# Usable Rules for Context-Sensitive Rewrite Systems*

Raúl Gutiérrez[1], Salvador Lucas[1], and Xavier Urbain[2]

[1] DSIC, Universidad Politécnica de Valencia, Spain
{rgutierrez , slucas}@dsic.upv.es
[2] Cédric-CNAM, ENSIIE, France
urbain@ensiie.fr

**Abstract.** Recently, the dependency pairs (DP) approach has been generalized to context-sensitive rewriting (CSR). Although the *context-sensitive dependency pairs (CS-DP) approach* provides a very good basis for proving termination of CSR, the current developments basically correspond to a ten-years-old DP approach. Thus, the task of adapting all recently introduced dependency pairs techniques to get a more powerful approach becomes an important issue. In this direction, *usable rules* are one of the most interesting and powerful notions. Actually usable rule have been investigated in connection with proofs of *innermost termination* of CSR. However, the existing results apply to a quite restricted class of systems. In this paper, we introduce a notion of usable rules that can be used in proofs of termination of CSR with arbitrary systems. Our benchmarks show that the performance of the CS-DP approach is much better when such usable rules are considered in proofs of termination of CSR.

**Keywords:** Dependency pairs, term rewriting, termination.

## 1 Introduction

During the last decade, the impressive advances in techniques for proving termination of rewriting (remarkably the dependency pairs approach [6,10,13,14]) have succeeded in solving termination problems that stood out of reach for a long time. Roughly speaking, given a Term Rewriting System (TRS) $\mathcal{R}$, the dependency pairs associated to $\mathcal{R}$ give rise to a new TRS $\mathsf{DP}(\mathcal{R})$ which (together with $\mathcal{R}$) determines the so-called *dependency chains* whose finiteness characterizes termination of $\mathcal{R}$. The dependency pairs can be presented as a *dependency graph*, where the absence of infinite chains can be analyzed by considering the *cycles* in the graph. Basically, given a *cycle* $\mathfrak{C} \subseteq \mathsf{DP}(\mathcal{R})$ in the dependency graph, we require $l \succsim r$ for *all* rules in the TRS $\mathcal{R}$, $u \succsim v$ or $u \sqsupset v$ for all dependency pairs $u \to v \in \mathfrak{C}$ and $u \sqsupset v$ for *at least one* $u \to v \in \mathfrak{C}$. Here, $\succsim$ is a stable

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and monotonic quasi-ordering on terms and $\sqsupset$ is a *well-founded* ordering; both of them can be different for the different cycles in the dependency graph.

Termination problems with many rules require more time for getting an answer. Even worse: since termination proofs are usually constrained to succeed within a given (often short) time-out, the proof could get lost due to a lack of time. For those reasons, techniques leading to increase the efficiency (and also the power) of the dependency pairs method, like *usable rules*, appear like a key issue. Usable rules $\mathcal{U}(\mathcal{R},\mathfrak{C}) \subseteq \mathcal{R}$ are associated to a given *cycle* $\mathfrak{C}$ of the dependency graph for $\mathcal{R}$. For particular (but widely used) classes of quasi-orderings $\succeq$, we can restrict the comparisons $l \succeq r$ to rules $l \to r$ in $\mathcal{U}(\mathcal{R},\mathfrak{C})$ instead of using $\mathcal{R}$. Since $\mathcal{U}(\mathcal{R},\mathfrak{C})$ is (usually) smaller than $\mathcal{R}$, proofs of termination often become easier in this way. Usable rules were introduced ten years ago by Arts and Giesl for proving termination of innermost rewriting [5]. The adaptation of the idea to (unrestricted) rewriting [14,17] took some years. A possible reason for that is that the proof of soundness for the innermost and for the unrestricted cases are totally different. The proof of soundness in [14,17] relies on a transformation in which all infinite (minimal) rewrite sequences can be simulated by using a *restricted* set of rules. This transformation was devised by Gramlich for a completely different purpose [15]. Later, Urbain [24] used it (with some modifications) to prove termination of rewriting modules. Finally, Hirokawa and Middeldorp [17] and (independently) Thiemann *et al.* [14] combined this idea with the idea of *usable rules* leading to an improved framework for proving termination of rewriting.

In this paper, we extend the notion of usable rule to the recently introduced dependency pairs approach for context-sensitive rewriting (CS-DPs [2,3]). Proving termination of *context-sensitive rewriting* (*CSR* [18,20]) is an interesting problem with many applications in the fields of term rewriting and programming languages (see [8,12,19,20,22] for further motivations). In *CSR*, a *replacement map* (i.e., a mapping $\mu : \mathcal{F} \to \wp(\mathbb{N})$ satisfying $\mu(f) \subseteq \{1,\ldots,k\}$, for each $k$-ary symbol $f$ of a signature $\mathcal{F}$) is used to discriminate the argument positions on which the rewriting steps are allowed; rewriting at the topmost position is always possible. The following example gives a first intuition of CSR and CS-DPs; full details are given below.

*Example 1.* Consider the following TRS $\mathcal{R}$ borrowed from [7, Example 4.7.37]. The program zips two lists of integers into a single one but instead of pairing the components it rather computes their quotients:

$$\mathtt{sel}(\mathtt{0},\mathtt{cons}(x,xs)) \to x \quad (1) \qquad \mathtt{sel}(\mathtt{s}(n),\mathtt{cons}(x,xs)) \to \mathtt{sel}(n,xs) \quad (7)$$

$$\mathtt{minus}(x,\mathtt{0}) \to x \quad (2) \qquad \mathtt{minus}(\mathtt{s}(x),\mathtt{s}(y)) \to \mathtt{minus}(x,y) \quad (8)$$

$$\mathtt{quot}(\mathtt{0},\mathtt{s}(y)) \to \mathtt{0} \quad (3) \qquad \mathtt{quot}(\mathtt{s}(x),\mathtt{s}(y)) \to \mathtt{s}(\mathtt{quot}(\mathtt{minus}(x,y),\mathtt{s}(y))) \quad (9)$$

$$\mathtt{zWquot}(\mathtt{nil},x) \to \mathtt{nil} \quad (4) \qquad \mathtt{from}(x) \to \mathtt{cons}(x,\mathtt{from}(\mathtt{s}(x))) \quad (10)$$

$$\mathtt{zWquot}(x,\mathtt{nil}) \to \mathtt{nil} \quad (5) \qquad \mathtt{tail}(\mathtt{cons}(x,xs)) \to xs \quad (11)$$

$$\mathtt{head}(\mathtt{cons}(x,xs)) \to x \quad (6)$$

$$\mathtt{zWquot}(\mathtt{cons}(x,xs),\mathtt{cons}(y,ys)) \to \mathtt{cons}(\mathtt{quot}(x,y),\mathtt{zWquot}(xs,ys)) \quad (12)$$

with $\mu(\texttt{cons}) = \{1\}$ and $\mu(f) = \{1, \ldots, ar(f)\}$ for all other symbols $f \in \mathcal{F}$. The set of CS-DPs of $\mathcal{R}$ is:

$$\begin{array}{rlrl}
\texttt{MINUS}(\texttt{s}(x), \texttt{s}(y)) \to \texttt{MINUS}(x, y) & & \texttt{SEL}(\texttt{s}(n), \texttt{cons}(x, xs)) \to \texttt{SEL}(n, xs) \\
\texttt{QUOT}(\texttt{s}(x), \texttt{s}(y)) \to \texttt{MINUS}(x, y) & & \texttt{ZWQUOT}(\texttt{cons}(x, xs), \texttt{cons}(y, ys)) \to \texttt{QUOT}(x, y) \\
\texttt{QUOT}(\texttt{s}(x), \texttt{s}(y)) \to \texttt{QUOT}(\texttt{minus}(x, y), \texttt{s}(y)) & & \texttt{SEL}(\texttt{s}(n), \texttt{cons}(x, xs)) \to xs \\
\texttt{TAIL}(\texttt{cons}(x, xs)) \to xs & & &
\end{array}$$

Note that non-$\mu$-replacing subterms in right-hand sides (e.g., $\texttt{from}(\texttt{s}(x))$ in rule (10)) are *not* considered to build the CS-DPs. Also, in sharp contrast with the unrestricted case, *collapsing* dependency pairs like $\texttt{TAIL}(\texttt{cons}(x, xs)) \to xs$ (where the right-hand side is a variable) are introduced.

Regarding proofs of termination of *innermost CSR*, the straightforward adaptation of usable rules to the context-sensitive setting only works for the so-called *conservative systems* (see [4]) where collapsing dependency pairs do not occur. In Section 3, we show that the standard adaptation does *not* work when proofs of termination of *CSR* are attempted. In Section 4, we provide a general notion of usable rules for proving termination of *CSR*. Although we follow the same proof style, our proof of soundness differs from those in [14,15,17,24] in several aspects that we clarify below. In Section 5, we prove that it is possible to use the *standard* (simpler) notion of usable rules [14,17] in proofs of termination of *CSR* for a restricted class of CS-TRSs: the *strongly* conservative systems. Section 6 provides experimental evaluations and Section 7 concludes. Complete proofs are given in [16].

## 2 Preliminaries

We assume knowledge about standard definitions and notations for term rewriting (including dependency pairs) as given in, e.g., [23]. In the following, we provide some definitions and notation on CSR [18,20] and CS-DPs [2,3].

