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Towards a notion of usable rules for context-sensitive rewrite systems*

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Abstract. During the last decade, the impressive advances in techniques for proving termination of rewriting have succeeded in solving termination problems that were out of reach for a long time. Since increment in the size of the problems affects directly the possibility of getting an answer, treating problems in a modular way appears like a key issue. The use of the notion of dependency among symbols and the notion of dependency pairs has helped to carry out this task leading to the notion of *usable rules*. Context-sensitive rewriting (*CSR*) is a restriction of rewriting which forbids reductions on selected arguments of functions. In this paper, we discuss how to use this notion in proofs of termination of *CSR*.

1 Introduction

Termination is a fundamental property in programming, which allows us to know if for every computation the system will return in a finite time. This property is undecidable, hence all techniques for proving termination are incomplete. Thus one looks for sound approaches that are as complete as possible.

During the last decade, the impressive advances in techniques for proving termination of rewriting (remarkably the dependency pairs approach [1, 7, 11, 12]) have succeeded in solving termination problems that stood out of reach for a long time. The increment in size of the problems may affect directly in the time for getting an answer, or worse: the success of a selected method in discovering such answer. For that reason, treating problems in a modular way appears like a key issue. The main problem dealing with termination and unions of systems is that this property is not modular: the union of two disjoint Term Rewriting Systems (TRSs [21]) can be a non-terminating TRS [23].

A more restricted notion of termination may be used to solve this problem: $\mathcal{C}_\varepsilon$-termination [10]. $\mathcal{C}_\varepsilon$-termination is a modular property [10] and, as claimed by Urbain [24], it is not a very harsh condition in practice. To prove termination in a modular and incremental way, Urbain relies on the dependency pairs

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approach [1]. Roughly speaking, given a TRS $\mathcal{R}$, the dependency pairs associated to $\mathcal{R}$ conform a new TRS $\mathsf{DP}(\mathcal{R})$ which (together with $\mathcal{R}$) determines the so-called *dependency chains* whose finiteness characterizes termination of $\mathcal{R}$. The dependency pairs can be presented as a *dependency graph*, where the absence of infinite $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chains can be analyzed by considering the *cycles* in the graph. Urbain's idea is using the (explicit or implicit) modular structure of rewrite systems for proving that all (minimal) infinite rewrite sequences can be simulated by using a restricted set of rules. This allows us to prune irrelevant rules and termination constraints. Hirokawa and Middeldorp [13] and (independenly) Thiemann et al. [22] combined this idea with the idea of *usable rules* (originally introduced by Arts and Giesl for proving termination of *innermost* rewriting [1]), leading to a more powerful framework for proving termination in a modular way. In these approaches, usable rules are associated to a given *cycle* $\mathfrak{C}$ of the dependency graph. The benefit of using them in proofs of termination stems from the fact that we take these rules (together with the dependency pairs in the corresponding cycle $\mathfrak{C}$) instead of considering *all rules* in the TRS. Proofs of termination often become easier in this way.

Proving termination of *context-sensitive rewriting* (*CSR* [16, 17]) is an interesting problem with several applications in the fields of term rewriting and programming languages (see [6, 9, 17, 19] for further motivation). In *CSR*, a *replacement map* (i.e., a mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ satisfying $\mu(f) \subseteq \{1, \ldots, k\}$, for each k-ary symbol f of a signature $\mathcal{F}$) is used to discriminate the argument positions on which the rewriting steps are allowed; rewriting at the topmost position is always possible. In this way, we can *achieve* a terminating behaviour by pruning (all) infinite rewrite sequences. In [2], the dependency pairs method has been adapted to be used in proofs of termination of *CSR*. In this work we investigate how to extend the previous ideas to *CSR*.

After some preliminaries in Section 2, Section 3 introduces the basic notion of *usable* rule in *CSR*. In Section 4 we show how to use this notion for proving termination of *CSR*. Section 5 concludes.

2 Preliminaries

Throughout the paper, $\mathcal{X}$ denotes a countable set of variables and $\mathcal{F}$ denotes a signature i.e., a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. Given positions p, q, we denote their concatenation as $p.q$. If p is a position, and Q is a set of positions, $p.Q = \{p.q \mid q \in Q\}$. We denote the empty chain by Λ. The set of positions of a term t is $\mathcal{P}os(t)$. The subterm at position p of t is denoted as $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced by s. We write $t \trianglerighteq s$ if $s = t|_p$ for some $p \in \mathcal{P}os(t)$ and $t \triangleright s$ if $t \trianglerighteq s$ and $t \neq s$. The symbol labelling the root of t is denoted as $root(t)$. A *context* is a term $C \in \mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{X})$ with zero or more 'holes' $\Box$ (a fresh constant symbol).

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ where R is a set of rewrite rules. Given $\mathcal{R} = (\mathcal{F}, R)$, we consider $\mathcal{F}$ as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$, called *constructors* and symbols $f \in \mathcal{D}$, called *defined functions*, where $\mathcal{D} = \{root(l) \mid l \to r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$. We often write $\mathcal{R}(\mathcal{F})$ to make explicit that $\mathcal{R}$ is a TRS over a signature $\mathcal{F}$.

