Termination – orderings

What kind of orderings ?

- semantical orderings (interpretations)
  - Integers
  - Polynomials
  - ...
- syntactical orderings (precedences extended to terms)
  - LPO
  - MPO
  - RPO
  - ...
- By transformation

Orderings – semantical

$D \neq \varnothing$ equipped with $\geq_D$ and $>_D = \geq_D - \leq_D$

$\varphi : t \in \mathcal{T}(\mathcal{F}, \varnothing) := d \in D$

$\geq_\varphi$ and $>_\varphi$:

$$t_1 \geq_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) \geq_D \varphi(t_2)$$
$$t_1 >_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) >_D \varphi(t_2)$$

Well-founded if $>_D$ well-founded (converse)

Extension to terms with variables : $\varphi : t \in \mathcal{T}(\mathcal{F}, X) := d \in (X \to D) \to D$

$\geq_\varphi$ and $>_\varphi$:

$$t_1 \geq_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) \geq_{D,X} \varphi(t_2) \quad (\forall \rho : (X \to D), \varphi(t_1)(\rho) \geq_D \varphi(t_2)(\rho))$$
$$t_1 >_\varphi t_2 \quad \text{iff} \quad \varphi(t_1) >_{D,X} \varphi(t_2) \quad (\forall \rho : (X \to D), \varphi(t_1)(\rho) >_D \varphi(t_2)(\rho))$$

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$>_\varphi \neq \geq_\varphi - \leq_\varphi$

Stable-strict part, stable on all ground instance (on all extension of $\mathcal{F}$)

Orderings – semantical

Homomorphic Interpretation φ

For all $f \in \mathcal{F}$ of arity n, function $[\![f]\!]_\varphi : D^n \to D$,

for all $\rho \in X \to D$,

$$\varphi(f(t_1, \ldots, t_n))(\rho) = [\![f]\!]_\varphi(\varphi(t_1)(\rho), \ldots, \varphi(t_n)(\rho))$$
$$\varphi(x)(\rho) = \rho(x)$$

Lemma.

$(\geq_\varphi, >_\varphi)$ stable

Lemma.

If $\forall f \in \mathcal{F}, [\![f]\!]_\varphi$ increasing (resp. strictly) in each parameter,

then $\geq_\varphi$ (resp. $>_\varphi$) monotone

Orderings – semantical

$\mathbb{Z}_\mu = \{n \in \mathbb{Z} | n \geq \mu\}$ with natural ordering

Definition

Polynomial interpretation : homomorphic on $\mathbb{Z}_\mu$ such that
$\forall f \in \mathcal{F}$, $[\![f]\!]$ polynomial function

To go back to $\mathbb{Z}_0$: $f_0(x_1, \ldots, x_n) = f_\mu(x_1 + \mu, \ldots, x_n + \mu) - \mu$,
$\leadsto$ building $(\geq^0_\varphi, >^0_\varphi)$ from $(\geq^\mu_\varphi, >^\mu_\varphi)$.

Rk. — Comparison of polynomials : undecidable (Hilbert 10)
$\leadsto$ techniques not complete, here absolute positivity ($\mu = 0$, coef. > 0).
Rk. — Size : polynomial interp. $[\![f]\!](x_1, \ldots, x_n) = 1 + x_1 + \cdots + x_n$

Orderings – semantical

Example

$$\left\{ \begin{array}{lcl} x + 0 & \to & x \\ x + s(y) & \to & s(x+y) \end{array} \right\} \qquad \begin{array}{lcl} [\![0]\!] & = & 1 \\ [\![s]\!](x) & = & x+1 \\ [\![+]\!](x,y) & = & x + 2y \end{array}$$

$$[\![x+0]\!] = x + 2 > [\![x]\!] = x$$
$$[\![x + s(y)]\!] = x + 2y + 2 > [\![s(x+y)]\!] = x + 2y + 1$$

Orderings – semantical

Rk. — Other 'reasonable' rings (integral parts) possible : matrices, tropical algebras $(\infty, +, \min)$, arctic. . .
Rk. — Same idea, other functions : exponential

Example

$$\left\{ \begin{array}{lcl} --x & \to & x \\ -(x \vee y) & \to & (-x) \wedge (-y) \\ -(x \wedge y) & \to & (-x) \vee (-y) \\ x \wedge (y \vee z) & \to & (x \wedge y) \vee (x \wedge z) \\ (y \vee z) \wedge x & \to & (x \wedge y) \vee (x \wedge z) \end{array} \right\} \qquad \begin{array}{lcl} [\![\text{cte}]\!] & = & 2 \\ [\![\vee]\!](x,y) & = & x + y + 1 \\ [\![\wedge]\!](x,y) & = & xy \\ [\![-]\!](x) & = & 2^x \end{array}$$