*Context-Sensitive Rewriting.* Given a TRS $\mathcal{R} = (\mathcal{F}, R)$, we consider the signature $\mathcal{F}$ as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of *constructors* symbols $c \in \mathcal{C}$ and *defined* symbols $f \in \mathcal{D}$ where $\mathcal{D} = \{root(l) \mid l \to r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$. A mapping $\mu : \mathcal{F} \to \wp(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if $\forall f \in \mathcal{F},\ \mu(f) \subseteq \{1, \ldots, ar(f)\}$ [18]. Let $M_{\mathcal{F}}$ be the set of all $\mathcal{F}$-maps ($M_{\mathcal{R}}$ for the $\mathcal{F}$-maps of a TRS $\mathcal{R} = (\mathcal{F}, R)$). A binary relation $R$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is $\mu$-monotonic if $tRs$ implies $f(t_1, \ldots, t_{i-1}, t, \ldots, t_n)\, R\, f(t_1, \ldots, t_{i-1}, s, \ldots, t_n)$ for all $f \in \mathcal{F}$, $i \in \mu(f)$, and $t, s, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. The set of *$\mu$-replacing positions* $\mathcal{P}os^{\mu}(t)$ of $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is: $\mathcal{P}os^{\mu}(t) = \{\epsilon\}$, if $t \in \mathcal{X}$ and $\mathcal{P}os^{\mu}(t) = \{\epsilon\} \cup \bigcup_{i \in \mu(root(t))} i.\mathcal{P}os^{\mu}(t|_i)$, if $t \notin \mathcal{X}$. The set of *$\mu$-replacing* variables of $t$ is $\mathcal{V}ar^{\mu}(t) = \{x \in \mathcal{V}ar(t) \mid \exists p \in \mathcal{P}os^{\mu}(t), t|_p = x\}$. The $\mu$-replacing subterm relation $\trianglerighteq_{\mu}$ is defined by $t \trianglerighteq_{\mu} s$ if there is $p \in \mathcal{P}os^{\mu}(t)$ such that $s = t|_p$. We write $t \rhd_{\mu} s$ if $t \trianglerighteq_{\mu} s$ and $t \neq s$. We write

$t \rhd_{\not\mu} s$ to denote that $s$ is a non-$\mu$-replacing strict subterm of $t$: $t \rhd_{\not\mu} s$ if there is $p \in \mathcal{P}os(t) - Pos^{\mu}(t)$ such that $s = t|_p$. We say that $f \in \mathcal{F}$ is a *hidden symbol* in $l \to r \in R$ if there exists a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ s.t. $r \rhd_{\not\mu} t$ and $root(t) = f$. We say that a variable $x$ is *migrating* in $l \to r \in R$ if $x \in \mathcal{V}ar^{\mu}(r) - \mathcal{V}ar^{\mu}(l)$. In *context-sensitive rewriting* (*CSR* [18]), we (only) rewrite terms at $\mu$*-replacing* positions: $t$ $\mu$-rewrites to $s$, written $t \hookrightarrow_{\mu} s$ (or $t \hookrightarrow_{\mathcal{R},\mu} s$), if $t \xrightarrow{p}_{\mathcal{R}} s$ and $p \in \mathcal{P}os^{\mu}(t)$. A TRS $\mathcal{R}$ is $\mu$-terminating if $\hookrightarrow_{\mu}$ is terminating. A term $t$ is $\mu$-terminating if there is no infinite $\mu$-rewrite sequence $t = t_1 \hookrightarrow_{\mu} t_2 \hookrightarrow_{\mu} \cdots$. A pair $(\mathcal{R}, \mu)$ (or triple $(\mathcal{F}, \mu, R)$) where $\mathcal{R} = (\mathcal{F}, R)$ is a TRS and $\mu \in M_{\mathcal{R}}$ is often called a CS-TRS. We denote $\mathcal{H}(\mathcal{R}, \mu)$ (or just $\mathcal{H}$, if there is no ambiguity) the set of all hidden symbols in $(\mathcal{R}, \mu)$.

*Context-Sensitive Dependency Pairs.* Given a TRS $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ and $\mu \in M_{\mathcal{R}}$, the set of context-sensitive dependency pairs (CS-DPs) is $\mathsf{DP}(\mathcal{R}, \mu) = \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu) \cup \mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu)$, where $\mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$ and $\mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu)$ are obtained as follows: let $f(t_1, \ldots, t_m) \to r \in R$ and $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ be such that $r \trianglerighteq_{\mu} s$. Then (1) if $s = g(s_1, \ldots, s_n)$, for some $g \in \mathcal{D}$, $s_1, \ldots, s_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $l \not\rhd_{\mu} s$, then $f^{\sharp}(t_1, \ldots, t_m) \to g^{\sharp}(s_1, \ldots, s_n) \in \mathsf{DP}_{\mathcal{F}}(R, \mu)$; (2) if $s = x \in \mathcal{V}ar^{\mu}(r) - \mathcal{V}ar^{\mu}(l)$, then $f^{\sharp}(t_1, \ldots, t_m) \to x \in \mathsf{DP}_{\mathcal{X}}(R, \mu)$. Here, $f^{\sharp}$ and $g^{\sharp}$ are new fresh symbols (called *tuple* symbols) associated to the symbols $f$ and $g$ respectively. The CS-DPs in $\mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu)$ are called the *collapsing* CS-DPs. Let $\mathcal{F}^{\sharp} = \mathcal{F} \cup \{f^{\sharp} \mid f \in \mathcal{F}\}$. We extend $\mu \in M_{\mathcal{F}}$ into $\mu^{\sharp} \in M_{\mathcal{F}^{\sharp}}$ by $\mu^{\sharp}(f) = \mu^{\sharp}(f^{\sharp}) = \mu(f)$ for each $f \in \mathcal{F}$. As usual, for $t = f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write $t^{\sharp}$ to denote the *marked* term $f^{\sharp}(t_1, \ldots, t_n)$. Let $\mathcal{T}^{\sharp}(\mathcal{F}, \mathcal{X}) = \{t^{\sharp} \mid t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) - \mathcal{X}\}$ be the set of marked terms. We will also use the set $\mathcal{P}^{\sharp}(\mathcal{F}, \mathcal{X}) = \mathcal{T}^{\sharp}(\mathcal{F}, \mathcal{X}) \times (\mathcal{T}^{\sharp}(\mathcal{F}, \mathcal{X}) \cup \mathcal{X})$. Given $t = f^{\sharp}(t_1, \ldots, t_k) \in \mathcal{T}^{\sharp}(\mathcal{F}, \mathcal{X})$, we write $t^{\natural}$ to denote the *unmarked* term $f(t_1, \ldots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. As usual, capital letters denote marked symbols in examples. A set of pairs $\mathcal{P} \subseteq \mathcal{P}^{\sharp}(\mathcal{F}, \mathcal{X})$ is decomposed into collapsing and non-collapsing pairs ($\mathcal{P}_{\mathcal{X}}$ and $\mathcal{P}_{\mathcal{F}}$, respectively): $\mathcal{P}_{\mathcal{X}} = \{u \to v \in \mathcal{P} \mid v \in \mathcal{X}\}$ and $\mathcal{P}_{\mathcal{F}} = \mathcal{P} - \mathcal{P}_{\mathcal{X}}$.

Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mathcal{P} \subseteq \mathcal{P}^{\sharp}(\mathcal{F}, \mathcal{X})$ and $\mu \in M_{\mathcal{F}}$. An $(\mathcal{R}, \mathcal{P}, \mu^{\sharp})$-chain is a finite or infinite sequence of pairs $u_i \to v_i \in \mathcal{P}$, for $i \geq 1$ such that there is a substitution $\sigma$ satisfying both:

1. $\sigma(v_i) \hookrightarrow^{*}_{\mathcal{R},\mu^{\sharp}} \sigma(u_{i+1})$, if $u_i \to v_i \in \mathcal{P}_{\mathcal{F}}$, and
2. if $u_i \to v_i = u_i \to x_i \in \mathcal{P}_{\mathcal{X}}$, then there is $s_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $\sigma(x_i) \trianglerighteq_{\mu} s_i$ and $s_i^{\sharp} \hookrightarrow^{*}_{\mathcal{R},\mu^{\sharp}} \sigma(u_{i+1})$.

where $\mathcal{V}ar(v_i) \cap \mathcal{V}ar(u_j) = \varnothing$ for all $i \neq j$ (renaming if necessary). Let $\mathcal{M}_{\infty,\mu}$ be the set of minimal non-$\mu$-terminating terms. Then, $t \in \mathcal{M}_{\infty,\mu}$ if $t$ is non-$\mu$-terminating and every strict $\mu$-replacing subterm of $t$ is terminating. We say that an $(\mathcal{R}, \mathcal{P}, \mu^{\sharp})$-chain is *minimal* if for all $i \geq 1$ $\sigma(v_i)$ (whenever $u_i \to v_i \in \mathcal{P}_{\mathcal{F}}$), $s_i^{\sharp}$ (whenever $u_i \to v_i \in \mathcal{P}_{\mathcal{X}}$) are $\mu$-terminating w.r.t. $\mathcal{R}$. A CS-TRS $\mathcal{R} = (\mathcal{F}, \mu, R)$ is $\mu$-terminating if and only if there is no infinite minimal $(\mathcal{R}, \mathsf{DP}(\mathcal{R}, \mu), \mu^{\sharp})$-chain. For finite CS-TRSs, the CS-DPs can be presented as a *context-sensitive dependency graph* (CS-DG); there is an arc from $u \to v \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$ to $u' \to$

$v' \in \mathsf{DP}(\mathcal{R},\mu)$ if there is a substitution $\sigma$ such that $\sigma(v) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u')$; and, there is an arc from $u \to v \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ to $u' \to v' \in \mathsf{DP}(\mathcal{R},\mu)$ if $root(u')^\natural \in \mathcal{H}$. We consider the *strongly connected components* in this graph. A $\mu$-reduction pair $(\succeq, \sqsupset)$ consists of a stable and weakly $\mu$-monotonic quasi-ordering $\succeq$, and a stable and well-founded ordering $\sqsupset$ satisfying $\succeq \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \succeq \subseteq \sqsupset$. From now on, we assume that all CS-TRSs are finite.