Context-sensitive rewriting. A mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if $\forall f \in \mathcal{F},\ \mu(f) \subseteq \{1, \ldots, ar(f)\}$ [16]. Let $M_{\mathcal{F}}$ be the set of all $\mathcal{F}$-maps (or $M_{\mathcal{R}}$ for the $\mathcal{F}$-maps of a TRS $(\mathcal{F}, R)$). A binary relation R on terms is μ-monotonic if $t\, R\, s$ implies $f(t_1, \ldots, t, \ldots, t_k)\, R\, f(t_1, \ldots, s, \ldots, t_k)$ for every $t, s, t_1, \ldots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. The set of *$\mu$-replacing positions* $\mathcal{P}os^{\mu}(t)$ of $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is: $\mathcal{P}os^{\mu}(t) = \{\Lambda\}$, if $t \in \mathcal{X}$ and $\mathcal{P}os^{\mu}(t) = \{\Lambda\} \cup \bigcup_{i \in \mu(root(t))} i.\mathcal{P}os^{\mu}(t|_i)$, if $t \notin \mathcal{X}$. The set of *replacing* variables of t is $\mathcal{V}ar^{\mu}(t) = \{x \in \mathcal{V}ar(t) \mid \exists p \in \mathcal{P}os^{\mu}(t), t|_p = x\}$. The μ-replacing subterm relation $\trianglerighteq_{\mu}$ is given by $t \trianglerighteq_{\mu} s$ if there is $p \in \mathcal{P}os^{\mu}(t)$ such that $s = t|_p$. We write $t \triangleright_{\mu} s$ if $t \trianglerighteq_{\mu} s$ and $t \neq s$. In *context-sensitive rewriting* (*CSR* [16]), we (only) contract *replacing* redexes: t μ-rewrites to s, written $t \hookrightarrow_{\mu} s$ (or $t \hookrightarrow_{\mathcal{R},\mu} s$), if $t \xrightarrow{p}_{\mathcal{R}} s$ and $p \in \mathcal{P}os^{\mu}(t)$. A TRS $\mathcal{R}$ is μ-terminating if $\hookrightarrow_{\mu}$ is terminating. A term t is μ-terminating if there is no infinite μ-rewrite sequence $t = t_1 \hookrightarrow_{\mu} t_2 \hookrightarrow_{\mu} \cdots$. A pair $(\mathcal{R}, \mu)$ where $\mathcal{R}$ is a TRS and $\mu \in M_{\mathcal{R}}$ is often called a CS-TRS.

Dependency pairs. Given a TRS $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ a new TRS $\mathsf{DP}(\mathcal{R}) = (\mathcal{F}^{\sharp}, D(R))$ of *dependency pairs* for $\mathcal{R}$ is given as follows: if $f(t_1, \ldots, t_m) \to r \in R$ and $r = C[g(s_1, \ldots, s_n)]$ for some defined symbol $g \in \mathcal{D}$ and $s_1, \ldots, s_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, then $f^{\sharp}(t_1, \ldots, t_m) \to g^{\sharp}(s_1, \ldots, s_n) \in D(R)$, where $f^{\sharp}$ and $g^{\sharp}$ are new fresh symbols (called *tuple* symbols) associated to defined symbols f and g respectively [1]. Let $\mathcal{D}^{\sharp}$ be the set of tuple symbols associated to symbols in $\mathcal{D}$ and $\mathcal{F}^{\sharp} = \mathcal{F} \cup \mathcal{D}^{\sharp}$. As usual, for $t = f(t_1, \ldots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write $t^{\sharp}$ to denote the *marked* term $f^{\sharp}(t_1, \ldots, t_k)$. Given $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$, $T^{\sharp}$ denotes $\{t^{\sharp} \mid t \in T\}$.

A reduction pair $(\succsim, \sqsupset)$ consists of a stable and weakly monotonic quasi-ordering $\succsim$, and a stable and well-founded ordering $\sqsupset$ satisfying either $\succsim \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \succsim \subseteq \sqsupset$. Note that *monotonicity is not required* for $\sqsupset$.

Let $\mathcal{T}_{\infty}$ be a set of minimal non-terminating terms: a term $t \in \mathcal{T}_{\infty}$ if t is non-terminating and every strict subterm is terminating.

$\mathcal{C}_{\varepsilon}$-termination. A TRS $\mathcal{R}(\mathcal{F})$ is $\mathcal{C}_{\varepsilon}$-terminating if $\mathcal{R} \uplus \pi$, where $\pi = \{G(x, y) \to x,\ G(x, y) \to y\}$ (with $G \notin \mathcal{F}$), is terminating.

This definition has been extended to CSR [8]: A TRS $\mathcal{R}(\mathcal{F})$ is $\mathcal{C}_{\varepsilon}$-$\mu$-terminating if $\mathcal{R} \uplus \pi$ is μ_{G}-terminating, where $\mu_{\mathsf{G}}(f) = \mu(f)$ if $f \in \mathcal{F}$ and $\mu_{\mathsf{G}}(\mathsf{G}) = \{1, 2\}$.

$\mathcal{C}_{\varepsilon}$-termination is modular for unions of

- disjoint TRSs [10,20],
- finitely branching and constructor sharing systems [10],
- finitely branching and composable systems [15].

This work consider finitely branching CS-TRSs where $\mathcal{C}_\varepsilon$-termination is modular.

2.1 Context-sensitive dependency pairs

Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. We define $\mathsf{DP}(\mathcal{R}, \mu) = \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu) \cup \mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu)$ to be the set of *context-sensitive* dependency pairs (CS-DPs) where: $\mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu) = \{l^\sharp \to s^\sharp \mid l \to r \in R, r \trianglerighteq_\mu s, root(s) \in \mathcal{D}, l \ntriangleright_\mu s\}$ and $\mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu) = \{l^\sharp \to x \mid l \to r \in R, x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)\}$. We extend $\mu \in M_{\mathcal{F}}$ into $\mu^\sharp \in M_{\mathcal{F}^\sharp}$ by $\mu^\sharp(f) = \mu(f)$ if $f \in \mathcal{F}$, and $\mu^\sharp(f^\sharp) = \mu(f)$ if $f \in \mathcal{D}$.