Orderings – semantical

Rk. — Other 'reasonable' rings (integral parts) possible : matrices, tropical algebras $(\infty, +, \min)$, arctic. . .
Rk. — Same idea, other functions : ordinals

Example

$$\left\{ \begin{array}{lcl} Dt \to & & 1 \\ D(\text{cte}) & \to & 0 \\ D(x+y) & \to & D(x) + D(y) \\ D(x \times y) & \to & (y \times D(x)) + (x \times D(y)) \\ D(x-y) & \to & D(x) - D(y) \end{array} \right\} \qquad \begin{array}{lcl} [\![D]\!](x) & = & \omega^x \\ [\![t]\!] & = & 1 \\ [\![\text{cte}]\!] & = & 1 \\ [\![*]\!](x,y) & = & x + y \end{array}$$

Orderings – semantical

Rk. — Other 'reasonable' rings (integral parts) possible : matrices, tropical algebras $(\infty, +, \min)$, arctic...

Rk. — Same idea, other functions : exponential, ordinals...

Rk. — DP $\rightsquigarrow$ weak monotony ? forget variables !

$$\left\{\begin{array}{llllll} \#0 & \to & \# & & & \\ \# + x & \to & x & x + \# & \to & x \\ x0 + y0 & \to & (x+y)0 & x1 + y0 & \to & (x+y)1 \\ x0 + y1 & \to & (x+y)1 & x1 + y1 & \to & ((x+y) + \#1)0 \end{array}\right\}$$

$[\![\#]\!] = 0$	$[\![0]\!](x) = x + 1$	$[\![\top 0]\!](x) = 0$	non mono
$[\![1]\!](x) = x + 1$	$[\![+]\!](x, y) = x$ non mono	$[\![\top +]\!](x, y) = x$	non mono

Orderings – syntactical

Orderings on symbols (precedences) extended to terms

$s < t$ if s consists of subterms smaller (for ordering),

in a structure of symbols smaller (for precedence)

Comparison of subterms $\rightsquigarrow$ lexicographic extension, multiset extension

Ordering pairs extended to lists and multisets of terms

Definition

Lexicographic extension : $s :: l >^{\text{lex}} t :: l' \begin{cases} s > t \text{ and } |l| = |l'|, \\ s = t \text{ and } l >^{\text{lex}} l' \end{cases}$

Theorem.

$>^{\text{lex}}$ well-founded if $>$ well-founded

Stable, monotone

Orderings – syntactical

Definition

Multiset M : application $M_E : E \to \mathbb{N}$ such that $\{e \in E | M_E(e) \neq 0\}$ finite

Notation :

- $e \in M$ if $M_E(e) \geq 1$,
- $M \subseteq N$ if $\forall e, M_E(e) \leq N_E(e)$,
- $M' = M \setminus N$ def. $M'_E(e) = \max(0, M_E(e) - N_E(e))$

Definition

$(\geq^{\text{mul}}, >^{\text{mul}})$

- $M \geq^{\text{mul}} M$,
- $M \geq^{\text{mul}} N \wedge e \geq e' \Rightarrow M \cup \{e\} \geq^{\text{mul}} N \cup \{e'\}$,
- $M \geq^{\text{mul}} N \wedge e > e_1, \ldots, e > e_{k \geq 0} \Rightarrow$ $M \cup \{e\} >^{\text{mul}} N \cup \{e_1, \ldots, e_k\}$,
- $M >^{\text{mul}} N \wedge e \geq e' \Rightarrow M \cup \{e\} >^{\text{mul}} N \cup \{e'\}$.

Orderings – syntactical

Orderings on symbols (precedences) extended to terms

Definition

Precedence : preordering on $\mathcal{F}$.

Admissible status function for precedence $\geq_{\mathrm{P}}$:

application $\mathrm{ST} : \mathcal{F} \to \{\text{lex}, \text{mul}\}$ such that

1. $f =_{\mathrm{P}} g \Rightarrow \mathrm{ST}(f) = \mathrm{ST}(g)$,
2. $f =_{\mathrm{P}} g$ and $\mathrm{ST}(f) = \text{lex} = \mathrm{ST}(g)$ then f and g same arity

Orderings – syntactical

RPO : $s \geq_{\text{RPO}} t$ if and only if

$s = x \in X$ and $t = x$ or

$s = f(s_1, \ldots, s_n)$ with $f \in \mathcal{F}$ and

- $s_i \geq_{\text{RPO}} t$ for an i, $1 \leq i \leq n$ or
- $t = g(t_1, \ldots, t_m)$ with $g \in \mathcal{F}$ and
  - $f > g$ and for all j, $1 \leq j \leq m$, $s >_{\text{RPO}} t_j$ or
  - $f \simeq g$ and
    - $\text{St}(f)$ = mul and $\{s_1, \ldots, s_n\}(\geq_{\text{RPO}})_{\text{mul}}\{t_1, \ldots, t_m\}$ or
    - $\text{St}(f)$ = lex thus $n = m$ and $(s_1, \ldots, s_n)(\geq_{\text{RPO}})_{\text{lex}}(t_1, \ldots, t_m)$ with for all j, $1 \leq j \leq m$, $s >_{\text{RPO}} t_j$