## 3 Basic Usable Rules

Consider a set of pairs $\mathcal{P}$ and a CS-TRS $(\mathcal{R},\mu)$. Then, the set of usable rules is the smallest set of rules from $\mathcal{R}$ which are needed to capture all the infinite minimal $(\mathcal{R},\mathcal{P},\mu^\sharp)$-chains. The rules that are responsible for generating the chains between pairs are those rules rooted by symbols that appear in the right-hand side of the pairs below the root symbol. This concept is captured by the definition of direct dependency [14,17,24]:

**Definition 1 (Direct Dependency [14,17]).** *Given a TRS $\mathcal{R} = (\mathcal{F}, R)$, we say that $f \in \mathcal{F}$* directly depends *on $g \in \mathcal{F}$, written $f \rhd_d g$, if there is a rule $l \to r \in R$ with $f = root(l)$ and $g$ occurs in $r$.*

The set of defined symbols in a term $t$ is $\mathcal{D}Fun(t) = \{f \mid \exists p \in \mathcal{P}os(t), f = root(t|_p) \in \mathcal{D}\}$. Let $\rhd_d^*$ be the transitive and reflexive closure of $\rhd_d$. Then, we have:

**Definition 2 (Usable Rules [14,17]).** *For a set $\mathcal{G}$ of symbols we denote by $\mathcal{R} \mid \mathcal{G}$ the set of rewriting rules $l \to r \in \mathcal{R}$ with $root(l) \in \mathcal{G}$. The set $\mathcal{U}(\mathcal{R},t)$ of* usable rules *of a term $t$ is defined as $\mathcal{R} \mid \{g \mid f \rhd_d^* g$ for some $f \in \mathcal{D}Fun(t)\}$. If $\mathcal{P}$ is a set of dependency pairs then $\mathcal{U}(\mathcal{R},\mathcal{P}) = \bigcup_{l \to r \in \mathcal{P}} \mathcal{U}(\mathcal{R},r)$.*

The set $\mathcal{U}(\mathcal{R},\mathcal{P})$ can be used instead of $\mathcal{R}$ when looking for a reduction pair that proves termination of $\mathcal{R}$ [14,17]. Let us now focus on CS-TRSs.

A first attempt to give a notion of usable rules for CSR is given in [4] (basic usable rules) for proofs of *innermost* termination. The results in [4] show that the straightforward generalization of Definition 2 to *CSR* (see Definition 4 below) only applies to *conservative* CS-TRSs and cycles (of CS-DPs), that is, systems having only conservative rules [22]: a rule $l \to r \in R$ is *conservative* if $\mathcal{V}ar^\mu(r) \subseteq \mathcal{V}ar^\mu(l)$. First, we adapt Definition 1 to the CSR setting as follows:

**Definition 3 (Basic $\mu$-Dependency).** *Given a CS-TRS $(\mathcal{F},\mu,R)$, we say that $f \in \mathcal{F}$ has a* basic $\mu$-dependency *on $g \in \mathcal{F}$, written $f \blacktriangleright_{d,\mu} g$, if there is $l \to r \in R$ with $f = root(l)$ and $g$ occurs in $r$ at a $\mu$-replacing position.*

This leads to a straightforward extension of Definition 2. The set of *$\mu$-replacing* defined symbols in a term $t$ is $\mathcal{D}Fun^\mu(t) = \{f \mid \exists p \in \mathcal{P}os^\mu(t), f = root(t|_p) \in \mathcal{D}\}$. Then, we have[1]:

[1] Note that, due to the focus on innermost CSR, [4, Def. 5] slightly differs from ours.

**Definition 4 (Basic Context-Sensitive Usable Rules).** *Let* $\mathcal{R} = (\mathcal{F}, R)$ *be a TRS and* $\mu \in M_{\mathcal{R}}$*. The set* $\mathcal{U}_B(\mathcal{R}, \mu, t)$ *of* basic context-sensitive usable rules *of a term* $t$ *is defined as* $\mathcal{R} \mid \{g \mid f \blacktriangleright^*_{d,\mu} g \text{ for some } f \in \mathcal{D}Fun^{\mu}(t)\}$*, where* $\blacktriangleright^*_{d,\mu}$ *is the transitive and reflexive closure of* $\blacktriangleright_{d,\mu}$*. If* $\mathcal{P} \subseteq \mathcal{P}^{\sharp}(\mathcal{F}, \mathcal{X})$*, then* $\mathcal{U}_B(\mathcal{R}, \mu^{\sharp}, \mathcal{P}) = \bigcup_{l \to r \in \mathcal{P}} \mathcal{U}_B(\mathcal{R}, \mu^{\sharp}, r)$*.*

*Example 2.* (Continuing Example 1) The cycles in the CS-DG are:

$$\{\mathtt{SEL}(\mathtt{s}(n), \mathtt{cons}(x, xs)) \to \mathtt{SEL}(n, xs)\} \qquad (C_1)$$

$$\{\mathtt{MINUS}(\mathtt{s}(x), \mathtt{s}(y)) \to \mathtt{MINUS}(x, y)\} \qquad (C_2)$$

$$\{\mathtt{QUOT}(\mathtt{s}(x), \mathtt{s}(y)) \to \mathtt{QUOT}(\mathtt{minus}(x, y), \mathtt{s}(y))\} \qquad (C_3)$$

Consider the cycle $C_3$; then, $\mathcal{U}_B(\mathcal{R}, \mu^{\sharp}, C_3)$ contains the following rules:

$$\mathtt{minus}(x, \mathtt{0}) \to x \qquad \mathtt{minus}(\mathtt{s}(x), \mathtt{s}(y)) \to \mathtt{minus}(x, y)$$

However, as we are going to see, and in sharp contrast with [4], Definition 4 does not lead to a correct approach for proving termination of *CSR*, *even for conservative TRSs.*

*Example 3.* Consider the TRS $\mathcal{R} = \{\mathtt{f}(\mathtt{c}(x), x) \to \mathtt{f}(x, x), \mathtt{b} \to \mathtt{c}(\mathtt{b})\}$ [4] together with $\mu(\mathtt{f}) = \{1, 2\}$ and $\mu(\mathtt{c}) = \varnothing$. Note that $(\mathcal{R}, \mu)$ is conservative (and innermost $\mu$-terminating, see [4]).

We have a single cycle $C = \{\mathtt{F}(\mathtt{c}(x), x) \to \mathtt{F}(x, x)\}$. According to Definition 4, we have no usable rules because $\mathtt{F}(x, x)$ contains no symbol in $\mathcal{F}$. We could wrongly conclude $\mu$-termination of $(\mathcal{R}, \mu)$, but we have the infinite minimal $(\mathcal{R}, C, \mu^{\sharp})$-chain $\underline{\mathtt{F}(\mathtt{c}(\mathtt{b}), \mathtt{b})} \to \mathtt{F}(\underline{\mathtt{b}}, \mathtt{b}) \hookrightarrow \underline{\mathtt{F}(\mathtt{c}(\mathtt{b}), \mathtt{b})} \to \cdots$.

In the following, we develop a correct definition of usable rules that can be applied to *arbitrary* CS-TRSs.

## 4 Termination of CS-TRSs with Usable Rules

As shown in [14,17], considering the set of *usable rules* instead of all the rules suffices for proving termination of $(\mathcal{R}, \mathcal{P})$-chains (or $\mathcal{P}$-minimal sequences in [17]). In [14,17], an *interpretation* of terms as sequences of their possible reducts is used[2]. The definition of the transformation requires adding new fresh (list constructor) symbols $\bot, \mathtt{g} \notin \mathcal{F}$ and the (projection) rules $\mathtt{g}(x, y) \to x$, $\mathtt{g}(x, y) \to y$ (the $\pi$-rules). In this way, infinite minimal $(\mathcal{R}, \mathcal{P})$-chains can be represented as infinite $(\mathcal{U}(\mathcal{R}, \mathcal{P}) \cup \pi, \mathcal{P})$-chains. We recall here the interpretation definition.