Let $(\mathcal{R}, \mu)$ be a CS-TRS. Given $\mathcal{P} \subseteq \mathsf{DP}(\mathcal{R}, \mu)$, an $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chain is a finite or infinite sequence of pairs $u_i \to v_i \in \mathcal{P}$, for $i \geq 1$ such that there is a substitution σ satisfying both:

1. $\sigma(v_i) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$, if $u_i \to v_i \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$, and
2. if $u_i \to v_i = u_i \to x_i \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu)$, then there is $s_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $\sigma(x_i) \trianglerighteq_\mu s_i$ and $s_i^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$.

Context-sensitive dependency graph. The context-sensitive dependency pairs can be presented as a *context-sensitive dependency graph*, where the absence of infinite chains can be analyzed by considering the *cycles* in the graph.

Given a TRS $\mathcal{R} = (\mathcal{F}, R)$ and $\mu \in M_{\mathcal{R}}$, we say that $f \in \mathcal{F}$ is a *hidden symbol* [3] if there are $l \to r \in R$ and $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ s.t. $r \triangleright_{\not\mu} t$ and $root(t) = f$. Let $\mathcal{H}(\mathcal{R}, \mu)$ (or just $\mathcal{H}$, if there is no ambiguity) be the set of all hidden symbols in $(\mathcal{R}, \mu)$.

Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. The context-sensitive dependency graph consists of the set $\mathsf{DP}(\mathcal{R}, \mu)$ of context-sensitive dependency pairs together with arcs which connect them as follows:

1. There is an arc from a dependency pair $u \to v \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$ to a dependency pair $u' \to v' \in \mathsf{DP}(\mathcal{R}, \mu)$ if there is a substitutions σ such that $\sigma(v) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u')$.
2. There is an arc from a dependency pair $u \to v \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu)$ to a dependency pair $u' \to v' \in \mathsf{DP}(\mathcal{R}, \mu)$ if $root(u')^\natural \in \mathcal{H}(\mathcal{R}, \mu)$.

Let $\mathcal{M}_{\infty,\mu}$ be a set of minimal non-μ-terminating terms: a term $t \in \mathcal{M}_{\infty,\mu}$ if t is non-μ-terminating and every strict μ-replacing subterm is terminating.

3 Usable Rules

In programming, the idea of *module* comes in a natural way. Programmers usually set in a module those functions which have common features or properties. Then, new modules which use these functions are written. This notion arises in the same way in the term rewriting context: when rules defining some function symbols f (by means of rules $f(l_1, \ldots, l_k) \to r$) are collected together and, they are used by other rules from other modules. Urbain exploits this modular decomposition approach to prove termination of rewriting in a modular and incremental way [24]. Although termination is not modular (in general), he suceeds thanks to imposing a harder termination condition for modules: the $\mathcal{C}_{\varepsilon}$-termination.

Recent papers improve the modular approach using the notion of usable rules [12, 14]. Usable rules were introduced by Arts and Giesl in [1] in connection with innermost termination. Hirokawa and Middeldorp [13] and (independenly) Thiemann et al. [22] showed that they can also be used to prove termination. Like in Urbain's approach, dependencies between symbols are used to dismiss rules in proofs of termination. The notion of usable rules concerns Strongly Connected Component (SCC) to get only the rules that can be involved in an infinite chain of dependency pairs included in the SCC. This rules are called *usable* (within this SCC).

Definition 1 (Dependency [14]). *Given a TRS $\mathcal{R}$ over a signature $\mathcal{F}$, we say that $f \in \mathcal{F}$ directly depends on $g \in \mathcal{F}$, written $f \rhd_d g$, if there is a rule $l \to r \in \mathcal{R}$ with $f = root(l)$ and g occurs in r.*

The set of defined function symbols in a term t is $\mathcal{D}Fun(t) = \{f \mid \exists p \in \mathcal{P}os(t), f = root(t|_p) \in \mathcal{D}\}$. Now we have:

Definition 2 (Usable rules [14]). *For a set $\mathcal{G}$ of defined function symbols we denote by $\mathcal{R} \mid \mathcal{G}$ the set of rewriting rules $l \to r \in \mathcal{R}$ with $root(l) \in \mathcal{G}$. The set of $\mathcal{U}(t)$ of usable rules of a term t is defined as $\mathcal{R} \mid \{g \mid f \rhd_d^* g$ for some $f \in \mathcal{D}Fun(t)\}$. If $\mathcal{P}$ is a set of dependency pairs then*

$$\mathcal{U}(\mathcal{P}) = \bigcup_{l \to r \in \mathcal{P}} \mathcal{U}(r)$$

Usable rules in Definition 2 can be used instead of $\mathcal{R}$ when looking for a reduction pair which proves termination of $\mathcal{R}$ [12, 14]. In the following sections we discuss suitable notions of usable rules for *CSR*.

3.1 Basic usable rules for CSR

A *conservative* system is a CS-TRS having only conservative rules [19].

