What if…

Unorientable equations ?

(C)   x + y   =   y + x

Here both sides renamings of the other : WF lost

Relation $\rightsquigarrow R \cup S$

with R rules, S unorientable

Rewriting modulo

Relation $\rightsquigarrow R \cup S$ with R rules, S unorientable

Apply R in classes modulo $=_S$ BL77

Definition

$s \xrightarrow[R/S]{} t$

$$\exists s',\ \exists t', s =_S s' \ \wedge\ t =_S t' \ \wedge\ s' \xrightarrow[R]{} t'$$

Unpractical (finiteness of classes + complexity) No S-step "above"

R-step, S-extended PS81

Definition

$s \xrightarrow[S/R]{} t$

$$\exists l \to r \in R,\ \exists \sigma,\ \exists p,\ s|_p =_S l\sigma \ \wedge\ t = s[r\sigma]_p$$

Rewriting modulo – properties

A a set, $\to_R$, $\to_S$ over A, $\to_{R^S}$ relation between $\to_R$ $\to_{R/S}$ (eg. $\to_{S/R}$)

Definition

- $\to_R$ $\to_{R^S}$-Church-Rosser modulo $\to_S$ iff

$$\forall x, y,\ \ x\ (\xleftrightarrow[R]{} \cup \xleftrightarrow[S]{})^*\ y \ \Rightarrow\ (\exists z, z',\ \ x \xrightarrow[R^S]{*} z \xleftrightarrow[S]{*} z' \xleftarrow[R^S]{*} y)$$

Rewriting modulo – properties

A a set, $\to_R$, $\to_S$ over A, $\to_{R^S}$ relation between $\to_R$ $\to_{R/S}$ (eg. $\to_{S/R}$)

Definition

- $\to_{R^S}$ confluent modulo $\to_S$ iff

$$\forall x, y, z,\ \ y \xleftarrow[R^S]{*} x \xrightarrow[R^S]{*} z \ \Rightarrow\ (\exists v, v',\ \ y \xrightarrow[R^S]{*} v \xleftrightarrow[S]{*} v' \xleftarrow[R^S]{*} z)$$

- $\to_{R^S}$ locally confluent with $\to_R$ modulo $\to_S$ iff

$$\forall x, y, z,\ \ y \xleftarrow[R^S]{} x \xrightarrow[R]{} z \ \Rightarrow\ (\exists v, v',\ \ y \xrightarrow[R^S]{*} v \xleftrightarrow[S]{*} v' \xleftarrow[R^S]{*} z)$$

Rewriting modulo – properties

A a set, $\to_R$, $\to_S$ over A, $\to_{R^S}$ relation between $\to_R$ $\to_{R/S}$ (eg. $\to_{S/R}$)

Definition

- $\to_{R^S}$ coherent modulo $\to_S$ iff

$$\forall x,y,z,\ y \xleftarrow[R^S]{*} x \xleftrightarrow[S]{*} z \Rightarrow (\exists v,v',\ y \xrightarrow[R^S]{*} v \xleftrightarrow[S]{*} v' \xleftarrow[R^S]{*} z)$$

- $\to_{R^S}$ locally coherent modulo $\to_S$ iff

$$\forall x,y,z,\ y \xleftarrow[R^S]{} x \xleftrightarrow[S]{} z \Rightarrow (\exists v,v',\ y \xrightarrow[R^S]{*} v \xleftrightarrow[S]{*} v' \xleftarrow[R^S]{*} z)$$

Rewriting modulo – properties

Linking all that…

Theorem.

A a set, $\to_R$, $\to_S$ over A, $\to_{R^S}$ relation between $\to_R$ $\to_{R/S}$. If WF($\leftarrow_{R/S}$) then :

1. $\to_R$ $\to_{R^S}$-Church-Rosser modulo $\to_S$
2. $\to_{R^S}$ confluent + coherent modulo $\to_S$
3. $\to_{R^S}$ locally coherent + locally confluent with $\to_R$ modulo $\to_S$
4. for all $x,\ y \in A,\ x \longleftrightarrow^*_{R\cup S} y$ iff for all $x\downarrow,\ y\downarrow$ $\to_{R^S}$ normal forms $x\downarrow \longleftrightarrow^*_S y\downarrow$