$s >_{\text{RPO}} t$ if $s \geq_{\text{RPO}} t$ and $t \not\geq_{\text{RPO}} s$

Stable, monotone, well-founded if precedence well-founded

Orderings – syntactical

Example

$$\left\{\begin{array}{lcl} \text{Ack}(0, x) & \to & s(x) \\ \text{Ack}(s(x), 0) & \to & \text{Ack}(x, s(0)) \\ \text{Ack}(s(x), s(y)) & \to & \text{Ack}(x, \text{Ack}(s(x), y)) \end{array}\right\}$$

RPO with Ack $>_{\text{P}}$ s and $\text{St}(\text{Ack})$ = lex

Rk. — RPO $\supseteq \rhd$ (simplification ordering) : benefit with DP ?

$\rightsquigarrow$ change the relation !

Consequences for E

Convergent = strongly normalizing + confluent

Theorem

R convergent, $s =_{E_R} t$ iff $s\downarrow_R \equiv t\downarrow_R$

If : OK

Only if : $s =_{E_R} t$ hence $s \longrightarrow^* u \longleftarrow^* t$

$s\downarrow_R \longleftarrow^* s \longrightarrow^* u \longleftarrow^* t$

R confluent : $s\downarrow_R \longleftarrow^* u$, and same for t

Now by confluence on u and unique NF. OK

Consequences for E

Example

$$\begin{array}{rcl} x \cdot e & = & x \\ x \cdot I(x) & = & e \\ (x \cdot y) \cdot z & = & x \cdot (y \cdot z) \end{array}$$

Consequences for E

Example

$$\begin{array}{rcl} x \cdot e & \to & x \\ x \cdot I(x) & \to & e \\ (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\ x \cdot (I(x) \cdot z) & \to & e \cdot z \end{array}$$

$$x \cdot (I(x) \cdot z) \longleftarrow (x \cdot I(x)) \cdot z \longrightarrow e \cdot z$$

Completion

Getting rid of $\leftrightarrow$ and $\longleftarrow \cdot \longrightarrow$ in proofs

... additionally of $l \to r$ when reducible

From $E \rightsquigarrow R$ convergent + $\forall l \to r \in R,\ r$ irr., and l irr. for $R \setminus \{l \to r\}$ (canonical)

Completion

Input : E + $>$ WF, stable + monotonic

- $>$ to orient $e \in E$ as new rules $\in R$
- CP as new equations $\in E$

$\rightsquigarrow$ evolution of (E_i, R_i)

Initial state :

$$\text{Initial} \quad \frac{}{E_0 \ ; \ \varnothing}$$

Completion

$$\text{Orient} \quad \frac{E \cup \{s = t\} \ ; \ R}{E \ ; \ R \cup \{s \to t\}} \quad \text{if } s > t$$

$$\text{Orient} \quad \frac{E \cup \{s = t\} \ ; \ R}{E \ ; \ R \cup \{t \to s\}} \quad \text{if } t > s$$

Completion

Critical Pair $$\frac{E\ ;\ R}{E \cup \{e\}\ ;\ R}$$ if e critical pair of R

Completion

Trivial $$\frac{E \cup \{s = s\}\ ;\ R}{E\ ;\ R}$$

Completion

Simplify $$\frac{E \cup \{s = t\}\ ;\ R}{E \cup \{s' = t\}\ ;\ R}$$ if $s \xrightarrow[R]{} s'$

Simplify $$\frac{E \cup \{s = t\}\ ;\ R}{E \cup \{s = t'\}\ ;\ R}$$ if $t \xrightarrow[R]{} t'$

Completion

Compose $$\frac{E\ ;\ R \cup \{l \to r\}}{E\ ;\ R \cup \{l \to r'\}}$$ if $r \xrightarrow[R]{} r'$

Completion

Collapse $\dfrac{E \; ; \; R \cup \{l \to r\}}{E \cup \{l' = r\} \; ; \; R}$ if $l \xrightarrow[g \to d]{} l', g \to d \in R$, and

- $l|_p = g\sigma$ with $p \neq \Lambda$ with σ not renaming
- or $l = g\sigma$ and $r \succ d\sigma$ with σ renaming

Completion

Preservation of equational theories

For all $\dfrac{E \; ; \; R}{E' \; ; \; R'}$, $=_{E \cup E_R} \equiv =_{E' \cup E'_R}$