**Definition 5 (Interpretation [14,17]).** *Let* $\mathcal{R} = (\mathcal{F}, R)$ *be a TRS and* $\mathcal{G} \subseteq \mathcal{F}$*. Let* $>$ *be an arbitrary total ordering over* $\mathcal{T}(\mathcal{F}^{\sharp} \cup \{\bot, \mathtt{g}\}, \mathcal{X})$ *where* $\bot$ *is a new constant symbol and* $\mathtt{g}$ *is a new binary symbol. The interpretation* $I_{\mathcal{G}}$ *is a mapping*

[2] This method goes back to [15].

*from terminating terms in* $\mathcal{T}(\mathcal{F}^\sharp, \mathcal{X})$ *to terms in* $\mathcal{T}(\mathcal{F}^\sharp \cup \{\bot, \mathsf{g}\}, \mathcal{X})$ *defined as follows:*

$$I_{\mathcal{G}}(t) = \begin{cases} t & \textit{if } \ t \in \mathcal{X} \\ f(I_{\mathcal{G}}(t_1), \ldots, I_{\mathcal{G}}(t_n)) & \textit{if } \ t = f(t_1 \ldots t_n) \textit{ and } f \notin \mathcal{G} \\ \mathsf{g}(f(I_{\mathcal{G}}(t_1), \ldots, I_{\mathcal{G}}(t_n)), t') & \textit{if } \ t = f(t_1 \ldots t_n) \textit{ and } f \in \mathcal{G} \end{cases}$$

*where* $\qquad t' = \textit{order}\,(\{I_{\mathcal{G}}(u) \mid t \to_{\mathcal{R}} u\})$

$$\textit{order}(T) = \begin{cases} \bot, & \textit{if } T = \varnothing \\ \mathsf{g}(t, \textit{order}(T - \{t\})) & \textit{if } t \textit{ is minimal in } T \textit{ w.r.t. } > \end{cases}$$

The set of symbols $\mathcal{G} \subseteq \mathcal{F}$ in Definition 5 is intended to represent the set of 'non-usable symbols', i.e., symbols which do not occur in the usable rules of the considered set of pairs $\mathcal{P}$. In rewriting, when considering infinite minimal $(\mathcal{R}, \mathcal{P})$-chains, we only deal with terminating terms over $\mathcal{R}$. The interpretation in Definition 5 is defined only for terminating terms because non-terminating terms would yield an infinite term which, actually, does *not* belong to $\mathcal{T}(\mathcal{F}^\sharp \cup \{\bot, \mathsf{g}\}, \mathcal{X})$.

Similarly, we aim at defining a $\mu$-interpretation $I_{\mathcal{G},\mu}$ that allows us to associate an infinite $(\mathcal{U}(\mathcal{R}, \mu^\sharp, \mathcal{P}) \cup \pi, \mathcal{P}, \mu^\sharp)$-chain to each infinite minimal $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chain. Actually, the main problem is that $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chains contain non-$\mu$-terminating terms in non-$\mu$-replacing positions which are potentially able to reach $\mu$-replacing positions: subterms at a $\mu$-replacing position are $\mu$-terminating, but we do not know anything about subterms at non-$\mu$-replacing positions. Hence, we have to define our $\mu$-interpretation $I_{\mathcal{G},\mu}$ both on $\mu$-terminating and non-$\mu$-terminating terms. In [3], we have investigated the structure of infinite $\mu$-rewriting sequences issued from minimal non-$\mu$-terminating terms. Intuitively, one of the main results in [3] states that terms at non-$\mu$-replacing positions in the right-hand side of the rules are essential to track infinite minimal $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chains involving collapsing CS-DPs (see [3, Proposition 3.6]). These terms, by definition, are formed by *hidden symbols*. This observation gives us the key to generalize Definition 5 properly. Following Definition 5, a $\mu$-terminating but non-terminating term generates an infinite list. For this reason, $I_{\mathcal{G}}$ (as a mapping from finite into finite terms) is *not* defined for non-terminating terms.

Regarding our $\mu$-interpretation, if we consider the rules headed by hidden symbols as *usable*, then we are avoiding such infinite $\mu$-interpretations of $\mu$-terminating terms. A non-$\mu$-terminating term $t$ (below a non-$\mu$-replacing position) is treated as if its root symbol does not belong to $\mathcal{G}$, because if it occurs in the $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chain at a $\mu$-replacing position, then $t \trianglerighteq_\mu s$ and $s^\sharp$ becomes the next term in the chain. To simulate all possible derivations of the terms over $(\mathcal{R}, \mu)$ we also need to add to the system the $\pi$-rules. Our new $\mu$-interpretation is:

**Definition 6 ($\mu$-Interpretation).** *Let* $\mathcal{R} = (\mathcal{F}, \mu, R)$ *be a CS-TRS,* $\mathcal{G} \subseteq \mathcal{F}$ *be such that* $\mathcal{G} \cap \mathcal{H} = \varnothing$*. Let* $>$ *be an arbitrary total ordering over* $\mathcal{T}(\mathcal{F}^\sharp \cup \{\bot, \mathsf{g}\}, \mathcal{X})$ *where* $\bot$ *is a new constant symbol and* $\mathsf{g}$ *is a new binary symbol (with* $\mu(\mathsf{g}) = \{1, 2\}$*). The* $\mu$*-interpretation* $I_{\mathcal{G},\mu}$ *is a mapping from arbitrary terms in* $\mathcal{T}(\mathcal{F}^\sharp, \mathcal{X})$

*to terms in* $\mathcal{T}(\mathcal{F}^\sharp \cup \{\bot, \mathsf{g}\}, \mathcal{X})$ *defined as follows:*

$$I_{\mathcal{G},\mu}(t) = \begin{cases} t & \text{if } t \in \mathcal{X} \\ f(I_{\mathcal{G},\mu}(t_1), \ldots, I_{\mathcal{G},\mu}(t_n)) & \text{if } t = f(t_1 \ldots t_n) \text{ and } f \notin \mathcal{G} \\ & \text{or } t \text{ is non-}\mu\text{-terminating} \\ \mathsf{g}(f(I_{\mathcal{G},\mu}(t_1), \ldots, I_{\mathcal{G},\mu}(t_n)), t') & \text{if } t = f(t_1 \ldots t_n) \text{ and } f \in \mathcal{G} \\ & \text{and } t \text{ is } \mu\text{-terminating} \end{cases}$$

$$\text{where} \qquad t' = \mathit{order}\left(\{I_{\mathcal{G},\mu}(u) \mid t \hookrightarrow_{(\mathcal{R},\mu)} u\}\right)$$

$$\mathit{order}(T) = \begin{cases} \bot, & \text{if } T = \varnothing \\ \mathsf{g}(t, \mathit{order}(T - \{t\})) & \text{if } t \text{ is minimal in } T \text{ w.r.t. } > \end{cases}$$

The set $\mathcal{G} \subseteq \mathcal{F}$ in Definition 6 corresponds to the set of non-usable symbols as discussed below. Now, we prove that $I_{\mathcal{G},\mu}$ is well-defined. The most important difference (and essential in our proof) among our $\mu$-interpretation and all previous ones [14,15,17,24] is that $I_{\mathcal{G},\mu}$ is well-defined both for $\mu$-terminating or non-$\mu$-terminating terms.

**Lemma 1.** *Let* $\mathcal{R} = (\mathcal{F}, R)$ *be a TRS,* $\mu \in M_{\mathcal{F}}$ *and let* $\mathcal{G} \subseteq \mathcal{F} - \mathcal{H}$*. Then,* $I_{\mathcal{G},\mu}$ *is well-defined.*

Now, we define an appropriate notion of direct $\mu$-dependency. This is not straightforward as shown in the next example.

*Example 4.* Consider the following conservative non-$\mu$-terminating CS-TRS $\mathcal{R} = \{\mathtt{a}(x,y) \to \mathtt{b}(x,x),\ \mathtt{d}(x,\mathtt{e}) \to \mathtt{a}(x,x),\ \mathtt{b}(x,\mathtt{c}) \to \mathtt{d}(x,x),\ \mathtt{c} \to \mathtt{e}\}$ with $\mu(\mathtt{a}) = \mu(\mathtt{d}) = \{1,2\}$, $\mu(\mathtt{b}) = \{1\}$ and $\mu(\mathtt{c}) = \mu(\mathtt{e}) = \varnothing$. The only cycle consists of the dependency pairs $C = \{\mathtt{A}(x,y) \to \mathtt{B}(x,x),\ \mathtt{D}(x,\mathtt{e}) \to \mathtt{A}(x,x),\ \mathtt{B}(x,\mathtt{c}) \to \mathtt{D}(x,x)\}$.