AC signature

Focus on AC

$f \in \mathcal{F}_{\text{AC}}$

$$\begin{array}{lrcl} (C) & f(x,y) & = & f(y,x) \\ (AC) & f(f(x,y),z) & = & f(x,f(y,z)) \end{array}$$

$s \xrightarrow[S/R]{} t$ $\quad \exists l \to r \in R,\ \exists\sigma,\ \exists p,\ s|_p =_S l\sigma \ \wedge\ t = s[r\sigma]_p$

$s \xrightarrow[\text{AC}/R]{} t$ $\quad \exists l \to r \in R,\ \exists\sigma,\ \exists p,\ s|_p =_{\text{AC}} l\sigma \ \wedge\ t = s[r\sigma]_p$

OK for local coherence with C

$R:\ \{a+b \to c\}$ $\quad c+c \xleftarrow[R/\text{AC}]{} (a+c)+b \xrightarrow[\text{AC}/R]{} ?$

PB for local coherence with AC (at 1)

OK if R extended by $(a+x)+b \to c+x$

Variadic terms, flattening

An easy $=_{\text{AC}}$ class witness ?

Definition

Set $\mathcal{T}_{\text{var}}(\mathcal{F},X)$ of variadic term :

- $x \in X$ then $x\mathcal{T}_{\text{var}}(\mathcal{F},X)$
- $f \in \mathcal{F} \setminus \mathcal{F}_{\text{AC}},\ \tau(f) = n,\ t_1,\ldots,t_n \in \mathcal{T}_{\text{var}}(\mathcal{F},X)$ then $f(t_1,\ldots,t_n) \in \mathcal{T}_{\text{var}}(\mathcal{F},X)$
- $f \in \mathcal{F}_{\text{AC}},\ t_1,\ldots,t_n \in \mathcal{T}_{\text{var}}(\mathcal{F},X)(n \geq 2)$ then $f(t_1,\ldots,t_n) \in \mathcal{T}_{\text{var}}(\mathcal{F},X)$

Variadic term, flattening

Definition

Flat form $\overline{t}$ of t :

$$\begin{array}{rll} \overline{x} & = x & \text{if } x \in X \\ \overline{f(t_1,\ldots,t_n)} & = f(\overline{t_1},\ldots,\overline{t_n}) & \text{if } f \notin \mathcal{F}_{AC} \\ \overline{f(t_1,t_2)} & = \begin{cases} f(\overline{t_1},\overline{t_2}) & f \in \mathcal{F}_{AC}, t_1(\Lambda) \neq f, t_2(\Lambda) \neq f \\ f(\overline{t_1},v_1,\ldots,v_n) & t_1(\Lambda) \neq f, \overline{t_2} = f(v_1,\ldots,v_n) \\ f(u_1,\ldots,u_m,\overline{t_2}) & \overline{t_1} = f(u_1,\ldots,u_m), t_2(\Lambda) \neq f \\ f(u_1,\ldots,u_m,v_1,\ldots,v_n) & \overline{t_1} = f(u_1,\ldots,u_m), \overline{t_2} = f(v_1,\ldots \end{cases} \end{array}$$

Now $=_{AC}$ only permutation congruence !

Rewriting on AC-flat

Definition

$s \xrightarrow[\overline{AC/Rext}]{} t$

- $s|_p =_{AC} \overline{l\sigma}, \quad t = \overline{s[r\sigma]_p}$,
- $s|_p =_{AC} \overline{f(l_1,\ldots,l_n,x)\sigma}, \quad t = \overline{s[f(r,x)\sigma]_p}$ $\qquad l = f(l_1,\ldots,l_n)$

Rk. — Extension needed for $l \to r$ only when

1. Obviously $l(\Lambda) \in \mathcal{F}_{AC}$
2. No variable x in l such that :
   1. ONE occ. of x, and only at 1.
   2. either $r = x$ or $r = f(x, r')$ with no x in r'

Example

No need : $x + 0 \to x$

Need ext. : $x + (-x) \to 0$, or $x + S(y) \to s(x + y)$

AC-termination

Theorem.
R/AC terminates iff $\overline{\text{AC}/R}$ terminates

Orderings : compatibility

$$\begin{array}{ccc} s & > & t \\ \|_{AC} & & \|_{AC} \\ s' & > & t' \end{array} \qquad \begin{array}{ccc} s & \geq & t \\ \|_{AC} & & \|_{AC} \\ s' & \geq & t' \end{array}$$