Definition 3 (Conservative rule [19]). *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. A rule $l \to r \in R$ is conservative if $\mathcal{V}ar^{\mu}(r) \subseteq \mathcal{V}ar^{\mu}(l)$.*

Dealing with conservative CS-TRSs, symbols below a non-μ-replacing position cannot be involved in any infinite $(\mathcal{R}, \mathcal{P}, \mu^{\sharp})$-chain. Starting with this idea, we are going to use the knowledge about non-replacing positions to dismiss certain rules from the set of usable rules in Definition 2. We can relax the previous dependency relation for some symbols thanks to the replacement map. Then, the μ-dependency relation is:

Definition 4. *Given a CS-TRS $(\mathcal{R}(\mathcal{F}),\mu)$, $f\in\mathcal{F}$ directly μ-depends on $g\in\mathcal{F}$, written $f \blacktriangleright_{d,\mu} g$, if there is $l\to r\in R$ with $f = root(l)$ and g occurs in r in a μ-replacing position.*

This leads to a straightforward extension of Definition 2. The set of *replacing* defined fuction symbols in a term t is $\mathcal{D}Fun^{\mu}(t) = \{f \mid \exists p\in\mathcal{P}os^{\mu}(t), f = root(t|_p)\in\mathcal{D}\}$. Then, we have:

Definition 5. *The set $\mathcal{U}_0(t,\mu)$ of basic context-sensitive usable rules of a term t is defined as $\mathcal{R} \mid \{g \mid f \blacktriangleright^*_{d,\mu} g$ for some $f\in\mathcal{D}Fun^{\mu}(t)\}$. If $\mathcal{P}$ is a set of dependency pairs then:*

$$\mathcal{U}_0(\mathcal{P},\mu) = \bigcup_{l\to r\in\mathcal{P}} \mathcal{U}_0(r,\mu)$$

The following example illustrates how Definition 5 allows us to obtain a better set of usable rules.

Example 1. Consider the following TRS $\mathcal{R}$:

```
f(a,X,X) → f(X,b,b)
b → a
```

together with $\mu(\mathtt{f}) = \varnothing$. We have the following SCC: `F(a,X,X)` $\to$ `F(X,b,b)`. Since `b` is in non-μ-replacing positions in the rhs of CS-DP, there is no usable rule.

However, Definition 4 does not lead to a correct approach for proving termination of *CSR*, even for conservative TRSs.

Example 2. Consider the following TRS $\mathcal{R}$:

```
f(c(X),X) → f(X,X)
b → c(b)
```

together with $\mu(\mathtt{f}) = \{1,2\}$ and $\mu(\mathtt{c}) = \varnothing$. Note that $\mathcal{R}$ is μ-conservative. We have only one cycle: `F(c(X),X)` $\to$ `F(X,X)`.

According to Definition 5, we have no usable rule because `F(X,X)` has no defined symbol. We would be tempted to conclude termination of $\mathcal{R}$, but we have the following infinite μ-rewriting sequence:

$\underline{\mathtt{f(c(b),b)}} \hookrightarrow \mathtt{f(\underline{b},b)} \hookrightarrow \underline{\mathtt{f(c(b),b)}} \hookrightarrow \cdots$

3.2 Strongly conservative TRSs

According to the discussion in the previous section, we are going to consider a more restrictive kind of conservative CS-TRSs: the *strongly conservative* CS-TRSs, in which the problem illustrated by Example 2 is not possible.

Definition 6. *Let $\mathcal{F}$ be a signature, $\mu\in M_{\mathcal{F}}$, and $t\in\mathcal{T}(\mathcal{F},\mathcal{X})$. We denote $N\mathcal{V}ar^{\mu}(t)$ the set of all $x\in\mathcal{V}ar(t)$ such that $t \rhd_{\not\mu} x$.*

Definition 7 (Strongly conservative TRS). *Let $\mathcal{F}$ be a signature and $\mu \in M_{\mathcal{F}}$. A rule $l \to r$ is strongly conservative if it is conservative and $\mathcal{V}ar^{\mu}(l) \cap N\mathcal{V}ar^{\mu}(l) = \varnothing$. A TRS $\mathcal{R} = (\mathcal{F}, R)$ is strongly conservative if all rules in R are strongly conservative.*

Left-linear CS-TRSs trivially satisfy $\mathcal{V}ar^{\mu}(l) \cap N\mathcal{V}ar^{\mu}(l) = \varnothing$. Hence, left-linear conservative CS-TRSs are strongly conservative. The CS-TRS $\mathcal{R}$ in Example 1 is strongly conservative, but $\mathcal{R}$ in Example 2 is not.

4 Proving termination of strongly conservative CS-TRSs

Theorem 1 below is the main result in this paper. It shows that basic usable rules in Definition 5 can be used to improve proofs of termination of *CSR* for strongly conservative CS-TRSs. In order to prove this theorem, we are going to provide an interpretation of terms like Hirokawa and Middeldorp's [13]. The difference between both representations is that we are going to treat $\mathcal{M}_{\infty,\mu}$ terms instead of $\mathcal{T}_{\infty}$ terms. For that reason, we have to pay special attention to non-μ-replacing positions because we don't know if there is a possible infinite μ-rewrite sequence in them.

Definition 8. *Let $(\mathcal{R}, \mu)$ be a CS-TRS over a signature $\mathcal{F}$, let $\mathcal{G} \subseteq \mathcal{F}$. Let $>$ be an arbitrary total ordering over $\mathcal{T}(\mathcal{F}^{\sharp} \cup \{\bot, \mathtt{G}\}, \mathcal{X})$ where $\bot$ is a new constant symbol and $\mathtt{G}$ is a new binary symbol. The interpretation $I_{\mathcal{G},\mu}$ is a mapping from μ-terminating terms in $\mathcal{T}^{\sharp}(\mathcal{F}, \mathcal{X})$ into $\mathcal{T}(\mathcal{F}^{\sharp} \cup \{\bot, \mathtt{G}\}, \mathcal{X})$ defined as follows:*