According to Definition 4, we have no basic usable rules because the right-hand sides of the dependency pairs have no defined symbols. Since we do not consider the rule $\mathtt{c} \to \mathtt{e}$ as usable, we would assume $\mathcal{G} = \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},\mathtt{e}\}$. Then, we *cannot* simulate the infinite minimal $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chain $\underline{\mathtt{A}(\mathtt{c},\mathtt{c})} \hookrightarrow \underline{\mathtt{B}(\mathtt{c},\mathtt{c})} \hookrightarrow \mathtt{D}(\mathtt{c},\underline{\mathtt{c}}) \hookrightarrow \underline{\mathtt{D}(\mathtt{c},\mathtt{e})} \hookrightarrow \underline{\mathtt{A}(\mathtt{c},\mathtt{c})} \hookrightarrow \cdots$ because we have:

$$s = I_{\mathcal{G},\mu}(\mathtt{A}(\mathtt{c},\mathtt{c})) = \underline{\mathtt{A}(g(\mathtt{c},g(\mathtt{e},\bot)),g(\mathtt{c},g(\mathtt{e},\bot)))} \hookrightarrow \mathtt{B}(g(\mathtt{c},g(\mathtt{e},\bot)),g(\mathtt{c},g(\mathtt{e},\bot))) = t$$

The interpreted term $g(\mathtt{c},g(\mathtt{e},\bot))$ at the $\mu$-replacing position 1 of $s$ is 'moved' to a non-$\mu$-replacing position 2 of $t$. Hence, we cannot reduce $t$ on the second argument of $\mathtt{B}$ to obtain the term $\mathtt{B}(g(\mathtt{c},g(\mathtt{e},\bot)),\mathtt{c})$ required for applying the next CS-DP $(\mathtt{B}(x,\mathtt{c}) \to \mathtt{D}(x,x))$ which continues the previous $(\mathcal{R},\mathcal{P},\mu)$-chain.

In order to avoid this problem, we modify Definition 3 to take into account symbols occurring at non-$\mu$-replacing positions in the *left-hand sides* of the rules.

**Definition 7 ($\mu$-Dependency).** *Given a CS-TRS* $\mathcal{R} = (\mathcal{F}, \mu, R)$*, we say that* $f \in \mathcal{F}$ directly $\mu$-depends *on* $g \in \mathcal{F}$*, written* $f \rhd_{d,\mu} g$*, if there is a rule* $l \to r \in R$ *with* $f = \mathit{root}(l)$ *and (1)* $g$ *occurs in* $r$ *at a* $\mu$*-replacing position or (2)* $g$ *occurs in* $l$ *at a non-*$\mu$*-replacing position.*

Remarkably, condition (2) in Definition 7 is not very problematic in practice because most programs are *constructor systems*, which means that no defined symbols occur below the root in the left-hand side of the rules.

Now we are ready to define our notion of usable rules. The set of *non-$\mu$-replacing* defined symbols in a term $t$ is $\mathcal{ND}Fun^\mu(t) = \{f \mid \exists p \in \mathcal{P}os(t)$ and $p \notin \mathcal{P}os^\mu(t), f = root(t|_p) \in \mathcal{D}\}$.

**Definition 8 (Context-Sensitive Usable Rules).** *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_\mathcal{R}$, and $\mathcal{P} \subseteq \mathcal{P}^\sharp(\mathcal{F}, \mathcal{X})$. The set $\mathcal{U}(\mathcal{R}, \mu^\sharp, \mathcal{P})$ of* context-sensitive usable rules *for $\mathcal{P}$ is given by $\mathcal{U}(\mathcal{R}, \mu^\sharp, \mathcal{P}) = \mathcal{U}_\mathcal{H}(\mathcal{R}, \mu) \cup \bigcup_{l \to r \in \mathcal{P}} \mathcal{U}_E(\mathcal{R}, \mu^\sharp, l \to r)$.*

*where* $\mathcal{U}_E(\mathcal{R}, \mu, l \to r) = \mathcal{R} \mid \{g \mid f \rhd^*_{d,\mu} g$ *for some* $f \in \mathcal{D}Fun^\mu(r) \cup \mathcal{ND}Fun^\mu(l)\}$
$\mathcal{U}_\mathcal{H}(\mathcal{R}, \mu) = \mathcal{R} \mid \{g \mid f \rhd^*_{d,\mu} g$ *for some* $f \in \mathcal{H}\}$

Note that $\mathcal{U}_E$ *extends* the notion of usable rules in Definition 2, by taking into account not only dependencies with symbols on the right-hand sides of the rules, but *also with some symbols in proper subterms of the left-hand sides*. We call $\mathcal{U}_E(\mathcal{R}, \mu)$ the set of *extended* usable rules. On the other hand, $\mathcal{U}_\mathcal{H}$ is the set of usable rules corresponding to the hidden symbols. Now, we are ready to formulate and prove our main result in this section.

**Theorem 1.** *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mathcal{P} \subseteq \mathcal{P}^\sharp(\mathcal{F}, \mathcal{X})$, and $\mu \in M_\mathcal{F}$. If there exists a $\mu$-reduction pair $(\gtrsim, \sqsupset)$ such that $\mathcal{U}(\mathcal{R}, \mu^\sharp, \mathcal{P}) \cup \pi \subseteq \gtrsim$, $\mathcal{P} \subseteq \gtrsim \cup \sqsupset$, and*

1. *If $\mathcal{P}_\mathcal{X} = \varnothing$, then $\mathcal{P} \cap \sqsupset \neq \varnothing$*
2. *If $\mathcal{P}_\mathcal{X} \neq \varnothing$, then $\trianglerighteq_\mu \subseteq \gtrsim$, and*
   *(a) $\mathcal{P} \cap \sqsupset \neq \varnothing$ and $f(x_1, \ldots, x_k) \gtrsim f^\sharp(x_1, \ldots, x_k)$ for all $f^\sharp$ in $\mathcal{P}$, or*
   *(b) $f(x_1, \ldots, x_k) \sqsupset f^\sharp(x_1, \ldots, x_k)$ for all $f^\sharp$ in $\mathcal{P}$.*

*Let $P_\sqsupset = \{u \to v \in \mathcal{P} \mid u \sqsupset v\}$. Then there are no infinite minimal $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chains whenever:*

1. *there are no infinite minimal $(\mathcal{R}, \mathcal{P} \setminus \mathcal{P}_\sqsupset, \mu^\sharp)$-chains in in case (1) and in case (2a).*
2. *there are no infinite minimal $(\mathcal{R}, (\mathcal{P} \setminus \mathcal{P}_\mathcal{X}) \setminus \mathcal{P}_\sqsupset, \mu^\sharp)$-chains in case (2b).*

*Proof (Sketch).* By contradiction. Assume that there exists an infinite minimal $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chain $\mathcal{A}$ but there is no infinite minimal $(\mathcal{R}, \mathcal{P} \setminus \mathcal{P}_\sqsupset, \mu^\sharp)$-chains in case (1) and (2a), or there is no infinite minimal $(\mathcal{R}, (\mathcal{P} \setminus \mathcal{P}_\mathcal{X}) \setminus \mathcal{P}_\sqsupset, \mu^\sharp)$-chains in case (2b). We can assume that there is a $\mathcal{P}' \subseteq \mathcal{P}$ such that $\mathcal{A}$ has a tail $\mathcal{B}$ where all pairs are used infinitely often:

$$t_1 \hookrightarrow^*_{\mathcal{R},\mu} u_1 \to_{\mathcal{P}'} \circ \trianglerighteq^\sharp_\mu t_2 \hookrightarrow^*_{\mathcal{R},\mu} u_2 \to_{\mathcal{P}'} \circ \trianglerighteq^\sharp_\mu \cdots$$

where $s \trianglerighteq^\sharp_\mu t$ for $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $t \in \mathcal{T}^\sharp(\mathcal{F}, \mathcal{X})$ means that $s \trianglerighteq_\mu t^\natural$.

Let $\sigma$ be a substitution, we denote by $\sigma_{I_{\mathcal{G},\mu}}$ the substitution that assigns to each variable $x$ the term $I_{\mathcal{G},\mu}(\sigma(x))$ and let $\mathcal{G}$ be the set of defined symbols of $\mathcal{R} \backslash \mathcal{U}(\mathcal{R}, \mu^\sharp, \mathcal{P})$. We show that after applying $I_{\mathcal{G},\mu}$ we get an infinite $(\mathcal{U}(\mathcal{R}, \mu^\sharp, \mathcal{P}) \cup \pi, \mathcal{P}', \mu^\sharp)$-chain. All terms in the infinite chain are $\mu$-terminating w.r.t. $(\mathcal{R}, \mu)$. We proceed by induction. Let $i \geq 1$.

– If we consider the step $u_i \to_{\mathcal{P}'} \circ \trianglerighteq_\mu^\sharp t_{i+1}$, we have two possibilities:
  1. There is $l \to r \in \mathcal{P}'_{\mathcal{F}}$, then we get:
$$I_{\mathcal{G},\mu}(u_i) \hookrightarrow_\pi^* \sigma_{\mathcal{I}_{\mathcal{G},\mu}}(l) \to_{\mathcal{P}'_{\mathcal{F}}} \sigma_{\mathcal{I}_{\mathcal{G},\mu}}(r) = I_{\mathcal{G},\mu}(r) = I_{\mathcal{G},\mu}(t_{i+1})$$
  2. There is an $l \to x \in \mathcal{P}'_{\mathcal{X}}$, then we get:
$$I_{\mathcal{G},\mu}(u_i) \hookrightarrow_\pi^* \sigma_{\mathcal{I}_{\mathcal{G},\mu}}(l) \to_{\mathcal{P}'_{\mathcal{X}}} \sigma_{\mathcal{I}_{\mathcal{G},\mu}}(x) = I_{\mathcal{G},\mu}(\sigma(x))$$
$$\text{and } I_{\mathcal{G},\mu}(\sigma(x)) \trianglerighteq_\mu I_{\mathcal{G},\mu}(t_{i+1}^\natural)$$

– If we consider $t_i \hookrightarrow_{\mathcal{R},\mu}^* u_i$. We get $I_{\mathcal{G},\mu}(t_i) \hookrightarrow_{\mathcal{U}(\mathcal{R},\mu^\sharp,\mathcal{P})\cup\pi}^* I_{\mathcal{G},\mu}(u_i)$.