AC-termination

Polynomials

$$\begin{array}{rcl} P(X,Y) & = & P(Y,X) \\ P(P(X,Y),Z) & = & P(X,P(Y,Z)) \end{array}$$

Over commutative ring : $aXY + b(X+Y) + c$ with $b^2 = b + ac$

In practice : simple/linear polynomials, possibly simple-mixed

$$\begin{cases} x + 0 & \to & x \\ x + s(y) & \to & s(x+y) \end{cases} \qquad \mathcal{F}_{AC} = \{+\} \qquad \begin{array}{rcl} [\![+]\!](x,y) & = & yx + 2(y+x) + 2 \\ [\![s]\!](x) & = & x + 1 \\ [\![0]\!] & = & 0 \end{array}$$

AC-termination

Path orderings

$f(x,y) >_{RPO} g(x,y) \qquad f >_{\text{prec}} g$

alas $f(x,y,z) <_{RPO} f(g(x,y),z)$

Not monotonic ! $\rightsquigarrow$ normalising systems, new orderings...

APO [Bachmair & Plaisted]

Associative Path Cond. : $\forall f \in \mathcal{F}_{AC} \begin{cases} f \text{ minimal in } \mathcal{F} \quad \text{or} \\ \exists g \in \mathcal{F}_{AC} \text{ s.t. } f \text{ minimal } \in \mathcal{F} \setminus \{g\} \end{cases}$

Normalising system D convergent :

$\forall f >_{\text{prec}} g \in \mathcal{F}_{AC} \quad f(g(x,y),z) \to g(f(x,z),f(y,z))$

APO : $s >_{APO} t \iff \overline{s\downarrow_D} >_{RPO} \overline{t\downarrow_D}$ (monotonic, simplifying)

Other systems $\rightsquigarrow$ EAPO, MAPO... [Delor & Puel]

Without normalising system ? $\rightsquigarrow$ ACRPO [Rubio]

AC-termination – DP

Definition

For $f(t_1,\ldots,t_n) \to r$,

AC-dependecy pairs : "classical" DP + DP from $f(t_1,\ldots,t_n,x) \to \overline{f(r,x)}$

with fresh x (when necessary)

$x+0 \to x$		$x \times 0 \to 0$
$x+s(y) \to s(x+y)$		$x \times s(y) \to (x \times y)+x$
$\langle x+s(y), x+y\rangle$		$\langle x+s(y)+z, x+y\rangle$
$\langle x \times s(y), x \times y\rangle$		$\langle x+s(y)+z, s(x+y)+z\rangle$
$\langle x \times s(y), (x\times y)+x\rangle$		$\langle x \times 0 \times z, 0 \times z\rangle$
		$\langle x \times s(y) \times z, x \times y\rangle$
		$\langle x \times s(y) \times z, ((x\times y)+x)\rangle$
		$\langle x \times s(y) \times z, ((x\times y)+x)\times z\rangle$

Terminaison AC : DP

With flat terms : minimal ? $\quad u = f(u_1,\cdots,u_n) \trianglelefteq t$ immortal

- u immortal
- u_i all mortal
- f defined
- if $f \in \mathcal{F}_{AC}$, $u = f(u',u'')$ with $f(u')$ immortal and $f(u' \setminus \{u_i\})$ mortal

$\rightsquigarrow$ ensuring necessary arguments

AC-termination – DP

Lemma.

t immortal for R (modulo AC) then $\exists p,p',C,C',\langle s_1,s_2\rangle,\sigma$ s.t.

$$t \xrightarrow[R]{>p}{}^{\star} C[s_1\sigma]_p \xrightarrow[R]{p} C[C'[s_2\sigma]_{p'}]_p \text{ with } s_2\sigma \text{ immortal.}$$

Proof.

t immortal, $t = C[f(u)]_p$ with $f(u)$ immortal and u mortal

- $f \in \mathcal{F}_F$ as usual (exo)
- $f \in \mathcal{F}_{AC}$ then $f(u) = f(u',u'')$ with $f(u')$ minimal (hence step at Λ)

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v,u'')]_p \xrightarrow[l\to r]{p} C[t']_p \text{ with } t' \text{ immortal}$$

AC-termination – DP

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v,u'')]_p \xrightarrow[l\to r]{p} C[t']_p \text{ with } t' \text{ immortal}$$

- $f(v,u'') = \overline{l\sigma}$
- $f(v,u'') = \overline{f(l,x)\sigma}$ (with x fresh)