$$I_{\mathcal{G},\mu}(t) = \begin{cases} t & \text{if } t \in \mathcal{X} \\ f(I_{\mathcal{G},\mu,f,1}(t_1), \ldots, I_{\mathcal{G},\mu,f,n}(t_n)) & \text{if } t = f(t_1 \ldots t_n) \text{and } f \notin \mathcal{G} \\ \mathtt{G}(f(I_{\mathcal{G},\mu,f,1}(t_1), \ldots, I_{\mathcal{G},\mu,f,n}(t_n)), t') & \text{if } t = f(t_1 \ldots t_n) \text{and } f \in \mathcal{G} \end{cases}$$

where

$$I_{\mathcal{G},\mu,f,i}(t) = \begin{cases} I_{\mathcal{G},\mu}(t) & \text{if } i \in \mu(f) \\ t & \text{if } i \notin \mu(f) \end{cases}$$

$$t' = order\left(\{I_{\mathcal{G},\mu}(u) \mid t \hookrightarrow_{\mathcal{R},\mu} u\}\right)$$

$$order(T) = \begin{cases} \bot, & \text{if } T = \varnothing \\ \mathtt{G}(t, order(T - \{t\}) & \text{if } t \text{ is the minimum element of } T \end{cases}$$

$Red(t, \mu)$ denotes the set of interpreted one-step-μ-reducts of t. We assume a fixed and total ordering on $\mathcal{T}(\mathcal{F}^{\sharp} \cup \{\bot, \mathtt{G}\}, \mathcal{X})$. The difference between interpretations in [10, 13, 24] and our is that we do not interpret non-μ-replacing positions. The idea behind is that we don't need to interpret terms below non-μ-replacing positions when we are treating strongly conservative CS-TRSs.

The interpretation of a term $t = f(t_1, \ldots, t_n)$, where $f \in \mathcal{G}$, is a sequence of its interpreted one-step-reducts. It is possible to reach any of them by using a suitable $\hookrightarrow^{*}_{\pi_2,\mu_{\mathtt{G}}} \circ \hookrightarrow_{\pi_1,\mu_{\mathtt{G}}}$-sequence. In particular, we have:

Proposition 1. *Let $(\mathcal{R},\mu)$ be a CS-TRS over a signature $\mathcal{F}$ and let $\mathcal{G} \subseteq \mathcal{F}$. For all terms $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$, $I_{\mathcal{G},\mu}(t) \hookrightarrow^*_{\pi,\mu_{\mathtt{G}}} t$.*

Proof. By structural induction.

Lemma 1. *For each μ-terminating term t, $I_{\mathcal{G},\mu}(t)$ is finite.*

Proof. By well-founded induction based on the fact that μ-replacing subterms are terminating. the interpretation of non-μ-replacing positions is always finite (they are not developed). μ-replacing positions generate finite interpretations since there is no infinite μ-reduction for these subterms. Hence $I_{\mathcal{G},\mu}(t)$ is finite.

Definition 9. *Let $(\mathcal{R},\mu)$ be a CS-TRS and σ be a substitution. We denote by $\sigma_{I_{\mathcal{G},\mu}}$ the function from terms to terms that, given a term t and $x \in \mathcal{V}ar(t)$ replaces a variable occurrence x at position p in t by either $I_{\mathcal{G},\mu}(\sigma(x))$ if $p \in \mathcal{P}os^\mu(t)$, or $\sigma(x)$ if $p \notin \mathcal{P}os^\mu(t)$.*

The following result is obvious from Definition 9 and will be used later.

Proposition 2. *Let $(\mathcal{R},\mu)$ be a CS-TRS and σ be a substitution. Let t be a term such that $\mathcal{V}ar^\mu(t) \cap N\mathcal{V}ar^\mu(t) = \varnothing$ and $\overline{\sigma}_{I_{\mathcal{G},\mu}}$ be a substitution given by*

$$\overline{\sigma}_{I_{\mathcal{G},\mu}}(x) = \begin{cases} I_{G,\mu}(\sigma(x)) & \text{if } x \in \mathcal{V}ar^\mu(t) \\ \sigma(x) & \text{if } x \in N\mathcal{V}ar^\mu(t) \end{cases}$$

Then, $\overline{\sigma}_{I_{\mathcal{G},\mu}}(t) = \sigma_{I_{\mathcal{G},\mu}}(t)$.

Lemma 2. *Let $(\mathcal{R},\mu)$ be a CS-TRS over a signature $\mathcal{F}$ and let $\mathcal{G} \subseteq \mathcal{F}$. Let t be a term and σ be a substitution. If $\sigma(t)$ is terminating, then $I_{\mathcal{G},\mu}(\sigma(t)) \hookrightarrow^*_{\pi,\mu_{\mathtt{G}}} \sigma_{I_{\mathcal{G},\mu}}(t)$. If t does not contain $\mathcal{G}$-symbols, then $I_{\mathcal{G},\mu}(\sigma(t)) = \sigma_{I_{\mathcal{G},\mu}}(t)$.*

Proof. By structural induction on t:

- If t is a variable then $I_{\mathcal{G},\mu}(\sigma(t)) = \sigma_{I_{\mathcal{G},\mu}}(t)$.
- If $t = f(t_1,\ldots,t_n)$ then
  - If $f \notin \mathcal{G}$ then $I_{\mathcal{G},\mu}(\sigma(t)) = f(I_{\mathcal{G},\mu,f,1}(\sigma(t_1)),\ldots,I_{\mathcal{G},\mu,f,n}(\sigma(t_n)))$. Terms $\sigma(t_i)$ are μ-terminating for $i \in \mu(f)$. By the induction hypothesis, for all terms t_i such that $i \in \mu(f)$, we conclude $I_{\mathcal{G},\mu,f,i}(\sigma(t_i)) = I_{\mathcal{G},\mu}(\sigma(t_i)) \hookrightarrow^*_{\pi,\mu_{\mathtt{G}}} \sigma_{I_{\mathcal{G},\mu}}(t_i)$. And for all terms t_i such that $i \notin \mu(f)$, we have $I_{\mathcal{G},\mu,f,i}(\sigma(t_i)) = \sigma(t_i)$. By definition of $\sigma_{I_{\mathcal{G},\mu}}$, we have $\sigma_{I_{\mathcal{G},\mu}}(f(t_1,\ldots,t_n)) = f(t'_1,\ldots,t'_n)$, where $t'_i = I_{\mathcal{G},\mu}(t_i)$ if $i \in \mu(f)$ and $t'_i = \sigma(t_i)$ if $i \notin \mu(f)$. Therefore, $I_{\mathcal{G},\mu}(\sigma(t)) \hookrightarrow^*_{\pi,\mu_{\mathtt{G}}} \sigma_{I_{\mathcal{G},\mu}}(t)$.
  - If $f \in \mathcal{G}$, $I_{\mathcal{G},\mu}(\sigma(t)) = \mathtt{G}(f(I_{\mathcal{G},\mu,f,1}(\sigma(t_1)),\ldots,I_{\mathcal{G},\mu,f,n}(\sigma(t_n))),t'))$ for some t'. Using one step of π $(\hookrightarrow_{\pi,\mu_{\mathtt{G}}} f(I_{\mathcal{G},\mu,f,1}(\sigma(t_1)),\ldots,I_{\mathcal{G},\mu,f,n}(\sigma(t_n))))$ and the preceding result we get $I_{\mathcal{G},\mu}(\sigma(t)) \hookrightarrow^*_{\pi,\mu_{\mathtt{G}}} \sigma_{I_{\mathcal{G},\mu}}(t)$.

Then we conclude $I_{\mathcal{G},\mu}(\sigma(t)) \hookrightarrow^*_{\pi,\mu_{\mathsf{G}}} \sigma_{I_{\mathcal{G},\mu}}(t)$. The second part of the lemma is easily proved by structural induction and using Definition 8.

Lemma 3. *Let $(\mathcal{R}, \mu)$ be a CS-TRS over a signature $\mathcal{F}$, let $\mathcal{G} \subseteq \mathcal{F}$ and C a context with n replacing holes. If $t = C[t_1, \ldots, t_n]$ is terminating and the context C contains no $\mathcal{G}$-symbols then*

$$I_{\mathcal{G},\mu}(C[t_1, \ldots, t_n]) = C[I_{\mathcal{G},\mu}(t_1), \ldots, I_{\mathcal{G},\mu}(t_n)].$$

Proof. By structural induction on t:

- If C has not holes then $I_{\mathcal{G},\mu}(C[]) = C[]$.
- If the context $C = f(C_1[t_{11}, \ldots, t_{1n}], \ldots, C_m[t_{m1}, \ldots, t_{mn}])$ then $f \notin \mathcal{G}$ and if we interpret the term $I_{\mathcal{G},\mu}(f(C_1[t_{11}, \ldots, t_{1n}], \ldots, C_m[t_{m1}, \ldots, t_{mn}])) = f(I_{\mathcal{G},\mu}(C_1[t_{11}, \ldots, t_{1n}]), \ldots, I_{\mathcal{G},\mu}(C_m[t_{m1}, \ldots, t_{mn}]))$, by induction hypothesis we have $f(C_1[I_{\mathcal{G},\mu}(t_{11}), \ldots, I_{\mathcal{G},\mu}(t_{1n})], \ldots, C_m[I_{\mathcal{G},\mu}(t_{m1}), \ldots, I_{\mathcal{G},\mu}(t_{mn})])$. Our t_i terms are ones of the t_{ik} in one context C_k where $k \in [1..m]$ and $j \in [1..n]$, for that reason we can conclude saying that $I_{\mathcal{G},\mu}(C[t_1, \ldots, t_n]) = C[I_{\mathcal{G},\mu}(t_1), \ldots, I_{\mathcal{G},\mu}(t_n)]$

Lemma 4. *Let $(\mathcal{R}, \mu)$ be a strongly conservative CS-TRS and $\mathcal{P} \in \mathsf{CSDP}(\mathcal{R}, \mu)$. Let $\mathcal{G} \subseteq \mathcal{F}$. If s and t are μ-terminating terms and $s \hookrightarrow_{\mathcal{R},\mu} t$ then $I_{\mathcal{G},\mu}(s) \hookrightarrow^+_{\mathcal{U}_0(\mathcal{P},\mu)\cup\pi,\mu\cup\mu_{\mathsf{G}}} I_{\mathcal{G},\mu}(t)$.*

Proof. Let p the position of the rewrite step $s \hookrightarrow_{\mathcal{R},\mu} t$. There are two cases.