Persistence

Execution of completion

$$(E_0; \emptyset) \vdash (E_1; R_1) \vdash \ldots (E_i; R_i) \vdash (E_{i+1}; R_{i+1}) \vdash \ldots (E_n; R_n) \vdash \ldots$$

$$E_\infty = \bigcup_i E_i \qquad R_\infty = \bigcup_i R_i$$

Definition

Equation (resp. rule) e persistent from rank i if $e \in$

$$\bigcap_{i \le j} E_j \qquad \left(\text{resp.} \bigcap_{i \le j} R_j\right)$$

Definition

Equation (resp. rule) e persistent if $e \in$

$$E_\omega = \bigcup_i \bigcap_{i \le j} E_j \qquad \left(\text{resp.} R_\omega = \bigcup_i \bigcap_{i \le j} R_j\right)$$

Success, failure

Definition

Execution of completion

$$(E_0; \emptyset) \vdash (E_1; R_1) \vdash \ldots (E_i; R_i) \vdash (E_{i+1}; R_{i+1}) \vdash \ldots (E_n; R_n) \vdash \ldots$$

successful if $E_\omega = \emptyset$, failure otherwise

Definition

Execution fair if $CP(R_\omega) \subseteq E_\infty$

Completeness of completion

Theorem.
Fair and successful execution of completion

$$(E_0, \emptyset) \vdash (E_1, R_1) \vdash \dots (E_n, R_n) \vdash (E_{n+1}, R_{n+1}) \dots$$

- $\xleftrightarrow[E_\infty \cup R_\infty]{*}$ is equivalent to $\xrightarrow[R_\omega]{*} \xleftarrow[R_\omega]{*}$
- R_ω equivalent to E_0 (same theories)
- R_ω convergent
- (semi-)decision proc. for word problem on E_0 (dec. if R_ω finite)

Example : groups

Shamelessly stolen from É. Contejean

$$\left\{ \begin{array}{llcl} E_1 & e \cdot x & = & x \\ E_2 & I(x) \cdot x & = & e \\ E_3 & (x \cdot y) \cdot z & = & x \cdot (y \cdot z) \end{array} \right\}$$

RPO with $I > \cdot > e$,

Example : groups

Orient E_1

$$\begin{array}{llcl} E_1 & e \cdot x & = & x \\ E_2 & I(x) \cdot x & = & e \\ E_3 & (x \cdot y) \cdot z & = & x \cdot (y \cdot z) \\ R_4 & e \cdot x & \to & x \end{array}$$

Example : groups

Orient E_2

$$\begin{array}{llcl} E_2 & I(x) \cdot x & = & e \\ E_3 & (x \cdot y) \cdot z & = & x \cdot (y \cdot z) \\ R_4 & e \cdot x & \to & x \\ R_5 & I(x) \cdot x & \to & e \end{array}$$

Example : groups

Orient E_3

$$\begin{array}{llcl} E_3 & (x \cdot y) \cdot z & = & x \cdot (y \cdot z) \\ R_4 & e \cdot x & \to & x \\ R_5 & I(x) \cdot x & \to & e \\ R_6 & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \end{array}$$

Example : groups

Crit. Pair R_4 and R_6

$$\begin{array}{llcl} R_4 & e \cdot x & \to & x \\ R_5 & I(x) \cdot x & \to & e \\ R_6 & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\ E_7 & e \cdot (y \cdot z) & = & y \cdot z \end{array}$$

Crit. peak

$$e \cdot (y \cdot z) \xleftarrow[R_6]{\Lambda} (e \cdot y) \cdot z \xrightarrow[R_4]{1} y \cdot z$$

Example : groups

Simplify E_7 with R_4

$$\begin{array}{llcl} R_4 & e \cdot x & \to & x \\ R_5 & I(x) \cdot x & \to & e \\ R_6 & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\ E_7 & e \cdot (y \cdot z) & = & y \cdot z \\ E_8 & y \cdot z & = & y \cdot z \end{array}$$

Since

$$e \cdot (y \cdot z) \xrightarrow[R_4]{\Lambda} y \cdot z$$

Example : groups

. . . many steps later. . .

Example : groups

$$
\begin{array}{llcl}
R_4 & e \cdot x & \to & x \\
R_5 & I(x) \cdot x & \to & e \\
R_6 & (x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
R_{11} & I(x) \cdot (x \cdot z) & \to & z \\
R_{20} & x \cdot e & \to & x \\
R_{22} & I(I(x)) & \to & x \\
R_{24} & I(e) & \to & e \\
R_{26} & x \cdot (I(x) \cdot z) & \to & z \\
R_{28} & x \cdot I(x) & \to & e \\
R_{35} & I(x \cdot y) & \to & I(y) \cdot I(x)
\end{array}
$$

All critical pairs confl., no other rule applicable