AC-termination – DP

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v,u'')]_p \xrightarrow[l\to r]{p} C[t']_p \text{ with } t' \text{ immortal}$$

- $f(v,u'') = \overline{l\sigma}$

u'' empty ! $\rightsquigarrow$ reduction of $f(u')\ldots$

$$f(v) = \overline{l\sigma} \xrightarrow{\Lambda} t' = C[f'(w')]_{p'} \text{ with } w' = w\sigma \text{ (minimal : } p' \text{ outside } \sigma)$$

$\langle l, f'(w)\rangle \in$ DP

AC-termination – DP

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v,u'')]_p \xrightarrow[l\to r]{p} C[t']_p \text{ with } t' \text{ immortal}$$

- $f(v,u'') = \overline{l\sigma}$ OK
- $f(v,u'') = \overline{f(l,x)\sigma}$ (with x fresh)

AC-termination – DP

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v,u'')]_p \xrightarrow[l\to r]{p} C[t']_p \text{ with } t' \text{ immortal}$$

- $f(v,u'') = \overline{l\sigma}$ OK
- $f(v,u'') = \overline{f(l,x)\sigma}$ (with x fresh)

reduction of $f(v)$ hence u'' in $x\sigma$

$f(v,u'') = f(v', f(v'',u''))$ and $f(v) = f(v',v'')$

$\rightsquigarrow f(v') = \overline{l\sigma} \qquad f(v'',u'') = \overline{x\sigma}$

AC-termination – DP

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v,u'')]_p \xrightarrow[l\to r]{p} C[t']_p \text{ with } t' \text{ immortal}$$

- $f(v,u'') = \overline{l\sigma}$ OK
- $f(v,u'') = \overline{f(l,x)\sigma}$ (with x fresh)

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v',v'',u'')]_p \xrightarrow[l\to r]{p} C[f(r'\sigma,v'',u'')]_p \quad \begin{cases} r = r' \\ r = f(r') \end{cases}$$

immortal $f(r'\sigma,v'') = C'[\underbrace{f'(w')}]_{p'}$ $f'(w')$ immortal, etc.

AC-termination – DP

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v,u'')]_p \xrightarrow[l\to r]{p} C[t']_p \text{ with } t' \text{ immortal}$$

- $f(v,u'') = \overline{l\sigma}$ OK
- $f(v,u'') = \overline{f(l,x)\sigma}$ (with x fresh)

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v',v'',u'')]_p \xrightarrow[l\to r]{p} C[f(r'\sigma,v'',u'')]_p \quad \begin{cases} r = r' \\ r = f(r') \end{cases}$$

immortal $f(r'\sigma,v'') = C'[\underbrace{f'(w')}]_{p'}$

- $C' \neq []$ or $f \neq f'$
- $C' = []$ and $f = f'$

AC-termination – DP

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v,u'')]_p \xrightarrow[l\to r]{p} C[t']_p \text{ with } t' \text{ immortal}$$

- $f(v,u'') = \overline{l\sigma}$ OK
- $f(v,u'') = \overline{f(l,x)\sigma}$ (with x fresh) OK

$$t = C[f(u',u'')]_p \xrightarrow[R]{>p}{}^{\star} C[f(v',v'',u'')]_p \xrightarrow[l\to r]{p} C[f(r'\sigma,v'',u'')]_p \quad \begin{cases} r = r' \\ r = f(r') \end{cases}$$

immortal $f(r'\sigma,v'') = C'[\underbrace{f'(w')}]_{p'}$

- $C' \neq []$ or $f \neq f'$ v'' mortal $\rightsquigarrow$ f' in r' : $\langle f(l,x), \overline{f'(w)}\rangle$
- $C' = []$ and $f = f'$ $\langle f(l,x), \overline{f(r,x)}\rangle$

AC-termination – DP

Lemma.
If t immortal for R (modulo AC) then $\exists p, p', C, C', \langle s_1, s_2\rangle, \sigma$ such that

$$t \xrightarrow[R]{>p}{}^{\star} C[s_1\sigma]_p \xrightarrow[R]{p} C[C'[s_2\sigma]'_p]_p \text{ with } s_2\sigma \text{ immortal.}$$

Theorem.
R terminates modulo AC iff $\twoheadrightarrow_{\mathrm{DP}_{\mathrm{AC}(R)},R}$ terminates modulo AC.