_i \dots\}$ SN ?

Termination

Inclusion $\to_R \subseteq >$: $s > t$ for all $s \to_R t$



Infinitely many $s \to t \rightsquigarrow$ automation ?

Test finite ?

Termination – Manna-Ness

Idea : ordering on rules stable through closures giving relation

Theorem. (Lankford)

- R a TRS $\{l_1 \to r_1, \ldots, l_n \to r_n\}$,
- $<$ such that WF($<$), $<$ stable and monotone,

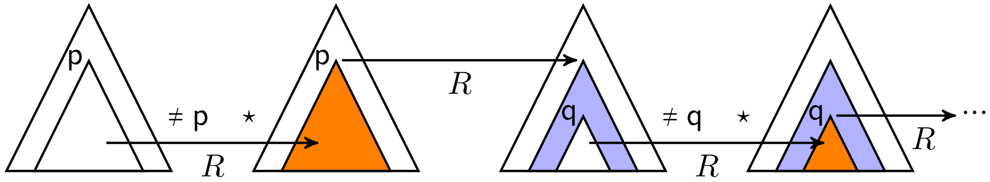
then $(\forall i, l_i > r_i) \Rightarrow$ SN($\to_R$).

Example

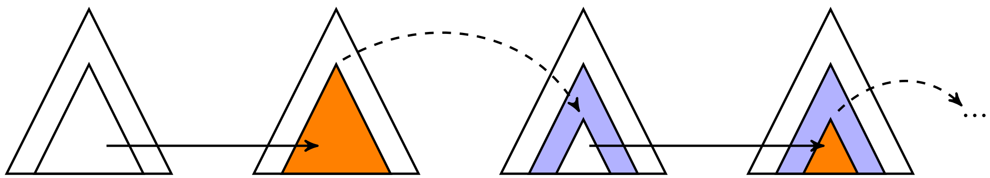
$$\left\{ \begin{array}{lcl} x + 0 & \to & x \\ x + s(y) & \to & s(x+y) \end{array} \right\} \quad [\![s]\!] >_e [\![t]\!] \text{ with } \begin{array}{lcl} [\![0]\!] & = & 1 \\ [\![s]\!](x) & = & x+1 \\ [\![+]\!](x,y) & = & x + 2y \end{array}$$

$$[\![x+0]\!] = x + 2 >_e [\![x]\!] = x$$
$$[\![x+s(y)]\!] = x + 2y + 2 >_e [\![s(x+y)]\!] = x + 2y + 1$$

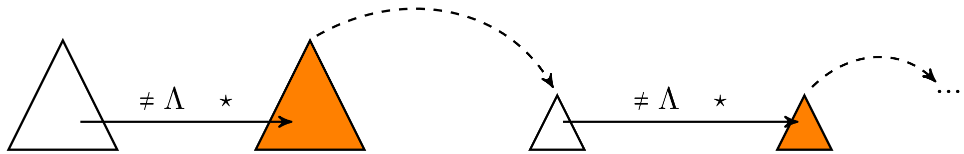
Termination – DP



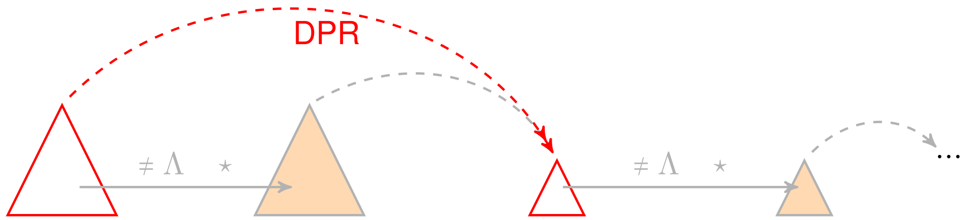
Termination – DP



Termination – DP



Termination – DP



Termination – DP

Definition

Given a TRS R,

$\mathcal{F} = D \uplus C \quad D : \{f \in \mathcal{F} \mid \exists (l \to r) \in R, l(\Lambda) = f\}$

D : defined (functions) C : constructors (data)

Definition

Given a rule $l \to r$,

Dependency pair : couple $\langle u, v \rangle$

- $u = l$
- $v = r|_p$ such that $r(p) \in D$

Set of dependency pairs of a TRS R : DP(R)