Therefore we get the infinite $(\mathcal{U}(\mathcal{R},\mu^\sharp,\mathcal{P}),\mathcal{P}',\mu^\sharp)$-chain:

$$I_{\mathcal{G},\mu}(t_1) \hookrightarrow_{\mathcal{U}(\mathcal{R},\mu^\sharp,\mathcal{P})\cup\pi}^* I_{\mathcal{G},\mu}(u_1) \to_{\mathcal{P}'} \circ \trianglerighteq_\mu^\sharp I_{\mathcal{G},\mu}(t_2) \hookrightarrow_{\mathcal{U}(\mathcal{R},\mu^\sharp,\mathcal{P})\cup\pi}^* I_{\mathcal{G},\mu}(u_2) \to_{\mathcal{P}'} \cdots$$

Using the premises of the theorem, by monotonicity and stability of $\gtrsim$, we would have that $I_{\mathcal{G},\mu}(t_i) \gtrsim I_{\mathcal{G},\mu}(u_i)$ for all $i \geq 1$. By stability of $\sqsupset$ (and of $\gtrsim$), we have that $I_{\mathcal{G},\mu}(u_i)(\gtrsim \cup \sqsupset)I_{\mathcal{G},\mu}(t_{i+1})$ for all $i \geq 1$ and $I_{\mathcal{G},\mu}(u_i) \sqsupset I_{\mathcal{G},\mu}(t_{i+1})$ for all $j \in J$ for an infinite set $J = \{j_1,\ldots,j_n,\ldots\}$ of natural numbers $j_1 < j_2 < \ldots < j_n < \ldots$. Now, since $\gtrsim \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \gtrsim \subseteq \sqsupset$, we would obtain an infinite sequence consisting of infinitely many $\sqsupset$-steps. We obtain a contradiction to the well-foundedness of $\sqsupset$. □

*Remark 1.* Notice that (as expected) $\mathcal{U}(\mathcal{R},\mathcal{P},\mu_\top) = \mathcal{U}(\mathcal{R},\mathcal{P})$, i.e., our usable rules for CS-TRSs $(\mathcal{R},\mu)$ coincide with the standard definition (see Definition 2) when $\mu = \mu_\top$ is considered (here, $\mu_\top(f) = \{1,\ldots,ar(f)\}$ for all symbols $f \in \mathcal{F}$, i.e., no replacement restriction is associated to any symbol).

Thanks to Theorem 1, we do not need to make all rules in $\mathcal{R}$ compatible with the weak component $\gtrsim_{\mathcal{P}}$ of a reduction pair $(\gtrsim_{\mathcal{P}},\sqsupset_{\mathcal{P}})$ associated to a given set of pairs $\mathcal{P}$. We just need to consider $\mathcal{U}(\mathcal{R},\mu^\sharp,\mathcal{P})$ (together with the $\pi$-rules).

*Example 5.* (Continuing Examples 1 and 2) Since $\mathcal{H} \cap \mathcal{D} = \{\mathtt{from},\mathtt{zWquot}\}$, we have that $\mathcal{U}(\mathcal{R},\mu^\sharp,C_1)$ is:

$$\begin{array}{rr}
\mathtt{minus}(x,\mathtt{0}) \to x & \mathtt{minus}(\mathtt{s}(x),\mathtt{s}(y)) \to \mathtt{minus}(x,y)\\
\mathtt{quot}(\mathtt{0},\mathtt{s}(y)) \to \mathtt{0} & \mathtt{quot}(\mathtt{s}(x),\mathtt{s}(y)) \to \mathtt{s}(\mathtt{quot}(\mathtt{minus}(x,y),\mathtt{s}(y)))\\
\mathtt{zWquot}(\mathtt{nil},x) \to \mathtt{nil} & \mathtt{from}(x) \to \mathtt{cons}(x,\mathtt{from}(\mathtt{s}(x)))\\
\mathtt{zWquot}(x,\mathtt{nil}) \to \mathtt{nil} &
\end{array}$$
$$\mathtt{zWquot}(\mathtt{cons}(x,xs),\mathtt{cons}(y,ys)) \to \mathtt{cons}(\mathtt{quot}(x,y),\mathtt{zWquot}(xs,ys))$$

According to Theorem 1, the following polynomial interpretation (computed by MU-TERM [1,21]) shows the absence of infinite $(\mathcal{R},C_1,\mu^\sharp)$-chains.

$$\begin{array}{rrr}
[\mathtt{s}](x) = x+1 & [\mathtt{quot}](x,y) = x+y & [\mathtt{minus}](x,y) = 0\\
[\mathtt{from}](x) = 0 & [\mathtt{sel}](x,y) = 0 & [\mathtt{zWquot}](x,y) = x+y\\
[\mathtt{cons}](x,y) = 0 & [\mathtt{0}](x,y) = 0 & [\mathtt{nil}](x,y) = 1\\
[\mathtt{SEL}](x,y) = x & &
\end{array}$$

Note that, if the rules for `sel` were present, we could not find a linear polynomial interpretation for solving this cycle.

*Remark 2.* When considering Definition 8 (usable rules for *CSR*) and Definition 2 (standard usable rules), one can observe that, despite the fact that *CSR* is a *restriction* of rewriting, we can obtain *more* usable rules in the context-sensitive case. Examples 3 and 4 show that this is because rules associated to hidden symbols that do *not* occur in the right-hand sides of the dependency pairs in the considered cycle can play an essential role in capturing infinite $\mu$-rewrite sequences. Thus, for terminating TRSs $\mathcal{R}$, it could be sometimes easier to find a proof of $\mu$-termination of the CS-TRS $(\mathcal{R}, \mu)$ if we ignore the replacement map $\mu$.

## 5 Improving Usable Rules

According to the discussion in Section 3, the notion of basic usable rules is not correct even for conservative systems. Still, since $\mathcal{U}_B(\mathcal{R}, \mu, \mathcal{P})$ is contained in (and is usually smaller than) $\mathcal{U}(\mathcal{R}, \mu, \mathcal{P})$, it is interesting to identify a class of CS-TRSs where basic usable rules can be safely used. Then, we consider a more restrictive kind of conservative CS-TRSs: the *strongly conservative* CS-TRSs.

**Definition 9.** *Let $\mathcal{F}$ be a signature, $\mu \in M_{\mathcal{F}}$ and $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. We denote $\mathcal{V}ar^{\not\mu}(t)$ the set of variables in $t$ occurring at non-$\mu$-replacing positions, i.e., $\mathcal{V}ar^{\not\mu}(t) = \{x \in \mathcal{V}ar(t) \mid t \rhd_{\not\mu} x\}$.*

**Definition 10 (Strongly Conservative).** *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. A rule $l \to r$ is strongly conservative if it is conservative and $\mathcal{V}ar^{\mu}(l) \cap \mathcal{V}ar^{\not\mu}(l) = \mathcal{V}ar^{\mu}(r) \cap \mathcal{V}ar^{\not\mu}(r) = \varnothing$; and $\mathcal{R}$ is strongly conservative if all rules in $\mathcal{R}$ are strongly conservative.*

Linear CS-TRSs trivially satisfy $\mathcal{V}ar^{\mu}(l) \cap \mathcal{V}ar^{\not\mu}(l) = \mathcal{V}ar^{\mu}(r) \cap \mathcal{V}ar^{\not\mu}(r) = \varnothing$. Hence, linear conservative CS-TRSs are strongly conservative. Note that the CS-TRSs in Examples 1 and 3 are not strongly conservative.

Theorem 2 below is the other main result of this paper. It shows that basic usable rules in Definition 4 can be used to improve proofs of termination of *CSR* for strongly conservative CS-TRSs. As discussed in Section 4, if we consider minimal $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chains, then we deal with $\mu$-terminating terms w.r.t. $(\mathcal{R}, \mu)$. We know that any $\mu$-replacing subterm is $\mu$-terminating, but we do not know anything about non-$\mu$-replacing subterms. However, dealing with strongly conservative CS-TRSs, we ensure that non-$\mu$-replacing subterms cannot become $\mu$-replacing after $\mu$-rewriting(s) above them. Hence, we develop a new basic $\mu$-interpretation $I'_{\mathcal{G},\mu}$ where non-$\mu$-replacing positions are not interpreted. In contrast to $I'_{\mathcal{G},\mu}$ (but closer to $I_{\mathcal{G}}$) our new basic $\mu$-interpretation is defined now for $\mu$-terminating terms only.