- If there is a function symbol from $\mathcal{G}$ at a position above p, then we can write $s = C[s_1, \ldots, s_i, \ldots, s_n]$ and $t = C[s_1, \ldots, t_i, \ldots, s_n]$, with $s_i \hookrightarrow_{\mathcal{R},\mu} t_i$, where s_i and t_i are in a replacing hole, $root(s_i) \in \mathcal{G}$ and the context contains no $\mathcal{G}$-symbols. We have $I_{\mathcal{G},\mu}(s_i) \hookrightarrow_{\pi,\mu_{\mathsf{G}}} Comb(\bigcup_{s_i \to u} I_{\mathcal{G},\mu}(u))$. Since $s_i \hookrightarrow_{(\mathcal{R},\mu)} t_i$, we can extract $I_{\mathcal{G},\mu}(t_i)$ from the term $Comb(\bigcup_{s_i \to u} I_{\mathcal{G},\mu}(u))$ by appropiate π-steps, so $I_{\mathcal{G},\mu}(s_i) \hookrightarrow^+_{\pi,\mu_{\mathsf{G}}} I_{\mathcal{G},\mu}(t_i)$. Using Lemma 3 we get $I_{\mathcal{G},\mu}(s) \hookrightarrow^+_{\mathcal{U}_0(\mathcal{P},\mu)\cup\pi,\mu\cup\mu_{\mathsf{G}}} I_{\mathcal{G},\mu}(t)$.
- Otherwise, we can write $s = C[s_1, \ldots, s_i, \ldots, s_n]$ and $t = C[s_1, \ldots, t_i, \ldots, s_n]$ with $s_i \overset{\epsilon}{\hookrightarrow}_{\mathcal{R},\mu} t_i$, where $root(s_i) \notin \mathcal{G}$ and the context C contains no $\mathcal{G}$-symbols. Since $root(s_i) \notin \mathcal{G}$, the applied rewriting rule $l \to r$ actually belongs to $\mathcal{U}_0(\mathcal{P}, \mu)$. Using lemma 2 we obtain $I_{\mathcal{G},\mu}(s_i) = I_{\mathcal{G},\mu}(\sigma(l)) \hookrightarrow^*_{\pi,\mu_{\mathsf{G}}} \sigma_{I_{\mathcal{G},\mu}}(l)$. Because right-hand sides of rules in $\mathcal{U}_0(\mathcal{P}, \mu)$ do not contain $\mathcal{G}$-symbols, the same lemma yields $I_{\mathcal{G},\mu}(t_i) = \sigma_{I_{\mathcal{G},\mu}}(r)$. By strong conservativity of the rule $l \to r$ and Proposition 2, $\sigma_{I_{\mathcal{G},\mu}}(l) = \overline{\sigma}_{I_{\mathcal{G},\mu}}(l) \hookrightarrow_{\mathcal{U}_0(\mathcal{P},\mu),\mu} \overline{\sigma}_{I_{\mathcal{G},\mu}}(r)$. By Proposition 1, $\overline{\sigma}_{I_{\mathcal{G},\mu}}(r) \hookrightarrow^*_{\pi,\mu_{\mathsf{G}}} \sigma_{I_{\mathcal{G},\mu}}(r)$ and thus $I_{\mathcal{G},\mu}(s_i) \hookrightarrow^+_{\mathcal{U}_0(\mathcal{P},\mu)\cup\pi,\mu\cup\mu_{\mathsf{G}}} I_{\mathcal{G},\mu}(t_i)$. Using Lemma 3 we get $I_{\mathcal{G},\mu}(s) \hookrightarrow^+_{\mathcal{U}_0(\mathcal{P},\mu)\cup\pi,\mu\cup\mu_{\mathsf{G}}} I_{\mathcal{G},\mu}(t)$.

Theorem 1. *Let $(\mathcal{R}, \mu)$ be a CS-TRS and $\mathcal{P}$ be a cycle in $\mathsf{CSDP}(\mathcal{R}, \mu)$. If $\mathcal{P} \cup \mathcal{U}_0(\mathcal{P}, \mu)$ is strongly conservative and there exists a triple $(\gtrsim, \geq, >)$ such that $\mathcal{U}_0(\mathcal{P}, \mu) \cup (\pi, \mu_{\mathsf{G}}) \subseteq \gtrsim$, $\mathcal{P} \subseteq \geq$, and $\mathcal{P} \cap > \neq \varnothing$, then there are no infinite $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chains.*

Proof. By contradiction. Assume that there is a $\mathcal{P}$-minimal μ-rewriting sequence:

$$t_1 \hookrightarrow^*_{\mathcal{R},\mu} u_1 \to_{\mathcal{P}} t_2 \hookrightarrow^*_{\mathcal{R},\mu} u_2 \to_{\mathcal{P}} \cdots$$

Let $\mathcal{G}$ be the set of defined symbols of $\mathcal{R}\backslash\mathcal{U}_0(\mathcal{P},\mu)$. We show that after applying the interpretation $I_{\mathcal{G},\mu}$ we obtain an infinite rewrite sequence in $\mathcal{U}_0(\mathcal{P},\mu) \cup \pi \cup \mathcal{P}$ in which every rule of $\mathcal{P}$ is used infinitely often. Since all terms in the infinite μ-rewriting sequence belong to $\mathcal{M}_{\infty,\mu}$, they are terminating with respect to $(\mathcal{R},\mu)$ and hence we can indeed apply the interpretation $I_{\mathcal{G},\mu}$. Let $i \geq 1$.