Termination – DP

Example

$D = \{0, +\} \quad C = \{\#, 1\} \qquad \mathcal{F} = \{\#, 0, 1, +\}$

$$\left\{ \begin{array}{lllllll} \#0 & \to & \# & & & & \\ \# + x & \to & x & & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & & x1 + y1 & \to & ((x+y) + \#1)0 \end{array} \right\}$$

$\langle x1 + y1, x + y \rangle \qquad \langle x1 + y0, x + y \rangle$

$\langle x0 + y1, x + y \rangle \qquad \langle x0 + y0, x + y \rangle$

$\langle x0 + y0, (x+y)0 \rangle \qquad \langle x1 + y1, (x+y) + \#1 \rangle$

$\langle x1 + y1, ((x+y) + \#1)0 \rangle$

Termination – DP

Example

$\{f(f(x)) \to f(g(f(x)))\} \quad D = \{f\} \quad C = \{g\}$

$\langle f(f(x)), f(g(f(x))) \rangle \quad \langle f(f(x)), f(x) \rangle$

what about derivations ?

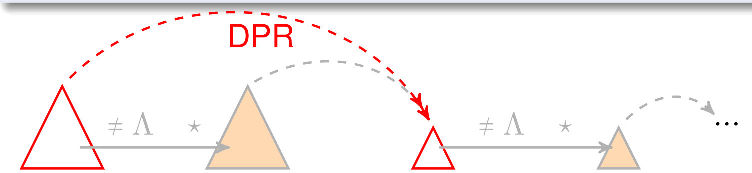
Termination – DP

Definition

Dependency chain : sequence of DP, subst. σ such that

$$\cdots \quad \langle u_i, \quad v_i\rangle \qquad \langle u_{i+1}, \quad v_{i+1}\rangle \quad \cdots$$

$$v_i\sigma \xrightarrow{\neq\Lambda\star} u_{i+1}\sigma \quad v_{i+1}\sigma$$



Termination – DP

Definition

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$$v_i\sigma \xrightarrow{\neq\Lambda\star} u_{i+1}\sigma$$

Theorem. (A & G)

SN($\to_R$) $\Leftrightarrow$ no infinite chain over DP(R)

Rephrased : SN($\to_R$) $\Leftrightarrow$ SN($\twoheadrightarrow_{\mathrm{DP}(R),R}$)

Termination – DP

Rk. — $R = \{f(f(x)) \to h(f(x)),\ g(x) \to f(x)\}$ $\qquad D = \{f, g\} \quad C = \{h\}$

DP(R) = $\{\langle f(f(x)), f(x)\rangle,\ \langle g(x), f(x)\rangle\}$

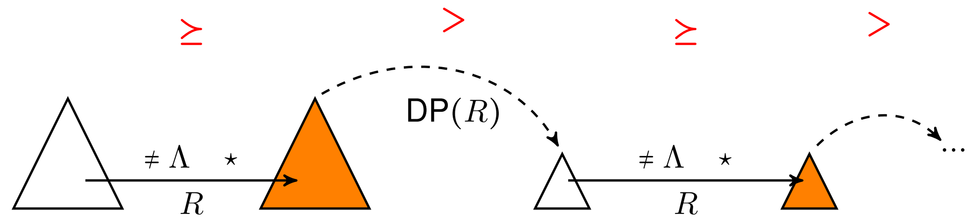
SN($\to_R$) ? $\rightsquigarrow$ SN($\xrightarrow[R]{\neq\Lambda\star} \cdot \xrightarrow[\mathrm{DP}(R)]{}$) ? $\qquad$ ($\rightsquigarrow$ SN($\twoheadrightarrow_{\mathrm{DP}(R),R}$) ?)

With minimal chains. . .

$f(f(x))\sigma$ NOT minimal : irrelevant

SN($\to_R$) $\Leftrightarrow$ SN($\twoheadrightarrow_{\langle g(x),f(x)\rangle,R}$)

Termination – DP, control



Theorem. (A & G)

If $(\geq, >)$ such that

1. $\geq \cdot > \subseteq >$, WF($<$), $\geq$ monotone, stable, $>$ stable, (monotony useless)
2. $l \geq r$ for each $l \to r \in R$,
3. $u > v$ for each $\langle u, v\rangle \in$ DP(R),

then SN($\to_R$)