**Definition 11 (Basic $\mu$-Interpretation).** *Let $(\mathcal{F}, \mu, R)$ be a CS-TRS and $\mathcal{G} \subseteq \mathcal{F}$. Let $>$ be an arbitrary total ordering over $\mathcal{T}(\mathcal{F}^\sharp \cup \{\bot, \mathsf{g}\}, \mathcal{X})$ where $\bot$ is a new constant symbol and $\mathsf{g}$ is a new binary symbol. The basic $\mu$-interpretation $I'_{\mathcal{G},\mu}$ is* *a mapping from $\mu$-terminating terms in $\mathcal{T}(\mathcal{F}^\sharp,\mathcal{X})$ to terms in $\mathcal{T}(\mathcal{F}^\sharp \cup \{\bot,\mathsf{g}\},\mathcal{X})$ defined as follows:*

$$I'_{\mathcal{G},\mu}(t) = \begin{cases} t & \text{if } t \in \mathcal{X} \\ f(I'_{\mathcal{G},\mu,f,1}(t_1),\ldots,I'_{\mathcal{G},\mu,f,n}(t_n)) & \text{if } t = f(t_1\ldots t_n) \text{ and } f \notin \mathcal{G} \\ \mathsf{g}(f(I'_{\mathcal{G},\mu,f,1}(t_1),\ldots,I'_{\mathcal{G},\mu,f,n}(t_n)),t') & \text{if } t = f(t_1\ldots t_n) \text{ and } f \in \mathcal{G} \end{cases}$$

$$\text{where} \quad I'_{\mathcal{G},\mu,f,i}(t) = \begin{cases} I'_{\mathcal{G},\mu}(t) & \text{if } i \in \mu(f) \\ t & \text{if } i \notin \mu(f) \end{cases}$$

$$t' = order\left(\{I'_{\mathcal{G},\mu}(u) \mid t \hookrightarrow_{\mathcal{R},\mu} u\}\right)$$

$$order(T) = \begin{cases} \bot, & \text{if } T = \varnothing \\ \mathsf{g}(t, order(T - \{t\})) & \text{if } t \text{ is minimal in } T \text{ w.r.t. } > \end{cases}$$

It is easy to prove that the basic $\mu$-interpretation is well-defined (finite) for all $\mu$-terminating terms.

**Lemma 2.** *For each $\mu$-terminating term $t$, the term $I'_{\mathcal{G},\mu}(t)$ is finite.*

For the proof of our next theorem, we need some auxiliary definitions and results.

**Definition 12.** *Let $(\mathcal{R},\mu)$ be a CS-TRS and $\sigma$ be a substitution and let $\mathcal{G} \subseteq \mathcal{F}$. We denote by $\sigma_{I'_{\mathcal{G},\mu}} : \mathcal{T}(\mathcal{F},\mathcal{X}) \to \mathcal{T}(\mathcal{F},\mathcal{X})$ a function that, given a term $t$ replaces occurrences of $x \in \mathcal{V}ar(t)$ at position $p$ in $t$ by either $I'_{\mathcal{G},\mu}(\sigma(x))$ if $p \in \mathcal{P}os^\mu(t)$, or $\sigma(x)$ if $p \notin \mathcal{P}os^\mu(t)$.*

**Proposition 1.** *Let $(\mathcal{R},\mu)$ be a CS-TRS and $\sigma$ be a substitution and let $\mathcal{G} \subseteq \mathcal{F}$. Let $t$ be a term such that $\mathcal{V}ar^\mu(t) \cap \mathcal{V}ar^{\not\mu}(t) = \varnothing$. Let $\overline{\sigma}_{I'_{\mathcal{G},\mu},t}$ be a substitution given by*

$$\overline{\sigma}_{I'_{\mathcal{G},\mu},t}(x) = \begin{cases} I'_{\mathcal{G},\mu}(\sigma(x)) & \text{if } x \in \mathcal{V}ar^\mu(t) \\ \sigma(x) & \text{otherwise} \end{cases}$$

*Then, $\overline{\sigma}_{I'_{\mathcal{G},\mu},t}(t) = \sigma_{I'_{\mathcal{G},\mu}}(t)$.*

The following theorem shows that we can safely consider the basic usable rules (with the $\pi$-rules) for proving termination of strongly conservative CS-TRSs.

**Theorem 2.** *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mathcal{P} \subseteq \mathcal{P}^\sharp(\mathcal{F},\mathcal{X})$, and $\mu \in M_\mathcal{F}$. If $\mathcal{P} \cup \mathcal{U}_B(\mathcal{R},\mu^\sharp,\mathcal{P})$ is strongly conservative and there exists a $\mu$-reduction pair $(\gtrsim,\sqsupset)$ such that $\mathcal{U}_B(\mathcal{R},\mu^\sharp,\mathcal{P}) \cup \pi \subseteq \gtrsim$, $\mathcal{P} \subseteq \gtrsim$, and $\mathcal{P} \cap \sqsupset \neq \varnothing$. Let $\mathcal{P}_\sqsupset = \{u \to v \in \mathcal{P} \mid u \sqsupset v\}$. Then there are no infinite minimal $(\mathcal{R},\mathcal{P},\mu^\sharp)$-chains whenever there are no infinite minimal $(\mathcal{R},\mathcal{P} \setminus \mathcal{P}_\sqsupset,\mu^\sharp)$-chains.*

*Proof (Sketch).* By contradiction. Assume that there exists an infinite minimal $(\mathcal{R},\mathcal{P},\mu^\sharp)$-chain $\mathcal{A}$ but there is no infinite minimal $(\mathcal{R},\mathcal{P} \setminus \mathcal{P}_\sqsupset,\mu^\sharp)$-chains. We can assume that there is a $\mathcal{P}' \subseteq \mathcal{P}$ such that $\mathcal{A}$ has a tail $\mathcal{B}$ where all pairs are used infinitely often:

$$t_1 \hookrightarrow^*_{\mathcal{R},\mu} u_1 \to_{\mathcal{P}'} t_2 \hookrightarrow^*_{\mathcal{R},\mu} u_2 \to_{\mathcal{P}'} \cdots$$

After applying the basic $\mu$-interpretation $I'_{\mathcal{G},\mu}$ we obtain an infinite $(\mathcal{U}_B(\mathcal{R},\mu^\sharp,\mathcal{P}) \cup \pi, \mathcal{P}', \mu^\sharp)$-chain. Since all terms in the infinite $(\mathcal{R},\mathcal{P}',\mu^\sharp)$-chain are $\mu$-terminating w.r.t. $(\mathcal{R},\mu)$, we can indeed apply the basic $\mu$-interpretation $I'_{\mathcal{G},\mu}$. Let $i \geq 1$.

- If we consider the pair step $u_i \to_{\mathcal{P}'} t_{i+1}$ we can obtain the following sequence:

$$I'_{\mathcal{G},\mu}(u_i) \hookrightarrow^*_\pi \sigma_{I'_{\mathcal{G},\mu}}(l) \hookrightarrow^*_\pi \overline{\sigma}_{I'_{\mathcal{G},\mu},r}(l) \to_{\mathcal{P}'} \overline{\sigma}_{I'_{\mathcal{G},\mu},r}(r) = \sigma_{I'_{\mathcal{G},\mu}}(r) = I'_{\mathcal{G},\mu}(t_{i+1})$$

- If we consider the rewrite sequence $t_i \hookrightarrow^*_{\mathcal{R},\mu} u_i$. All terms in it are $\mu$-terminating, then we get $I'_{\mathcal{G},\mu}(t_i) \hookrightarrow^*_{\mathcal{U}_B(\mathcal{R},\mu^\sharp,\mathcal{P})\cup\pi} I'_{\mathcal{G},\mu}(u_i)$.

So we obtain the infinite $\mu$-rewrite sequence:

$$I'_{\mathcal{G},\mu}(t_1) \hookrightarrow^*_{\mathcal{U}_B(\mathcal{R},\mu^\sharp,\mathcal{P})\cup\pi} I'_{\mathcal{G},\mu}(u_1) \hookrightarrow^*_\pi \circ \to_{\mathcal{P}'} I'_{\mathcal{G},\mu}(t_2) \hookrightarrow^*_{\mathcal{U}_B(\mathcal{R},\mu^\sharp,\mathcal{P})\cup\pi} \cdots$$

Using the premise of the theorem, it is transformed into an infinite sequence consisting of $\gtrsim$ and infinitely many $\sqsupset$ steps. Using the stability condition, this contradicts the well-foundedness of $\sqsupset$. $\square$

*Example 6.* (Continuing Examples 1, 2 and 5) Cycle $C_1$ is not strongly conservative, but cycles $C_2$ and $C_3$ are strongly conservative. Thus, we can use their basic usable rules. Cycle $C_2$ has no usable rules and we can easily find a polynomial interpretation to show the absence of infinite minimal $(\mathcal{R}, C_2, \mu^\sharp)$-chains:

$$[\mathtt{s}](x) = x+1 \qquad [\mathtt{MINUS}](x,y) = y$$

The basic usable rules $\mathcal{U}_B(\mathcal{R},\mu^\sharp,C_3)$ for $C_3$ are strongly conservative (see Example 2). The following polynomial interpretation proves the absence of infinite $(\mathcal{R}, C_3, \mu^\sharp)$-chains:

$$[\mathtt{0}] = 0 \qquad [\mathtt{s}](x) = x+1 \qquad [\mathtt{minus}](x,y) = x \qquad [\mathtt{QUOT}](x,y) = x$$

Since we dealt with cycle $C_1$ in Example 5, $\mu$-termination of $\mathcal{R}$ is proved. Until now, no tool for proving termination of *CSR* could find a proof for this $\mathcal{R}$ in Example 1. Thanks to the results in this paper, which have been implemented in MU-TERM, we can easily prove $\mu$-termination of $\mathcal{R}$ now.