- First consider the dependency pair step $u_i \to_{\mathcal{P}} t_{i+1}$. There is a context-sensitive dependency pair $l \to r \in \mathcal{P}$ and a substitution σ such that $u_i = \sigma(l)$ and $t_{i+1} = \sigma(r)$. We may assume that $\mathcal{D}om(\sigma) \subseteq \mathcal{V}ar(l)$. Since $u_i \in \mathcal{M}^\sharp_{\infty,\mu}$, $\sigma(x)$ is terminating for every variable $x \in \mathcal{V}ar^\mu(l)$. Since right-hand sides of rules in $\mathcal{U}_0(\mathcal{P},\mu)$ lack $\mathcal{G}$-symbols, we have $I_{\mathcal{G},\mu}(\sigma(r)) = \sigma_{I_{\mathcal{G},\mu}}(r)$ using Lemma 2. The same lemma also yield $I_{\mathcal{G},\mu}(\sigma(l)) \hookrightarrow^*_{\pi,\mu_{\mathsf{G}}} \sigma_{I_{\mathcal{G},\mu}}(l)$. Hence, by strong conservativity of pair $l \to r$ and Propositions 2 and 1,

$$I_{\mathcal{G},\mu}(u_i) \hookrightarrow^*_{\pi,\mu_{\mathsf{G}}} \sigma_{I_{\mathcal{G},\mu}}(l) = \overline{\sigma}_{I_{\mathcal{G},\mu}}(l) \to_{\mathcal{P}} \overline{\sigma}_{I_{\mathcal{G},\mu}}(r) \hookrightarrow^*_{\pi,\mu_{\mathsf{G}}} \sigma_{I_{\mathcal{G},\mu}}(r) = I_{\mathcal{G},\mu}(t_{i+1})$$

- Next consider the rewrite sequence $t_i \hookrightarrow^*_{\mathcal{R},\mu} u_i$. Because all terms in this sequence are terminating, we obtain $I_{\mathcal{G},\mu}(t_i) \hookrightarrow^*_{\mathcal{U}_0(\mathcal{P},\mu)\cup\pi} I_{\mathcal{G},\mu}(u_i)$ by repeated applications of Lemma 4.

So we obtain the infinite μ-rewrite sequence

$$I_{\mathcal{G},\mu}(t_1) \hookrightarrow^*_{\mathcal{U}_0(\mathcal{P},\mu)\cup\pi,\mu\cup\mu_{\mathsf{G}}} I_{\mathcal{G},\mu}(u_1) \hookrightarrow^*_{\pi,\mu_{\mathsf{G}}} \circ \to_{\mathcal{P}} I_{\mathcal{G},\mu}(t_2) \hookrightarrow^*_{\mathcal{U}_0(\mathcal{P},\mu)\cup\pi,\mu\cup\mu_{\mathsf{G}}} \cdots$$

in which all rules in $\mathcal{P}$ are infinitely ofen applied. Using the assumption of the theorem, the latter sequence is transformed into a infinite sequence consisting of $\gtrsim$,$\geq$, and infinitely many $>$ steps. Using the compatibility condition, we obtain a contradiction with the well-foundedness of $>$.

Example 3. Consider the following TRS $\mathcal{R}$ [5, Example 4.7.37]:

```
from(X) -> cons(X,from(s(X)))
sel(0,cons(X,XS)) -> X
sel(s(N),cons(X,XS)) -> sel(N,XS)
minus(X,0) -> 0
minus(s(X),s(Y)) -> minus(X,Y)
quot(0,s(Y)) -> 0
quot(s(X),s(Y)) -> s(quot(minus(X,Y),s(Y)))
zWquot(XS,nil) -> nil
zWquot(nil,XS) -> nil
zWquot(cons(X,XS),cons(Y,YS)) -> cons(quot(X,Y),zWquot(XS,YS))
```

together with $\mu(\texttt{cons}) = \{1\}$ and $\mu(f) = \{1,\ldots,ar(f)\}$ for all other symbols f. Note that $\mathcal{R}$ is not μ-conservative (due to the third rule). There are three cycles in the context-sensitive dependency graph:

(C1) {`SEL(s(N),cons(X,XS)) -> SEL(N,XS)`}
(C2) {`MINUS(s(X),s(Y)) -> MINUS(X,Y)`}
(C3) {`QUOT(s(X),s(Y)) -> QUOT(minus(X,Y),s(Y))`}

Whereas cycles C1 and C2 are easily shown harmless by using the subterm criterion (see [2, Section 5]), this is not possible with cycle C3. Since this cycle is conservative and left-linear, it is strongly conservative. Furthermore, the set $\mathcal{U}_0(C3, \mu^\sharp)$ of basic usable rules for C3 contains the following rules

```
minus(X,0) -> 0
minus(s(X),s(Y)) -> minus(X,Y)
```

which are strongly conservative as well. The following polynomial interpretation:

$$[\mathtt{minus}](x, y) = 0 \qquad [\mathtt{0}] = 0 \qquad [\mathtt{s}](x) = 1 \qquad [\mathtt{QUOT}](x, y) = x$$

proves the absence of infinite $(\mathcal{R}, \mathrm{C3}, \mu^\sharp)$-chains. Thus, $\mathcal{R}$ is proved μ-terminating.

5 Conclusions

We have investigated how *usable rules* can be used to improve termination proofs of CSR by dismissing certain rules when the (context-sensitive) dependency pairs approach is used to achieve the proof. The straightforward extension of the standard notion of usable rule (that we have called *basic* usable rules, see Definition 5) does not work for *CSR* even for the quite restrictive class of conservative CS-TRSs. We have introduced the notion of strongly conservative CS-TRS (Definition 7) to fix this problem. Theorem 1 shows that basic usable rules can be used in proofs of termination of strongly conservative CS-TRSs.

We have implemented the techniques described in this paper as part of the tool MU-TERM [4, 18]. Our first experience shows that, even for the restricted framework investigated in this paper, basic usable rules are helpful to improve proofs of termination of *CSR*. For instance, the μ-termination of $\mathcal{R}$ in Example 3 could not be proved by using the previous version of MU-TERM which did not implement any notion of usable rules. The new version of MU-TERM, however, is able to give a simple proof of termination of the example; in fact the proof reported in Example 3 has been automatically obtained by MU-TERM.

An extensive evaluation of them will be obtained after the participation of MU-TERM in the 2007 Termination Competition[3] which will be held soon.

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