## 6 Experiments

The techniques described in the previous sections have been implemented as part of the tool MU-TERM [1,21]. In order to make clear the real contribution of the new technique to the performance of the tool, we have implemented three different versions of MU-TERM: (1) a basic version without any kind of usable rules, (2) a second version implementing the results about usable rules described in [4], and (3) a final version that implements the usable rules described in this paper (we do not use the notion in [4] even if the TRS is conservative and innermost equivalent). Version (2) of MU-TERM proves termination of *CSR* as termination

**Table 1.** Comparative among the three MU-TERM versions

| Tool Version | Proved | Total Time | Average Time |
|---|---|---|---|
| **No Usable Rules** | 44/90 | 6.11s | 0.14s |
| **Innermost Usable Rules** | 52/90 | 11.75s | 0.23s |
| **Usable Rules** | 64/90 | 8.91s | 0.14s |

**Table 2.** Comparative over the 44 examples

| Tool Version | Proved | Total Time | Average Time |
|---|---|---|---|
| **No Usable Rules** | 44/90 | 6.11s | 0.14s |
| **Innermost Usable Rules** | 44/90 | 5.03s | 0.11s |
| **Usable Rules** | 44/90 | 3.57s | 0.08s |

of innermost *CSR* when the TRS is orthogonal (see [4,11]), 37 systems, and as termination of *CSR without usable rules* in the rest of cases. In order to keep the set of experiments simple (but still meaningful), we only use linear interpretations with coefficients in $\{0, 1\}$. The usual practice shows that this is already quite powerful (see [9] for recent benchmarks in this sense). The benchmarks have been executed in a completely automatic way with a timeout of 1 minute on each of the 90 examples in the Context-Sensitive Rewriting subcategory of the 2007 Termination Competition[3]. A complete report of our experiments can be found in:

`http://www.dsic.upv.es/~rgutierrez/muterm/rta08/benchmarks.html`

Table 1 summarizes our results. Our notion of usable rules works pretty well: we are able to prove 20 more examples than without any usable rules, and 12 more than with the restricted notion in [4]. Furthermore, a comparison over the 44 examples solved by all the three versions of MU-TERM, we see that version (3) of MU-TERM is 43% faster than (1) and 27% faster than (2) (see Table 2).

## 7 Conclusions

We have investigated how *usable rules* can be used to improve termination proofs of CSR when the (context-sensitive) dependency pairs approach is used to achieve the proof. In contrast to [4], the straightforward extension of the standard notion of usable rules (called here *basic* usable rules, see Definition 4) does not work for *CSR* even for the quite restrictive class of conservative (cycles of) CS-TRSs. We have shown how to adapt the notion of usable rules for their use with arbitrary CS-TRSs (Definition 8). Theorem 1 shows that the new notion of usable rules can be used in proofs of termination of CS-TRSs. Here, although the proof uses a transformation in the very same style than [14,17], the definition of the transformation is quite different from the usual one in that it applies to

[3] See `http://www.lri.fr/~marche/termination-competition/2007`

arbitrary terms, not only terminating ones. To our knowledge, this is the first time that Gramlich's transformation [15] is adapted and used in that way. We have also introduced the notion of strongly conservative rule and CS-TRS (Definition 10). Theorem 2 shows that basic usable rules can be used in proofs of termination involving strongly conservative cycles and rules. Although we follow the proof scheme in [14,17], a number of subtleties have to be carefully addressed before getting a correct adaptation of the proof.

We have implemented our techniques as part of the tool MU-TERM [1,21]. Our experiments show that usable rules are helpful to improve proofs of termination of *CSR*. Regarding the previous work on usable rules for innermost *CSR* [4], this paper provides a fully general definition which is not restricted to conservative systems. Actually, as we show in our experiments, our framework is more powerful in practice than trying to prove termination of *CSR* as innermost termination of *CSR* with the restricted notion of usable rules in [4]. Actually, our results provide a basis for refining the notion of usable rules in the *innermost* setting, thus hopefully allowing a generalization of the results in [4].

Finally, usable rules were an essential ingredient for MU-TERM in winning the context-sensitive subcategory of the 2007 competition of termination tools.

## References

1. Alarcón, B., Gutiérrez, R., Iborra, J., Lucas, S.: Proving Termination of Context-Sensitive Rewriting with MU-TERM. ENTCS 188, 105–115 (2007)
2. Alarcón, B., Gutiérrez, R., Lucas, S.: Context-Sensitive Dependency Pairs. In: Arun-Kumar, S., Garg, N. (eds.) FSTTCS 2006. LNCS, vol. 4337, pp. 297–308. Springer, Heidelberg (2006)
3. Alarcón, B., Gutiérrez, R., Lucas, S.: Improving the Context-Sensitive Dependency Graph. ENTCS 188, 91–103 (2007)
4. Alarcón, B., Lucas, S.: Termination of Innermost Context-Sensitive Rewriting Using Dependency Pairs. In: Konev, B., Wolter, F. (eds.) FroCos 2007. LNCS (LNAI), vol. 4720, pp. 73–87. Springer, Heidelberg (2007)
5. Arts, T., Giesl, J.: Proving Innermost Normalisation Automatically. In: Comon, H. (ed.) RTA 1997. LNCS, vol. 1232, pp. 157–171. Springer, Heidelberg (1997)
6. Arts, T., Giesl, J.: Termination of Term Rewriting Using Dependency Pairs. Theoretical Computer Science 236(1–2), 133–178 (2000)
7. Borralleras, C.: Ordering-Based Methods for Proving Termination Automatically. PhD thesis, Departament de Llenguatges i Sistemes Informàtics, UPC (2003)
8. Durán, F., Lucas, S., Meseguer, J., Marché, C., Urbain, X.: Proving Operational Termination of Membership Equational Programs. Higher-Order and Symbolic Computation (to appear, 2008)
9. Fuhs, C., Giesl, J., Middeldorp, A., Schneider-Kamp, P., Thiemann, R., Zankl, H.: SAT Solving for Termination Analysis with Polynomial Interpretations. In: Marques-Silva, J., Sakallah, K.A. (eds.) SAT 2007. LNCS, vol. 4501, pp. 340–354. Springer, Heidelberg (2007)
10. Giesl, J., Arts, T., Ohlebusch, E.: Modular Termination Proofs for Rewriting Using Dependency Pairs. Journal of Symbolic Computation 34(1), 21–58 (2002)
11. Giesl, J., Middeldorp, A.: Innermost Termination of Context-Sensitive Rewriting. In: Ito, M., Toyama, M. (eds.) DLT 2002. LNCS, vol. 2450, pp. 231–244. Springer, Heidelberg (2003)

12. Giesl, J., Middeldorp, A.: Transformation Techniques for Context-Sensitive Rewrite Systems. Journal of Functional Programming 14(4), 379–427 (2004)
13. Giesl, J., Thiemann, R., Schneider-Kamp, P.: The Dependency Pair Framework: Combining Techniques for Automated Termination Proofs. In: Baader, F., Voronkov, A. (eds.) LPAR 2004. LNCS (LNAI), vol. 3452, pp. 301–331. Springer, Heidelberg (2005)
14. Giesl, J., Thiemann, R., Schneider-Kamp, P., Falke, S.: Mechanizing and Improving Dependency Pairs. Journal of Automatic Reasoning 37(3), 155–203 (2006)
15. Gramlich, B.: Generalized Sufficient Conditions for Modular Termination of Rewriting. Applicable Algebra in Engineering, Communication and Computing 5, 131–151 (1994)
16. Gutiérrez, R., Lucas, S., Urbain, X.: Usable Rules for Context-Sensitive Rewrite System, DSIC-II/03/08. Technical report, UPV (2008)
17. Hirokawa, N., Middeldorp, A.: Tyrolean Termination Tool: Techniques and Features. Information and Computation 205(4), 474–511 (2007)
18. Lucas, S.: Context-Sensitive Computations in Functional and Functional Logic Programs. Journal of Functional and Logic Programming 1998(1), 1–61 (1998)
19. Lucas, S.: Termination of on-demand rewriting and termination of OBJ programs. In: Proc. of PPDP 2001, pp. 82–93. ACM Press, New York (2001)
20. Lucas, S.: Context-Sensitive Rewriting Strategies. Information and Computation 178(1), 293–343 (2002)
21. Lucas, S.: MU-TERM: A Tool for Proving Termination of Context-Sensitive Rewriting. In: van Oostrom, V. (ed.) RTA 2004. LNCS, vol. 3091, pp. 200–209. Springer, Heidelberg (2004), `http://zenon.dsic.upv.es/muterm/`
22. Lucas, S.: Proving Termination of Context-Sensitive Rewriting by Transformation. Information and Computation 204(12), 1782–1846 (2006)
23. Ohlebusch, E.: Advanced Topics in Term Rewriting. Springer, Heidelberg (2002)
24. Urbain, X.: Modular & Incremental Automated Termination Proofs. Journal of Automated Reasoning 32(4), 315–355 (2004)